On state spaces for perturbative QFT in curved spacetimes

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Contents

State spaces in perturbative QFT – 1

For perturbative QFT in a background gravitational field:

- The vacuum is replaced by the set $\mathcal{S}_H$ of Hadamard states. These are states of finite energy density.
- The extended algebra of Wick powers and time-ordered products is related to the subset $\mathcal{S}_{\mu SC} \subset \mathcal{S}_H$ of states satisfying a microlocal spectrum condition.

Questions:

- What are the relations between $\mathcal{S}_H$, $\mathcal{S}_{\mu SC}$ and the set $\mathcal{S}^{ext}$ of all states on the extended algebra?
- Can the construction of the extended algebra, involving $\mathcal{S}_{\mu SC}$, be generalised to an axiomatic framework?

State spaces in perturbative QFT – 2

We will find:

- Assuming (anti-)commutation relations:

$$\mathcal{S}_H = \mathcal{S}_{\mu SC} = \mathcal{S}^{ext}.$$

- Given a "nice" state space $\mathcal{S}$ on an algebra $\mathcal{A}$, we can canonically construct an extended algebra $\mathcal{A}^{ext}$ with $\mathcal{S} = \mathcal{S}^{ext}$.

Real scalar quantum fields

We define the *Borchers-Uhlmann algebra* $\mathcal{U}$ on a spacetime $M = (\mathcal{M}, g)$ such that:

- $\mathcal{U}$ is algebraically generated by a unit I and elements $\Phi(f)$ with $f \in C_0^\infty(M)$,
- Φ is $\mathbb{C}$-linear: $\Phi(f+g) = \Phi(f) + \Phi(g)$,
- Φ is real: $\Phi(f)^* = \Phi(\bar{f})$,
- there is some topology.

A *state* ω on $\mathcal{U}$ is a linear functional that is:

- positive: $\omega(A^*A) \geq 0$, $A \in \mathcal{U}$,
- normalised: $\omega(\mathrm{I}) = 1$,
- continuous, i.e. it is given by a sequence of *n-point distributions* ω_n on $M^{\times n}$:

$$\omega_n(x_n, \ldots, x_1) := \omega(\Phi(x_n) \cdots \Phi(x_1)).$$

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Truncated n-point distributions

The *truncated n-point distributions* ω_n^T for $n \geq 1$ are defined implicitly by:

$$\omega_n(x_n, \ldots, x_1) = \sum_P \prod_{r \in P} \omega_{|r|}^T(x_{r(|r|)}, \ldots x_{r(1)}),$$

where P is a partition of $\{1, \ldots, n\}$ into disjoint ordered subsets r with $|r|$ elements. In detail:

$$\begin{aligned}
\omega_1(x_1) &= \omega_1^T(x_1) \\
\omega_2(x_2, x_1) &= \omega_2^T(x_2, x_1) + \omega_1^T(x_2)\omega_1^T(x_1) \\
\omega_3(x_3, x_2, x_1) &= \omega_3^T(x_3, x_2, x_1) + \omega_2^T(x_3, x_2)\omega_1^T(x_1) \\
&\quad + \omega_2^T(x_3, x_1)\omega_1^T(x_2) + \omega_2^T(x_2, x_1)\omega_1^T(x_3) \\
&\quad + \omega_1^T(x_3)\omega_1^T(x_2)\omega_1^T(x_1)
\end{aligned}$$

A state is *quasi-free* if and only if $\omega_n^T = 0$ for $n \neq 2$.

Microlocal analysis – 1

To study the singularities of n-point distributions we use the notion of *wave front set* from microlocal analysis.

For a distribution (density) u on a manifold M

- the wave front set $WF(u)$ is a subset of T^*M,
- u is smooth near $x \in M$ iff $WF(u) \cap T^*_x M = \emptyset$,
- $WF(u+v) \subset WF(u) \cup WF(v)$,
- and many more nice properties...

For a distribution u with values in a Banach space B we define: $WF(u) := \overline{\cup_{l \in B'} WF(l \circ u)} \setminus \{(x,0)\}$.

- Many properties for these wave front sets follow from the scalar case.

Microlocal analysis – 2

For distributions u, v with values in a Hilbert space $\mathcal{H}$ one uses the Cauchy-Schwartz inequality to prove:

Theorem (Strohmaier-Verch-Wollenberg (2002))

$$(x,k) \in WF(u) \Leftrightarrow (x,-k;x,k) \in WF(\langle u,u\rangle).$$

and

$$(x,k;y,l) \in WF(\langle u,v\rangle) \Rightarrow (y,l) \in WF(v) \text{ or } l=0.$$

Microlocal spectrum condition – 1

Definition

A state ω on $\mathcal{U}$ satisfies the *microlocal spectrum condition* (μSC) iff $WF(\omega_n) \subset \Gamma_n$ for all n.

We call a state ω on $\mathcal{U}$ *Hadamard* iff $WF(\omega_2) \subset \Gamma_2$.

Here the sets $\Gamma_n \subset T^*M^{\times n}$ are physically motivated by embedding (Feynman) graphs into M:

$$(x_n, k_n; \ldots; x_1, k_1) \in \Gamma_n \quad \Leftrightarrow \quad \text{there are}$$

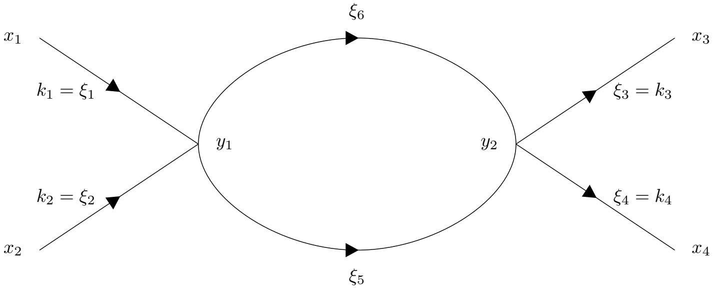
- finitely many points $y_1, \ldots, y_m \in M$,
- an ordering of $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ such that $x_1 < x_2 < \ldots < x_n$,
- finitely many piecewise smooth curves γ_e between a source $s(\gamma_e)$ and a target $t(\gamma_e) \geq s(\gamma_e)$ in $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$,
- on each γ_e a causal, future pointing covector field ξ_e such that $\nabla \xi_e = 0$ along γ_e,

Microlocal spectrum condition – 2

such that for each $i = 1, \ldots, n$, $j = 1, \ldots, m$:

$$k_i = \sum_{\gamma_e \colon s(\gamma_e) = x_i} \xi_e(x_i) - \sum_{\gamma_e \colon t(\gamma_e) = x_i} \xi_e(x_i),$$

$$0 = \sum_{\gamma_e \colon s(\gamma_e) = x_i} \xi_e(x_i) - \sum_{\gamma_e \colon t(\gamma_e) = x_i} \xi_e(x_i).$$



Microlocal spectrum condition – 3

Some useful properties of the Γ_n are:

- Each Γ_n is a convex cone and $\Gamma_n \cap \underline{0} = \emptyset$,
- $\Gamma_n \cap -\Gamma_n = \emptyset$,
- for every partition P of $\{1, \ldots, n\}$ as before with the permutation $\pi : (1, \ldots, n) \to (r_1, \ldots, r_m)$ we have

$$(\Gamma_{|r_1|} \cup \underline{0}) \cup \ldots \cup (\Gamma_{|r_m|} \cup \underline{0}) \subset \Gamma_{\pi(n)} \cup \underline{0},$$

where π reorders the factors in $(T^*M)^{\times n}$.

Corollary

A state ω enjoys the μSC if and only if $WF(\omega_n^T) \subset \Gamma_n$ for all $n \geq 1$.

Commutation relations – 1

The algebra $\mathcal{U}^E$ of a ”generalised free field” is defined like $\mathcal{U}$ with the additional requirement:

- Φ satisfies commutation relations:

$$[\Phi(f), \Phi(g)] = iE(f, g)\mathrm{I}$$

  for a scalar distribution E on $M^{\times 2}$.

I.e. we divide $\mathcal{U}$ by the ideal $\langle [\Phi(f), \Phi(g)] - iE(f, g)\mathrm{I} \rangle$.

Proposition

A state ω on $\mathcal{U}$ descends to a state on $\mathcal{U}^E$ for some E if and only if ω_n^T is fully symmetric in its arguments for all $n \neq 2$. Then, $iE(x_2, x_1) = \omega_2(x_2, x_1) - \omega_2(x_1, x_2)$.

(For anti-commutation relations the $\omega^T_{n \neq 2}$ are anti-symmetric and $iE(x_2, x_1) = \omega_2(x_2, x_1) + \omega_2(x_1, x_2)$.)

Commutation relations – 2

Proof.

Assume all ω_n^T are fully symmetric for $n \neq 2$. Expressing

$$\omega_n(x_n, \ldots, x_1) - \omega_n(x_n, \ldots, x_i, x_{i+1}, \ldots, x_1)$$

in terms of ω_n^T most terms cancel, namely:

- if the indices $i, i+1$ appear in the same $\omega_{n \geq 3}^T$,
- if $i, i+1$ don't appear in the same factor ω_n^T.

The remaining terms can be rewritten as

$$(\omega_2(x_{i+1}, x_i) - \omega_2(x_i, x_{i+1}))\, \omega_{n-2}(x_n, \ldots \widehat{x_{i+1}}, \widehat{x_i}, \ldots, x_1).$$

The opposite direction uses the same combinatorics. □

The state space for generalised free fields – 1

Corollary

A Hadamard state ω on $\mathcal{U}^E$ satisfies the μSC and each $\omega^T_{n\neq 2}$ is smooth. It follows that $\mathcal{S}_H = \mathcal{S}_{\mu SC}$.

Proof.

Using Hilbert space-valued distributions:

$$\begin{aligned}(x_n, k_n; \ldots; x_1, k_1) \in WF(\omega_n),\ k_1 \neq 0 \quad &\Rightarrow \\ (x_1, k_1) \in WF(\pi_\omega(\Phi(.))\Omega_\omega) \quad &\Rightarrow \\ (x_1, -k_1; x_1, k_1) \in WF(\omega_2), &\end{aligned}$$

so k_1 must be future pointing by the Hadamard condition. Similarly, k_n must be past pointing. This also holds for ω^T_n, so by symmetry each $\omega^T_{n\neq 2}$ is smooth! Hence $WF(\omega^T_n) \subset \Gamma_n$ for all n. □

The state space for generalised free fields – 2

Corollary (Brunetti-Fredenhagen-Köhler (1996))

The space $\mathcal{S}_H$ on $\mathcal{U}^E$ is closed under operations from the algebra.

Proof.

One checks that $WF(\omega(A^*\Phi(.)\Phi(.))A) \subset \Gamma_2$ for all $\omega \in \mathcal{S}_H = \mathcal{S}_{\mu SC}$. □

For a *free* real scalar field:

Corollary (cf. Hollands-Ruan (2002))

All Hadamard states on $\mathcal{A}$ extend to $\mathcal{A}^{ext}$, i.e. $\mathcal{S}_H = \mathcal{S}_{\mu SC} = \mathcal{S}^{ext}$.

Extended algebras in axiomatic QFT – 1

The choice of $\mathcal{S}_H$ on $\mathcal{A}$ and the definition of $\mathcal{A}^{ext}$ are related.

- Can this relation be generalised to an axiomatic framework?
- Can we always ensure that $\mathcal{S}^{ext}$ is the initial state space?
- What is a suitable topology on $\mathcal{A}^{ext}$?

These questions are related: a weaker topology on $\mathcal{A}^{ext}$ admits less (continuous) states.
The weakest l.c. topology that leaves all initial states continuous is generated by the semi-norms:

$$A \mapsto |\omega(A)|.$$

Extended algebras in axiomatic QFT – 2

Theorem

Let $\mathcal{A}$ be any topological *-algebra and $\mathcal{S}$ a state space such that

- *$\mathcal{S}$ is closed under operations from $\mathcal{A}$*
- *if a state ρ satisfies $|\rho(A)| \leq c|\omega(A)|$ for all $A \in \mathcal{A}$ and some $c > 0$, $\omega \in \mathcal{S}$, then $\rho \in \mathcal{S}$.*

Then one can canonically construct a (Hausdorff) topological *-algebra $\mathcal{A}^{ext}$ and a continuous algebraic homomorphism $\alpha : \mathcal{A} \to \mathcal{A}^{ext}$ such that:

- *$\alpha(\mathcal{A}) \subset \mathcal{A}^{ext}$ is dense,*
- *the full state space $\mathcal{S}^{ext} = \mathcal{S}$ (affine bijection),*
- *$\mathcal{A}^{ext}$ carries the weak topology induced by $\mathcal{S}^{ext}$.*

Extended algebras in axiomatic QFT – 3

Remarks:

- For a free scalar field our $\mathcal{A}^{ext}$ contains the extended algebra of Brunetti-Fredenhagen.
- $A \in \ker(\alpha) \quad \Leftrightarrow \quad \pi_\omega(A) = 0$ for all $\omega \in \mathcal{S}$.
- The state space $\mathcal{S}$ can be encoded entirely in the choice of topology (under the given assumptions).
- A similar theorem holds for locally covariant QFT's if the state spaces
  - are locally covariant,
  - satisfy local physical equivalence.

  Injective morphisms $\mathcal{A}_1 \to \mathcal{A}_2$ then extend to injective morphisms $\mathcal{A}_1^{ext} \to \mathcal{A}_2^{ext}$, but the algebras $\mathcal{A}^{ext}$ are not the same as above.
- In general $\mathcal{A}^{ext}$ is not (sequentially) complete.

Conclusions

- (Anti-)commutation relations are characterised by (anti)-symmetry of $\omega^T_{n\neq 2}$ in its arguments.
- For generalised free fields we have $\mathcal{S}_H = \mathcal{S}_{\mu SC}$, because $\omega^T_{n\neq 2}$ is smooth in this case.
- For free scalar fields it follows that $\mathcal{S}_H = \mathcal{S}_{\mu SC} = \mathcal{S}^{ext}$.
- In an axiomatic framework we can characterise "nice" state spaces $\mathcal{S}$ in terms of a topology on $\mathcal{A}$ and construct an extended algebra $\mathcal{A}^{ext}$.

Open problems

- Can we extend the equality $\mathcal{S}_H = \mathcal{S}_{\mu SC}$ to the axiomatic OPE approach?
- Is our $\mathcal{A}^{ext}$ bigger than that of Brunetti-Fredenhagen?
- How is our weak topology on $\mathcal{A}^{ext}$ related to the Hörmander topology?
- Can we choose a better topology on $\mathcal{A}^{ext}$ such that $\mathcal{S} = \mathcal{S}^{ext}$ *and* the algebra is (sequentially) complete?