Constructive Algebraic Quantum Field Theory

In AQFT the basic objects are

- algebras $\mathcal{A}(\mathcal{O})$ of observables associated with open $\mathcal{O} \subset \mathcal{M}^d$
- states ω on $\mathcal{A} = \bigvee_{\mathcal{O}} \mathcal{A}(\mathcal{O})$

subject to certain physically motivated conditions

From this structure we have learned how to extract physical information, *e.g.*

- superselection structure, charges and charge–carrying fields, statistics
- existence and properties of thermal equilibrium states
- particle interpretation (from infraparticles to ultraparticles), collision cross sections
- quantum fields locally associated with $\{\mathcal{A}(\mathcal{O})\}$, operator product expansions, short distance structure
- representation $U(\mathcal{P}_+^\uparrow)$ which acts covariantly on $\{\mathcal{A}(\mathcal{O})\}$ and satisfies the spectrum condition, dynamics
- the space–time itself

The conceptual advances made in AQFT in the past 50 years are truly admirable, but there is a fly in the ointment. We still do not have physically relevant interacting models for $d = 4$, much less models which reproduce the results of the computation schemes developed by elementary particle theorists. A crucial problem for the scientific program of AQFT (and, I believe, its survival) is to demonstrate the existence of such models.

One strategy (Glimm & Jaffe, *etc.*) for obtaining such models is to construct first suitable local quantum fields by beginning with certain well–defined cutoff theories and controlling the limits as the cutoffs are removed. The net $\{\mathcal{A}(\mathcal{O})\}$ is then generated from these local fields. These fields are constructed in a concrete representation, which then determines a folium of states on $\mathcal{A}$.

This strategy relies heavily on (quasi–)classical intuitions.

Purely Algebraic (and Quantum) Construction of Quantum Field Models

(Further details at 14:00 on Tuesday in the ITP.)

- Strategies
- Concrete implementation of a strategy

Purely Algebraic (and Quantum) Construction Strategies

These strategies go beyond the quasi–classical picture and have already resulted in models which either **cannot** be constructed by quasi–classical methods **at all**, or can only be done so (even in principle) with a **much greater** expenditure of effort than that required by the purely algebraic methods.

Crucial to all purely algebraic constructions so far have been algebras associated with wedge regions.

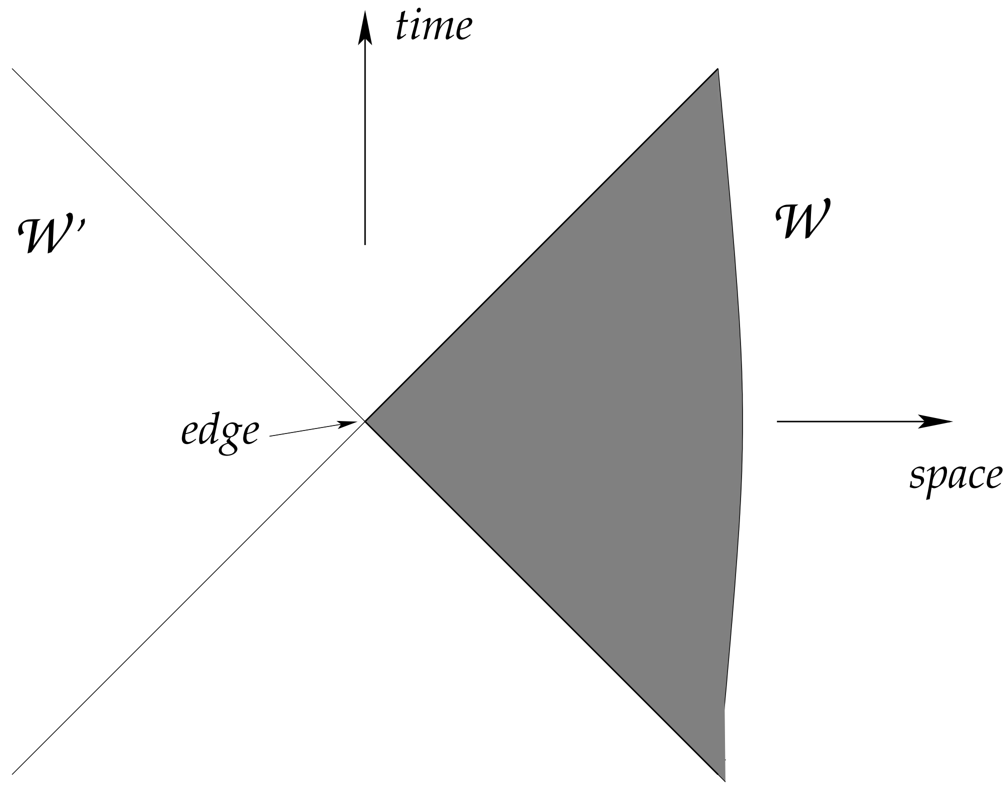

Figure 1: A wedge $\mathcal{W}$, its causal complement $\mathcal{W}'$ and their common edge

One purely algebraic strategy (Wiesbrock) is to construct a small number of von Neumann algebras satisfying certain relations (involving their modular structure with respect to a certain state vector) and from them to generate the net $\{\mathcal{A}(\mathcal{O})\}$ (in which the initial algebras are re-interpreted as certain wedge algebras).

Another algebraic strategy (Buchholz & Lechner, Buchholz & S., Longo & Rehren) is to construct a net of nonlocal wedge algebras and use suitable relative commutants of such algebras to construct a net $\{\mathcal{A}(\mathcal{O})\}$ of local algebras.

Yet another algebraic strategy (Buchholz & S.) is to construct a representation $U(\mathcal{P}_+^\uparrow)$ satisfying the spectrum condition (it would suffice to do so on Fock space) and for a fixed wedge $\mathcal{W}_0$ (*e.g.* $W_R \doteq \{x \in \mathcal{M}^4 \mid x_1 > |x_0|\}$) exhibit an algebra $\mathfrak{G}$ which satisfies the consistency conditions:

- $U(\lambda)\mathfrak{G}U(\lambda)^{-1} \subset \mathfrak{G}$ whenever $\lambda\mathcal{W}_0 \subset \mathcal{W}_0$ for $\lambda \in \mathcal{P}_+^\uparrow$.
- $U(\lambda')\mathfrak{G}U(\lambda')^{-1} \subset \mathfrak{G}'$ whenever $\lambda'\,\mathcal{W}_0 \subset \mathcal{W}_0'$ for $\lambda' \in \mathcal{P}_+^\uparrow$.

Any algebra $\mathfrak{G}$ satisfying these conditions is the germ of a quantum field theory in the following sense: setting

$$\mathcal{A}(\mathcal{W}) \doteq U(\lambda)\mathfrak{G}U(\lambda)^{-1}\,,$$

where $\lambda \in \mathcal{P}_+^\uparrow$ satisfies $\mathcal{W} = \lambda\mathcal{W}_0$ for given $\mathcal{W} \in \boldsymbol{\mathcal{W}}$, then the definition of the wedge algebras $\mathcal{A}(\mathcal{W})$ is consistent and satisfies the conditions of isotony, covariance and locality.

The algebras corresponding to arbitrary causally closed convex regions $\mathcal{O}$ can then consistently be defined by taking intersections of wedge algebras:

$$\mathcal{A}(\mathcal{O}) \doteq \bigcap_{\mathcal{W}\supset\mathcal{O}} \mathcal{A}(\mathcal{W})\,. \tag{1}$$

Conversely, any asymptotically complete quantum field theory with the given particle content fixes an algebra $\mathfrak{G}$ (namely the observable algebra associated with $\mathcal{W}_0$) with the above properties. Thus any decent quantum field theory can, in principle, be presented in this way. However, at present a dynamical principle by which the algebra $\mathfrak{G}$ can be selected is missing.

An important question for both the interpretation and the physical content of the model is: For which classes of regions $\mathcal{O}$ is the intersection (1) nontrivial, resp. is a vector invariant under $U(\mathcal{P}_+^\uparrow)$ cyclic for (1)?

Three Concrete, Purely Algebraic Construction Techniques

Modular Localization

Brunetti, Guido & Longo (2002) , Schroer (1997 . . .) ($d \geq 2$)

Initial data: strongly continuous (anti)unitary representation $U_1(\mathcal{P}_+)$ on $\mathcal{H}_1$ satisfying the spectrum condition. Yields $U(\mathcal{P}_+)$ on $\mathcal{H}$, bosonic Fock space with one–particle space $\mathcal{H}_1$.

Let $U_1(v_R(t)) = e^{itK_1}$ and define

$$\Delta_{W_R}^{1/2} \doteq e^{\pi K_1} \, , \; J_{W_R} \doteq U_1(\theta_R)$$

$$S_{W_R} \doteq U_1(\theta_R) e^{\pi K_1} = J_{W_R} \Delta_{W_R}^{1/2}$$

$$\mathcal{K}_{W_R} \doteq \{ f \in D(S_{W_R}) \mid S_{W_R} f = f \}$$

(real subspace of $\mathcal{H}_1$)

$$e^h \doteq \oplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} h^{\otimes n} \, , \, h \in \mathcal{H}_1$$

$$V(f) e^0 \doteq e^{-\frac{1}{4}\|f\|^2} e^{\frac{i}{\sqrt{2}} f} \, , \, f \in \mathcal{H}_1$$

$$V(f) V(g) \doteq e^{-\frac{i}{2}\mathrm{Im}\langle f,g \rangle} V(f+g) \, , \, f, g \in \mathcal{H}_1$$

(unitary operators on $\mathcal{H}$)

Define $\mathcal{A}(\mathcal{W}_R) \doteq \{V(f) \mid f \in \mathcal{K}_{W_R}\}''$.

Theorem 1 (Brunetti, Guido & Longo, 2002)**.** *The structure* $(U(\mathcal{P}_+^\uparrow), \mathcal{A}(\mathcal{W}_R))$ *satisfies the consistency conditions and determines a local, Poincaré covariant net* $\{\mathcal{A}(\mathcal{O})\}$*. The Fock vacuum vector* Ω *is cyclic for* $\mathcal{A}(\mathcal{W})$*, for any wedge* $\mathcal{W}$*, and for* $\mathcal{A}(\mathcal{C})$*, for any spacelike cone* $\mathcal{C}$*. In the case that* $U_1(\mathcal{P}_+)$ *is irreducible with half integer spin,* Ω *is also cyclic for* $\mathcal{A}(\mathcal{O})$*, for any double cone* $\mathcal{O}$*.*

Further results were obtained by Fassarella & Schroer (2002) and Mund, Schroer & Yngvason (2004) in the special case of massless, “infinite spin” representations U_1. In particular, Ω is not cyclic for double cone algebras. So there is no local quantum field associated with the net in this case. (Yngvason, 1970)

Factorizing S–Matrix Models — Using Nonlocal Fields to Define Local Nets

Schroer (1997, 1999), Lechner (2003–2008)

It turns out that this approach is very productive for $d = 2$, but it results in theories with trivial scattering for $d = 4$ (Borchers, Buchholz & Schroer (2001)).

Important Lesson: The construction is implemented using easily constructed nonlocal fields to obtain nonlocal wedge algebras $\mathcal{A}(\mathcal{W})$. For $d = 2$, if $\mathcal{W}_1, \mathcal{W}_2 \in \boldsymbol{\mathcal{W}}$ satisfy $\overline{\mathcal{W}_2} \subset \mathcal{W}_1$, then $\mathcal{O} = \mathcal{W}_1 \cap \mathcal{W}_2{}'$ is a double cone. And every double cone can be obtained in this manner. Defining for such a double cone

$$\mathcal{A}_0(\mathcal{O}) \doteq \mathcal{A}(\mathcal{W}_1) \cap \mathcal{A}(\mathcal{W}_2)'$$

results in a local, Poincaré covariant net, and the vacuum is cyclic for the resulting double cone algebras (Lechner (2008)). But arguments (Smirnov (1992), Schroer (1999)) and examples (McCoy, Tracy & Wu (1977)) indicate that the associated local fields must be infinite power series in the (simple) nonlocal fields.

Buchholz & S. studied a special case of these models for $d \geq 2$. Indeed, acting on antisymmetric Fock space is a scalar Fermi field ϕ satisfying (Jost, 1964)

$$\phi\big((\Box + m^2)f\big) = 0\,,$$

and

$$\big\{\phi(f), \phi(g)\big\} \doteq \phi(f)\phi(g) + \phi(g)\phi(f) = \big(\langle \overline{g} | f \rangle + \langle \overline{f} | g \rangle\big) \cdot \mathbf{1}\,.$$

These fields are used to obtain a maximally nonlocal wedge net $\{\mathcal{A}(\mathcal{W})\}_{\mathcal{W} \in \boldsymbol{\mathcal{W}}}$.

For $d = 4$, such an $\mathcal{O} = \mathcal{W}_1 \cap \mathcal{W}_2{}'$ is unbounded (spacelike cylinder). Defining

$$\mathcal{A}_0(\mathcal{O}) \doteq \mathcal{A}(\mathcal{W}_1) \cap \mathcal{A}(\mathcal{W}_2)' \,,$$

one has

Theorem 2 (Buchholz & S., 2007)**.** *For $d = 4$, the net of spacelike cylinder algebras determined by any fixed coherent family $\boldsymbol{\mathcal{W}}_0$ of wedges is local, nontrivial and is covariant under $U(\mathcal{P}_0)$, where $\mathcal{P}_0$ is the largest subgroup of $\mathcal{P}_+$ leaving the set $\boldsymbol{\mathcal{W}}_0$ fixed. Two body scattering can be defined, but is trivial. For $d = 2$, Haag–Ruelle scattering theory is applicable, and one has*

$$S = (-\mathbf{1})^{N(N-1)/2} \,.$$

Deformations

Grosse & Lechner (2007, 2008), Buchholz & S. (2008,2009) ($d \geq 2$)

In order to construct concrete models on noncommutative Minkowski space, Grosse and Lechner (2007) performed a certain deformation upon the free quantum field on Minkowski space. They remarked that the resultant net could be interpreted either on Minkowski space or on noncommutative Minkowski space. Buchholz and S. realized that the deformation of Grosse and Lechner was a special case of a deformation which could be applied to (essentially) any model, not just the free field.

Begin with an initial net $\{\mathcal{A}(\mathcal{O})\}$ and representation $U(\mathcal{P}_+^\uparrow)$ satisfying the spectrum condition. Consider the set $\mathfrak{F}$ of all operators F which are smooth under the adjoint action $\alpha_x(F) \doteq U(x)FU(x)^{-1}$, $x \in \mathbb{R}^4$. With

$$U(x) = e^{iPx} = \int e^{ipx} \, dE(p) \, , \quad x \in \mathbb{R}^d \, ,$$

and any skew–symmetric 4×4–matrix Q, define for any $F \in \mathfrak{F}$

$$_QF \doteq \int \alpha_{Qp}(F) \, dE(p) \, , \quad F_Q \doteq \int dE(p) \, \alpha_{Qp}(F) \, . \qquad (2)$$

The operators $_QF$ and F_Q are typically unbounded, even if F is bounded.

Theorem 3 (Buchholz & S., 2008)**.** *For all* $F \in \mathfrak{F}$ *one has:*

(a) ${}_{Q}F = F_Q$

(b) ${F^*}_Q \subset {F_Q}^*$

(c) $(F_{Q_1})_{Q_2} = F_{Q_1+Q_2}$ *for all skew–symmetric* Q_1, Q_2

(d) $\alpha_\lambda(F_Q) = \big(\alpha_\lambda(F)\big)_{\Lambda Q \Lambda^{-1}}$ *for all* $\lambda \in \mathcal{P}_+^\uparrow$

(e) ${}_{Q}F\Omega = F_Q\Omega = F\Omega$

Now consider Q which, with respect to the chosen proper coordinates (those for which $\mathcal{W}_0 = \mathcal{W}_R$), has the form

$$Q \doteq \begin{pmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \\ 0 & 0 & -\rho & 0 \end{pmatrix}$$

for some fixed $\kappa > 0$, $\rho \in \mathbb{R}$. Note that this matrix is skew symmetric with respect to the Lorentz inner product.

For such Q's Grosse and Lechner (2007) observed

(i) Let $\lambda = (\Lambda, x) \in \mathcal{P}_+^\uparrow$ be such that $\lambda \mathcal{W}_0 \subset \mathcal{W}_0$. Then $\Lambda Q \Lambda^{-1} = Q$.

(ii) Let $\lambda' = (\Lambda', x') \in \mathcal{P}_+^\uparrow$ be such that $\lambda' \mathcal{W}_0 \subset \mathcal{W}_0{}'$. Then $\Lambda' Q \Lambda'^{-1} = -Q$.

(iii) $QV_+ = \mathcal{W}_0$.

Definition 0.1. *Let* $\mathcal{W} \in \boldsymbol{\mathcal{W}}$ *and let* $\lambda = (\Lambda, x) \in \mathcal{P}_+^\uparrow$ *be such that* $\mathcal{W} = \lambda \mathcal{W}_0$*. Let* $\mathfrak{A}_Q(\mathcal{W})$ *be the polynomial algebra generated by all warped operators* $A_{\Lambda Q \Lambda^{-1}}$ *with* $A \in \mathfrak{A}(\mathcal{W}) \doteq \mathcal{A}(\mathcal{W}) \cap \mathfrak{F}$*.*

Theorem 4 (Buchholz & S., 2008, 2009)**.** *Let* $\mathcal{A}(\mathcal{W})$*,* $\mathcal{W} \in \boldsymbol{\mathcal{W}}$*, be a family of wedge algebras having the Reeh–Schlieder property and satisfying the conditions of isotony, covariance, and locality. Then the family of deformed algebras* $\mathfrak{A}_Q(\mathcal{W}) \subset \mathfrak{F}$*,* $\mathcal{W} \in \boldsymbol{\mathcal{W}}$*, also has these properties. In addition, if* $\mathcal{W}_1 \subset \mathcal{W}_2'$ *and* $B_1 \subset B_1^* \in \mathfrak{A}_Q(\mathcal{W}_1)$*,* $B_2 \subset B_2^* \in \mathfrak{A}_Q(\mathcal{W}_2)$*, then* B_1 *and* B_2 *commute strongly.*

Theorem 5 (Buchholz & S., 2009)**.** *Under the same assumptions as above, if* $\mathcal{A}_Q(\mathcal{W}_R) \doteq \{A_Q \mid A \in \mathfrak{A}(\mathcal{W}_R)\}''$*, then* $\mathcal{A}_Q(\mathcal{W}_R)$ *satisfies the consistency conditions, resulting in a local and covariant net* $\{\mathcal{A}_Q(\mathcal{O})\}$*.*

Moreover, if the original theory describes a scalar massive particle, then two body scattering is well defined in the deformed theory, resulting in the following relation between the deformed (improper) scattering states and the original (improper) scattering states:

$$|p \otimes_Q q\rangle^{\text{in}} = e^{i|pQq|}\, |p \otimes q\rangle^{\text{in}}$$
$$|p \otimes_Q q\rangle^{\text{out}} = e^{-i|pQq|}\, |p \otimes q\rangle^{\text{out}} .$$

Note that the deformed scattering states depend upon the choice of $\mathcal{W}_0$ through the choice of Q and thus break the Lorentz symmetry.

The kernels of the elastic scattering matrices in the deformed and undeformed theory are related by

$$^{out}\langle p \otimes_Q q | p' \otimes_Q q' \rangle^{in} = e^{i|pQq| + i|p'Qq'|} \; ^{out}\langle p \otimes q | p' \otimes q' \rangle^{in} \, .$$

Thus they differ from each other, showing that the initial and deformed theories are not isomorphic.

Observables With Localizations Smaller Than Wedges?

- Algebras associated with double cones are trivial.
- At least for a one parameter family of admissible Q's, algebras associated with spacelike cylinders are nontrivial.

Some References

- R. Brunetti, D. Guido and R. Longo, *Rev. Math. Phys.,* **14**, 759–785 (2002).
- D. Buchholz and G. Lechner, *Ann. Henri Poincaré,* **5**, 1065–1080 (2004).
- D. Buchholz and S.J. Summers, *J. Math. Phys.,* **45**, 4810–4831 (2004).
- D. Buchholz and S.J. Summers, *J. Phys. A,* **40**, 2147–2163 (2007).
- D. Buchholz and S.J. Summers, in: *Quantum Field Theory and Beyond*, edited by E. Seiler and K. Sibold (World Scientific, Singapore), pp. 107–121, 2008.
- H. Grosse and G. Lechner, *JHEP*, **0711**, 012 (2007).
- H. Grosse and G. Lechner, *JHEP, 09*, 131 (2008).
- G. Lechner, *Commun. Math. Phys.*, **277**, 821–860 (2008).
- B. Schroer, *Nucl. Phys. B,* **499**, 547–568 (1997).