UNIVERSITÄT LEIPZIG
INSTITUT FÜR THEORETISCHE PHYSIK

Theoretische Mechanik

Set of special problems

Solutions

1 Problem

The Lagrange function for a particle on a sphere with a radius R is

$$L = \frac{1}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right)$$

which leads to

$$H = \frac{1}{2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right).$$

We now look for the action variables

$$\begin{aligned} J_\varphi &= \oint K d\varphi = 2\pi K \\ J_\theta &= \oint \sqrt{2E - \frac{K^2}{\sin^2\theta}}\, d\theta, \end{aligned}$$

where $K = p_\varphi$ is a constant. In the θ-integral we substitute $u = \cot\theta$ and find

$$J_\theta = -iK \oint \sqrt{(u-A)(u+A)}\, \frac{du}{1+u^2},$$

where $A^2 + 1 = \frac{2E}{K^2}$. The contour of integration and branch cuts are shown in the Fig. 1. This

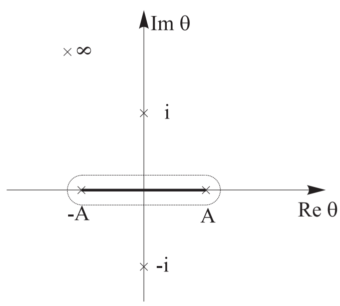

Figure 1: Complex θ integration with the choice of branch cut.

contour needs to be traversed in the positive direction, which is equivalent to traversing it in negative direction and regarding the cut as being outside of it. By the theorem of residues the integral is equal to $-2\pi i \sum(Res)$, where the residua at $u = \pm i$ and $u = \infty$ contribute. With $f(u) = [(u-A)(u+A)]^{\frac{1}{2}} \frac{1}{1+u^2}$ we find

$$Res(f,i) = i\frac{\sqrt{A^2+1}}{2i} = \frac{\sqrt{A^2+1}}{2}$$

$$Res(f,-i) = -i\frac{\sqrt{A^2+1}}{-2i} = \frac{\sqrt{A^2+1}}{2}$$

(the phase in positive/negative imaginary direction is $\pm\pi/2$). We note, that for $|u| > 1$ there is no discontinuity of the phase of $f(u)$ and by introducing $z = 1/u$ we find

$$Res(f, \infty) = -1.$$

Consequently $J_\theta = 2\pi(\sqrt{2E} - K$, so that

$$E = \frac{(J_\varphi + J_\theta)^2}{8\pi^2}$$

and both periods $T_\varphi = \partial E/\partial J_\varphi = \partial E/\partial J_\theta = T_\theta$ are equal.

2 Problem

The vector potential

$$A_x = -\frac{yB}{2r}, \qquad A_y = \frac{xB}{2r}$$

produces the field $B_z = BR/r$ (with $r = \sqrt{x^2+y^2}$). The Largrange-function

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{e}{c}\dot{x}^i A_i,$$

with $A_\varphi = -yA_x + xA_y = \frac{BR}{2}r$ leads to the Hamilton-function

$$H = \frac{1}{2m}\left[p_r^2 + \frac{(p_\varphi - \lambda r)^2}{r^2}\right],$$

where $\lambda = \frac{eBR}{2c}$. In the Hamilton-Jacobi formulation, the generating function is postulated to be

$$S = W(r) + K\varphi - \mathcal{E}t$$

$$W(r) = \int^r dr\sqrt{2E - \frac{(K-\lambda r)^2}{r^2}}$$

(we use $E = m\mathcal{E}$.) The multiperiodic motion occurs for the values of E, K and r where the radicand is positive, which happens only if $\lambda^2 - 2E > 0$. The turning points r_- and r_+ are the zeros of the radicand. We introduce the action variables:

$$J_\varphi = \oint \frac{\partial(K\varphi)}{\partial\varphi}\, d\varphi = 2\pi K$$

$$J_r = \oint dr \sqrt{2E - \frac{(K-\lambda r)^2}{r^2}},$$

with the last integral understood as over the path $r_- \to r_+ \to r_-$. We write it as

$$J_r = i\sqrt{\lambda^2 - 2E} \oint \frac{1}{r^2} \sqrt{(r-r_-)(r_+-r)}\, dr$$

with

$$\frac{r_- + r_-}{2} = \frac{K\lambda}{\lambda^2 - 2E}, \qquad r_- r_+ = \frac{K^2}{\lambda^2 - 2E}$$

given by the simplest Viete's formulas. The relevant contours and chosen branch cuts are illustrated in Fig. 2. The residues are

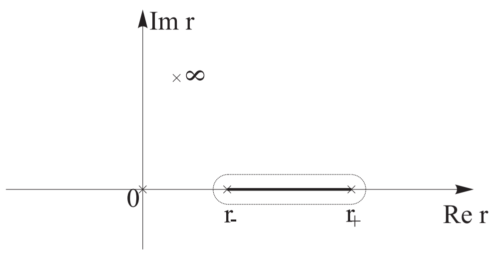

Figure 2: Complex integration with the choice of the branch cut.

$$Res(r=0) = \sqrt{r_- r_+}, \qquad Res(r=\infty) = \frac{r_- + r_+}{2}$$

We get

$$J_r = i\sqrt{\lambda^2 - 2E}(-2\pi i)\sum(Res) = 2\pi \left(\frac{K\lambda}{\sqrt{\lambda^2 - 2E}} + K \right)$$

from which it follows that

$$E = \frac{\lambda^2}{2}\left[1 - \frac{1}{(J_r/J_\varphi - 1)^2}\right].$$

Both periods can now be computed from

$$T_r = \frac{\partial E}{\partial J_r}, \qquad T_\varphi = \frac{\partial E}{\partial J_\varphi}.$$