Introduction to the Renormalization Group

Hands-on course to the basics of the RG

(based on: "Introduction to the Functional Renormalization Group" by P. Kopietz, L. Bartosch, and F. Schütz)

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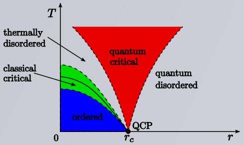
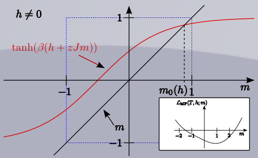
Outline

1. History of the RG

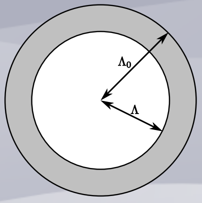
2. Phase Transitions and scaling



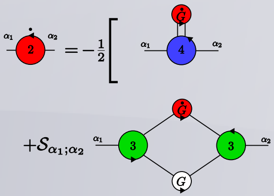


3. Mean-Field Theory

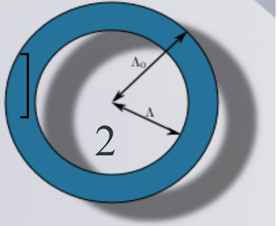
4. Wilsonian RG



5. Functional (exact) RG



6. Applications

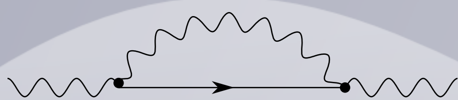


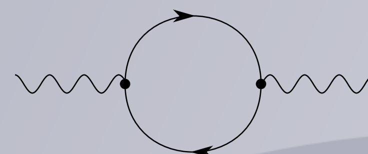
Introduction

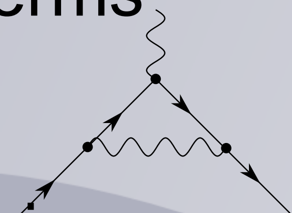
- What is the renormalization group?
  “All renormalization group studies have in common the idea of re-expressing the parameters which define a problem in terms of some other, perhaps simpler set, while keeping unchanged those physical aspects of a problem which are of interest.”
  (John Cardy, 1996)
- “Meta-Theory” about theories
- Make the problem as simple as possible, but not simpler.
- Describe general properties qualitatively, but not necessarily quantitatively.

Quantum Field Theory

- Problem: Perturbation theory in quantum electrodynamics gives rise to infinite terms





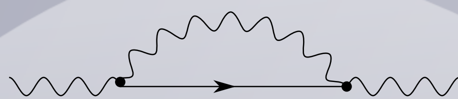


- Solution: all infinities can be absorbed in redefinition (=renormalization) of the parameters which have to be fixed by the experiment
- Renormalizable theories: finite number n of parameters sufficient, n experiments needed to fix them, predict all other experiments

Renormalization of QED

- 3 divergent diagramms → 3 parameters
- Electron self energy

$$\Sigma(P) = \frac{e^2}{8\pi^2\epsilon}(4m - \not{P})$$

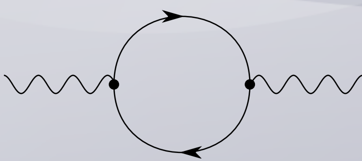


$$m = \frac{Z_m m_r}{Z_2}$$

electron mass renormalization

- Photon self energy

$$\Pi^{\mu\nu}(P) = \frac{e^2}{6\pi^2\epsilon}(K^\mu K^\nu - g^{\mu\nu}K^2)$$

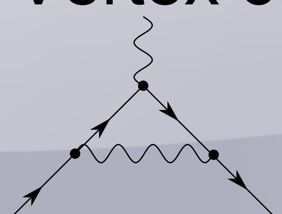


$$Z_3 = 1 - \frac{e_r^2}{6\pi^2\epsilon}$$

field renormalization

- Vertex correction

$$\Lambda(P, P+Q, Q)^\mu = \frac{e^2}{8\pi^2\epsilon}\gamma^\mu$$



$$e_r^2 = \frac{e^2}{1 - \frac{e^2}{6\pi^2} ln\mu/\mu_0}$$

charge renormalization

History

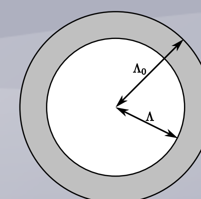
- Kenneth Wilson (1971/1972) calculation of critical exponents which are universal for a class of models

$$C(t) = |t|^{-\alpha} \quad \text{specific heat}$$

$$m(t) \sim (-t)^{\beta} \quad \text{magnetization}$$

$$t = \frac{T - T_c}{T_c}$$

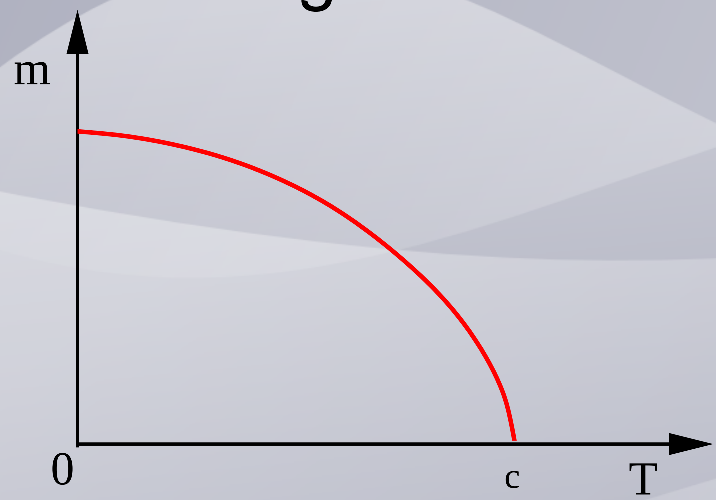
- new formulation of the RG idea (Wilsonian RG)



- Nobel Prize in Physics 1982:
  "...for his theory of critical phenomena in connection with phase transitions..."

Phase transitions and scaling hypothesis

- Phase transitions: examples
- Paramagnet-Ferromagnet Transition



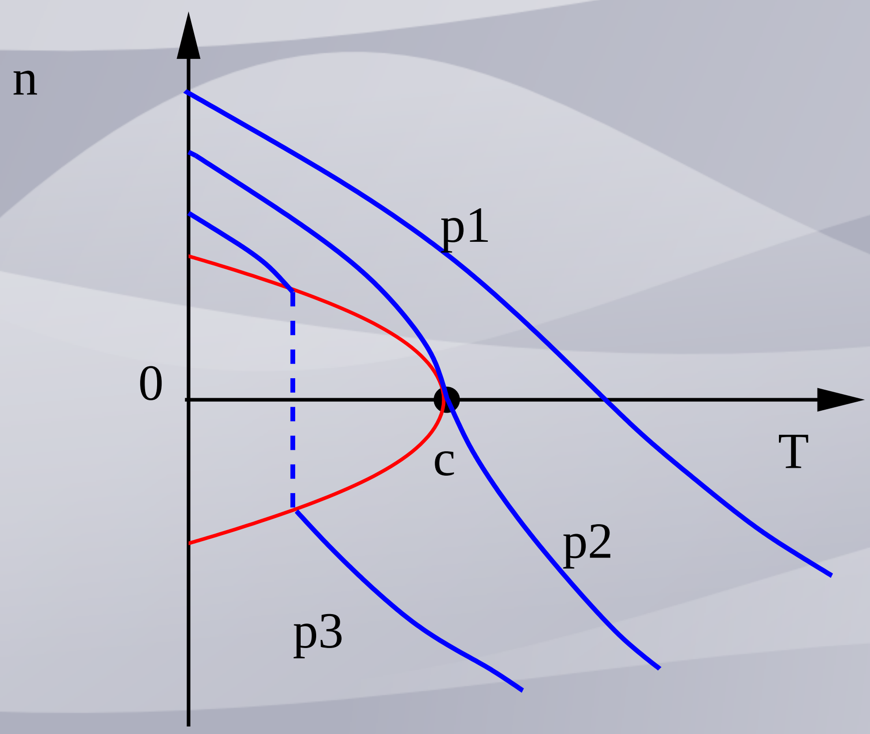
ordering parameter: magnetization

$$m = -\lim_{h \to 0} \frac{\partial f}{\partial h} \propto (T_c - T)^{\beta}$$

critical exponent: general for systems characterized by symmetry and dimensionality

Phase transitions and scaling hypothesis

- Liquid-Gas Transition



order parameter: density

$$n - n_c \propto (T - T_c)^\beta$$

same symmetry class as Ising model (classical spins in a magnetic field)
→ same critical exponent

$$H = -J \sum_{ij} s_i s_j - h \sum_i s_i$$

$$s_i = \pm 1$$

Universality classes

- Ising model, gas-liquid transition

$$H = -J\sum_{ij} s_i s_j \qquad Z_2 : \quad s_i \to -s_i$$

- XY_3 , Bose gas, (magnets in magnetic fields)

$$H = -J\sum_{ij}\left[S_i^x S_j^x + S_i^y S_j^y + (1+\lambda) S_i^z S_j^z\right]$$

$$O(2) : \vec{S} \to \vec{S}' = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{S}$$

- Heisenberg

$$H = -J\sum_{ij} \vec{S}_i \cdot \vec{S}_j \qquad O(3) : \quad \vec{S} \to \vec{S}'$$

Critical exponents

- specific heat $C(t) \propto |t|^{-\alpha}$
- spontaneous magnetization $m(t) \propto (-t)^{\beta}$ $\quad t = \frac{T - T_c}{T_c}$
- magnetic susceptibility $\chi(t) \propto |t|^{-\gamma}$
- critical isotherm $m(h) \propto |h|^{1/\delta}\mathrm{sgn}(h) \quad t = 0$
- correlation length $G(\vec{r}) \propto \frac{e^{-|r|/\xi}}{\sqrt{\xi^{D-3}|r|^{D-1}}} \quad \xi \propto |t|^{-\nu}$
- anomalous dimension

$$G(\vec{k}) \sim |\vec{k}|^{-2+\eta} \quad T = T_c$$

Scaling Hypothesis

- only two of six exponents are independent
- consider free energy density

$$f(t,h) = f_{\mathrm{sing}}(t,h) + f_{\mathrm{reg}}(t,h)$$

- singular part satisfies homogeneity relation

$$f_{\mathrm{sing}} = |t|^{D/y_t}\Phi_{\pm}\left(\frac{h}{|t|^{y_h/y_t}}\right) \quad \Phi_{\pm}(x) = f_{\mathrm{sing}}(\pm 1, x)$$

- critical exponents from derivatives

$$C = \frac{1}{T_c}\left.\frac{\partial^2 f}{\partial t^2}\right|_{h=0} \propto |t|^{-\alpha}$$

$$m = -\left.\frac{\partial f}{\partial h}\right|_{h=0} \propto (-t)^{\beta}$$

Scaling Hypothesis

- relations between exponents

$$2 - \alpha = 2\beta + \gamma = \beta(\delta + 1)$$

- scaling hypothesis for correlation function delivers two additional relations

$$2 - \alpha = D\nu \qquad \gamma = (2 - \eta)\nu$$

- relation between thermodynamic exponents and correlation function exponents

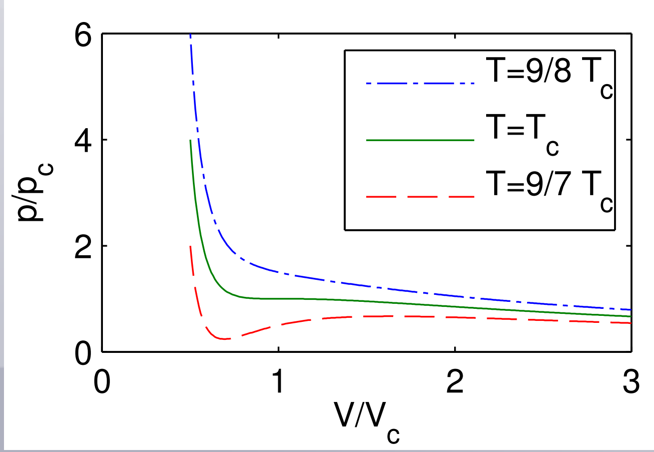
$$\begin{aligned} \alpha &= 2 - D\nu \\ \beta &= \frac{\nu}{2}(D - 2 + \eta) \\ \gamma &= \nu(2 - \eta) \\ \delta &= \frac{D + 2 - \eta}{D - 2 + \eta} \end{aligned}$$

Exercise 1: van der Waals Gas

- equation of state

$$\left(p + a\left(\frac{N}{V}\right)^2\right)(V - Nb) = NT$$

- sketch of isotherms



Exercise 1: Critical properties

- Thermodynamics: calculate free energy from pressure

$$F(T,V) = -\int_{V_0}^{V} p(V')dV' + \text{const.}(T)$$

$$F(T,V)_{\text{ideal}} = Nk_BT\ln\frac{h^3}{(2\pi m k_B T)^{3/2}V} + Nk_BT$$

- obtain quantities from derivatives of the free energy

$$C_v = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V \propto |t|^{-\alpha}$$ specific heat

Exercise 1: Critical exponents (continued)

- use equation of state $p(V) = \frac{Nk_BT}{V - V_c/3} - 3p_c\frac{V_c^2}{V^2}$

susceptibility → compressibility

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = -\frac{1}{V}\left(\frac{\partial p}{\partial V}\right)^{-1}_{T,V=V_c} \propto t^{-\gamma}$$

- rewrite at the critical temperature

$$n = n_c + \Delta n$$

$$p(\Delta n) = p_c + \text{const.}\Delta n^\delta \quad \Rightarrow (n - n_c) \propto (p - p_c)^{1/\delta}$$

Mean Field Theory

- Example: Ising model $$H = -J\sum_{ij} s_i s_j - h\sum_i s_i$$
- partition function $$Z(T,h) = \sum_{\{s_i\}} e^{-\beta H}$$
- magnetization $$m = \langle s_i \rangle = \frac{\sum_{\{s_i\}} s_i e^{-\beta H}}{Z}$$
- simplify term in Hamiltonian
$$s_i s_j = (m + \delta s_i)(m + \delta s_j) = -m^2 + m(s_i + s_j) + \delta s_j + \delta s_j$$
- free spins in field
$$H_{\mathrm{MF}} = N\frac{zJ}{2}m^2 - \sum_i (h + zJm)s_i$$

Mean Field Theory

- calculate partition function and free energy

$$Z_{\mathrm{MF}}(t,h) = e^{-\beta N Z J m^2/2}\left[2\cosh[\beta(h+zJm)]\right]^N$$

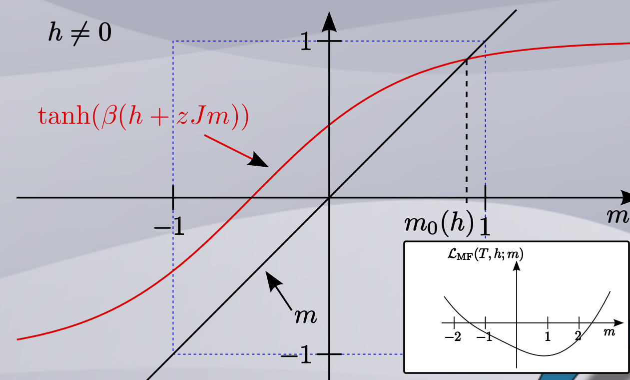
magnetization $m(t,h)$

- self consistency equation for magnetization

$$Z_{\mathrm{MF}}(t,h) = e^{-\beta N \mathcal{L}}$$

$$\frac{\partial \mathcal{L}}{\partial m} = 0$$

$$m_0 = \tanh[\beta(h+zjm_0)]$$



- free energy

$$f(T,h) = \mathcal{L}(T,h,m_0)$$

Mean Field Theory: critical exponents (only correct at D>4)

- minimum condition $\frac{\partial \mathcal{L}}{\partial m} = (T - T_c)m_0 + \frac{T_c}{3} m_0^3 - h = 0$
- magnetization $m_0 = \sqrt{\frac{T_c - T}{2T_c}} \propto (-t)^{1/2}$ $\beta = 1/2$
- susceptibility $\chi = \left.\frac{\partial m_0}{\partial h}\right|_{h=0} \propto \frac{1}{T - T_c}$ $\gamma = 1$
- critical isotherm $m_0(h) \propto h^{1/3}$ $\delta = 3$
- specific heat $C = -T \frac{\partial^2 f(T,h)}{\partial T^2}$

$C \approx T_c \frac{\partial^2 (T \ln 2)}{\partial T^2}$ $T > T_c$

$C \approx T_c \frac{\partial^2 (T \ln 2)}{\partial T^2} - \frac{3(T - T_c)^2}{4T_c}$ $T > T_c$

$\alpha = 0$

Wilsonian RG

- Basic idea: take into account interactions iteratively in small steps
- Formulation in terms of functional integrals: example Ising model (φ^4 theory)

$$S_{\Lambda_0}[\varphi] = \frac{1}{2}\int_{\vec{k}} [r_0 + c_0 \vec{k}^2]\varphi(-\vec{k})\varphi(\vec{k})$$

free action: particle is characterized by the the values of two parameters

$$S_1 = \frac{n_0}{4!}\int_{\vec{k}_1} \cdots \int_{\vec{k}_4} \delta(\vec{k}_1 + \ldots + \vec{k}_4)\varphi(\vec{k}_1)\varphi(\vec{k}_2)\varphi(\vec{k}_3)\varphi(\vec{k}_4)$$

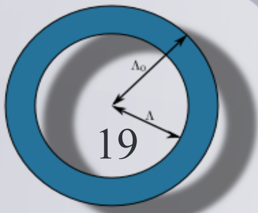
- cutoff $|\vec{k}| < \Lambda_0$

interaction between (scalar) particles: characterized by the coupling constant

only particles allowed up to a momentum
for example due to a lattice in a condensed matter system

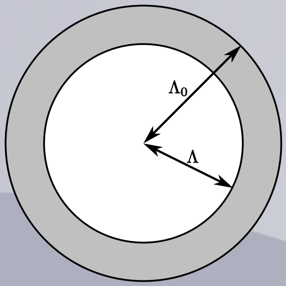
- partition function

$$\mathcal{Z} = \int \mathcal{D}[\varphi] e^{S_{\Lambda_0} + S_1}$$



Step 1: Mode elimination

- integrate out degrees of freedom associated with fluctuations at high energies



$$\mathcal{Z} = \int \mathcal{D}[\varphi] e^{-S_{\Lambda_0} - S_1} = \int \mathcal{D}[\varphi^<] \int \mathcal{D}[\varphi^>] e^{-S_{\Lambda_0} - S_1}$$

$$e^{-S'_\Lambda + S'_1} = \int \mathcal{D}[\varphi^>] e^{-S_{\Lambda_0} - S_1}$$

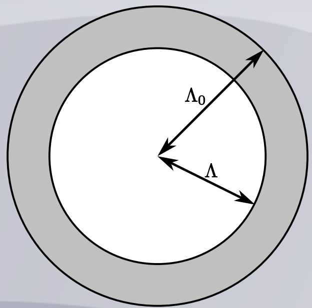
end up: theory with modified couplings due to interactions

$$S'_\Lambda[\varphi] = \frac{1}{2} \int_{\vec{k}} [r^< + c^< \vec{k}^2] \varphi(-\vec{k}) \varphi(\vec{k})$$

$$S'_1 = \frac{n^<}{4!} \int_{\vec{k}_1} \cdots \int_{\vec{k}_4} \delta(\vec{k}_1 + \ldots + \vec{k}_4) \varphi(\vec{k}_1) \varphi(\vec{k}_2) \varphi(\vec{k}_3) \varphi(\vec{k}_4)$$

Step 2: Rescaling

- Fields: defined on reduced space



  → blow up again the momentum space
- rescale wave vectors to get action with the same form as before (free and interaction part)

$$\vec{k}' = \Lambda_0/\Lambda \vec{k}$$

$$\varphi' = \zeta_b^{-1} \varphi^{<}$$

$$\zeta_b = b^{1+D/2} \sqrt{c_0/c^{<}} \quad b = \Lambda_0/\Lambda$$

Step 3: Iterative Procedure

- get relations for mode elimination and rescaling (semi-group)

$$r'(r_0,n_0) = b^2 Z_b \left[r_0 + \frac{n_0}{2} \int_{\Lambda}^{\Lambda_0} \frac{d^D k}{(2\pi)^D} \frac{1}{r_0 + c_0 \vec{k}^2} \right]$$

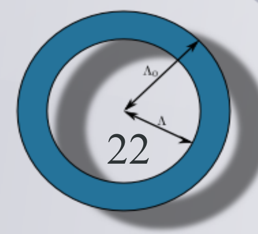
$$n'(r_0,n_0) = b^{4-D} Z_b^2 \left[n_0 - \frac{3n_0^2}{2} \int_{\Lambda}^{\Lambda_0} \frac{d^D k}{(2\pi)^D} \frac{1}{(r_0 + c_0 \vec{k}^2)^2} \right]$$

- iteration in infinitesimal steps (differential equations: flow equations)

$$\Lambda = \Lambda_0 e^{-\delta l} \approx \Lambda_0 (1 + \delta l)$$

$$\partial_l r(l) = 2r(l) + \frac{1}{2} \frac{n(l)}{1 + r(l)}$$

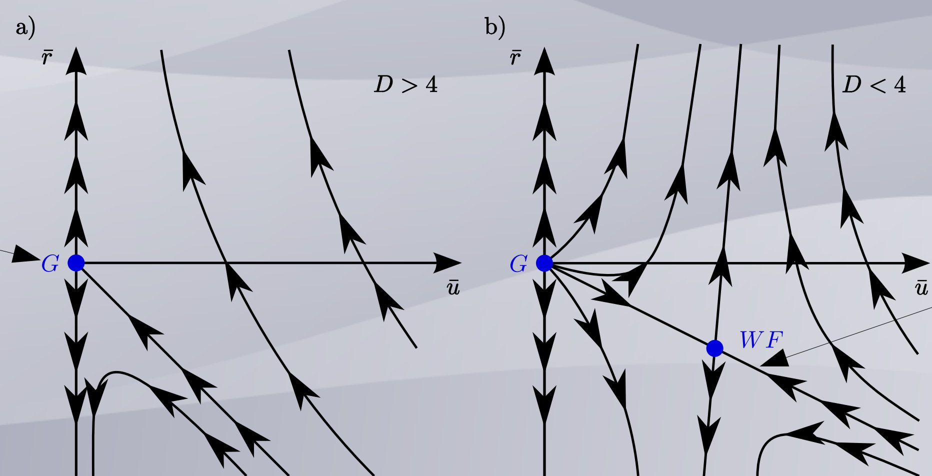
$$\partial_l n(l) = (4 - D) n(l) + \frac{3}{2} \frac{n(l)^2}{(1 + r(l))^2}$$



Flow diagrams

- solve coupled differential equations for different initial conditions (parameters of real system)
- critical surface: critical systems determined by the values of the coupling constants

one fixed point: Gaussian fixed point (mean field)

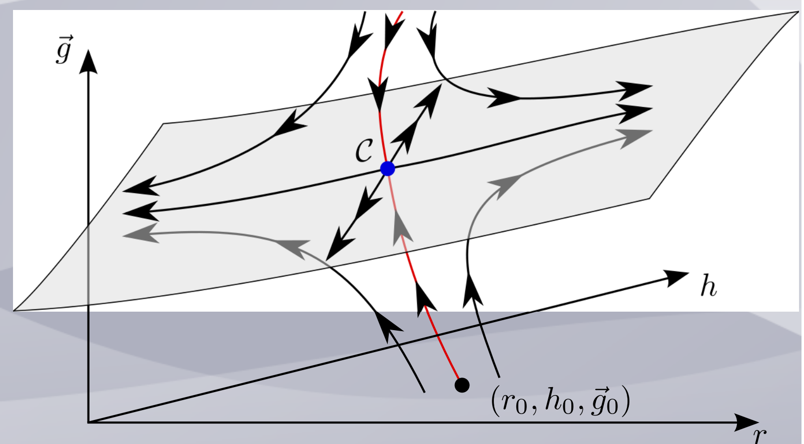


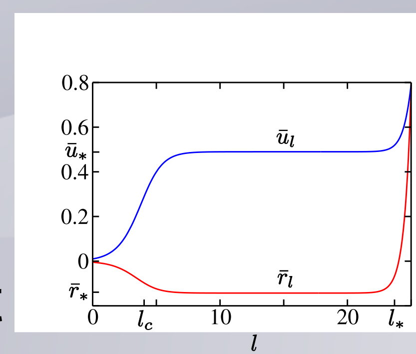
two fixed points: Gaussian, Wilson-Fisher

critical surface

RG fixed points and critical exponents

- RG fixed points: describe scale invariant system
- critical fixed points
  - relevant / irrelevant directions
  - correlation length: infinite
  - critical manifold (surface describes system at critical point)
- critical exponents
  - eigenvalues of linearised flow equations near to fixed point





Functional renormalization group

- basic idea: Express Wilsonian mode elimination in terms of formally exact functional differential equations
- generating functional of Green functions

$$G^{(n)}_{\alpha_1\ldots\alpha_n} = \frac{\int \mathcal{D}[\Phi] e^{-S[\Phi]} \Phi_{\alpha_1} \cdots \Phi_{\alpha_n}}{\int \mathcal{D}[\Phi] e^{-S[\Phi]}}$$

$$\mathcal{G}[J] = \frac{\int \mathcal{D}[\Phi] e^{-S[\Phi]+(J,\Phi)} \Phi_{\alpha_1} \cdots \Phi_{\alpha_n}}{\int \mathcal{D}[\Phi] e^{-S[\Phi]}}$$

derive RG equations for generating functional Green functions then given as derivatives

$$G^{(n)}_{\alpha_1\cdots\alpha_n} = \frac{\delta^n \mathcal{G}[J]}{\delta J_{\alpha_1} \cdots \delta J_{\alpha_n}}$$

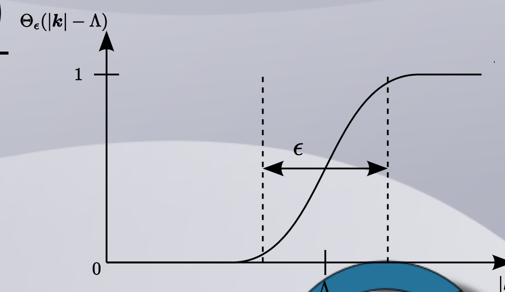
example: two point function $G^{(2)}(\vec{k}) \propto \frac{1}{|\vec{k}|^{2-\eta}}$ of Ising model

Exact renormalization group

- introduce cutoff: modify Gaussian propagator:
  allow only propagation of modes up to a certain momentum

$$S_0 = \frac{1}{2}\int (r_0 + c_0\vec{k}^2)\varphi(-\vec{k})\varphi(\vec{k}) = \frac{1}{2}\int G_0^{-1}(\vec{k})\varphi(-\vec{k})\varphi(\vec{k})$$

$$G_0(\vec{k}) = \frac{1}{r_0 + c_0\vec{k}^2} \to \frac{1-\Theta_c(|\vec{k}|-\Lambda)}{r_0 + c_0\vec{k}^2}$$



- take derivative of generating functional with respect to cutoff
  → FRG flow equation

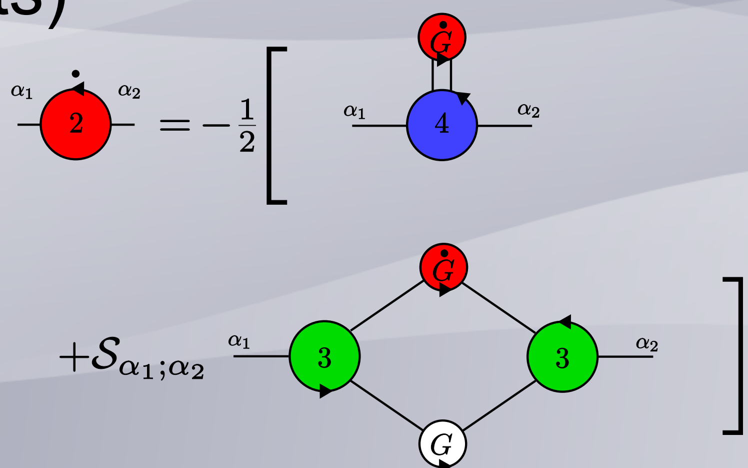
Exact FRG flow equation

- Wetterich equation

cutoff function

$$\partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{1}{2}\mathrm{Tr}\left[\frac{\partial_\Lambda R_\Lambda}{\frac{\partial^2 \Gamma_\Lambda[\Phi]}{\partial \Phi^2} + R_\Lambda}\right]$$

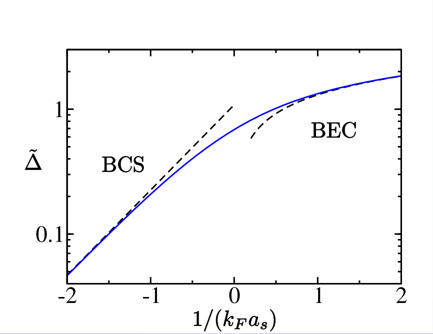
- flow equations for vertex functions (coupling constants)



Applications

- BCS-BEC crossover (electron gas with attractive interactions)
  - mean field theory (BCS-theory)

$$H = \sum_{\vec{k}\sigma} \epsilon_{\vec{k}} c^\dagger_{\vec{k}\sigma} c_{\vec{k}\sigma} - \frac{g_0}{V} \sum_{\vec{k},\vec{k}',\vec{p}} c^\dagger_{\vec{k}+\vec{p}} c^\dagger_{-\vec{k}} c_{-\vec{k}'} c_{\vec{k}'+\vec{p}}$$



- flow equations of order parameter
- FRG needs additionally Ward identities (relations between vertex functions

Applications

- interacting fermions

$$H = \sum_{\vec{k}\sigma} \epsilon_{\vec{k}} c^\dagger_{\vec{k}\sigma} c_{\vec{k}\sigma} + \frac{1}{2V} \sum_{\vec{k},\vec{k}',\vec{p}} V_{\vec{k},\vec{k}',\vec{p}} c^\dagger_{\vec{k}+\vec{p}} c^\dagger_{-\vec{k}} c_{-\vec{k}'} c_{\vec{k}'+\vec{p}}$$

$$\mathcal{Z} = \int \mathcal{D}[\psi, \bar{\psi}] e^{-S[\psi,\bar{\psi}] - S_1[\psi,\bar{\psi}]} \qquad S_1[\psi, \bar{\psi}] = \frac{1}{2} \int_{k,k',q} V \bar{\psi}\bar{\psi}\psi\psi$$

- Hubbard-Stratonovich transformation

$$e^{-S_1[\psi,\bar{\psi}]} = \int \mathcal{D}[\phi, \phi^*] e^{-S_0[\phi,\phi^*] - S'[\psi,\bar{\psi},\phi,\phi^*]}$$

$$S_0[\phi, \phi^*] = \frac{1}{2} \int_k V^{-1} \phi^*_k \phi_k$$

$$S'[\psi, \bar{\psi}, \phi, \phi^*] = i \int_k \int_q \psi_{k+q} \psi_k \phi_q$$

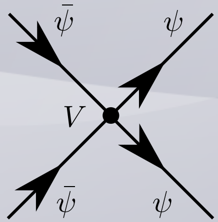
compare:

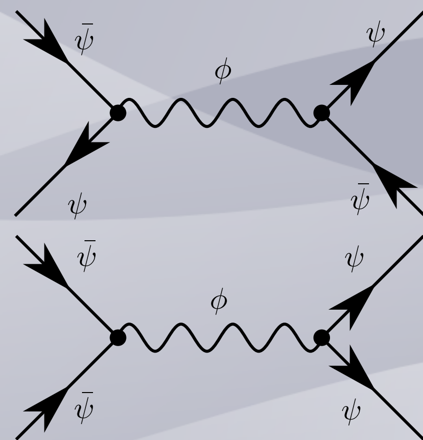
$$e^{-\frac{x^4}{2!}} = \int_{-\infty}^{\infty} dx\; e^{-\frac{x^2}{2} - ix^2 y}$$

Hubbard Stratonovich transformation

- interacting fermions → bosons and fermions with Yukawa type interaction

$$S_1[\psi, \bar{\psi}] = \frac{1}{2} \int_{k,k',q} V \bar{\psi}\bar{\psi}\psi\psi \longrightarrow S'[\psi, \bar{\psi}, \phi, \phi^*] = i \int_k \int_q \psi_{k+q}\psi_k\phi_q$$





$$S_0[\phi, \phi^*] = \frac{1}{2} \int_k V^{-1} \phi^*_k \phi_k$$

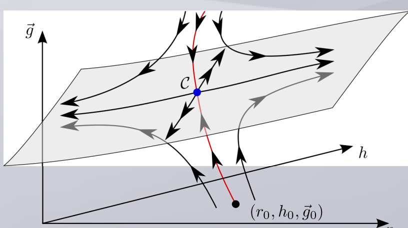
- FRG of coupled bosons and fermions

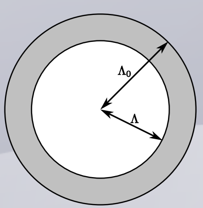
Summary

- phase transitions → critical exponents

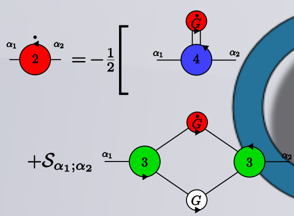
$$m = -\lim_{h\to 0}\frac{\partial f}{\partial h} \propto (T_c - T)^\beta$$

- universality classes (same symmetry → same properties at critical point)
- mean field theory
- Wilsonian Renormalisation Group





- functional Renormalization group



Exercise 2: Real-space RG of the 1D Ising model

- model

$$H = -J \sum_{i=1}^{N} s_i s_{i+1}$$

- transfer matrix method to calculate partition function

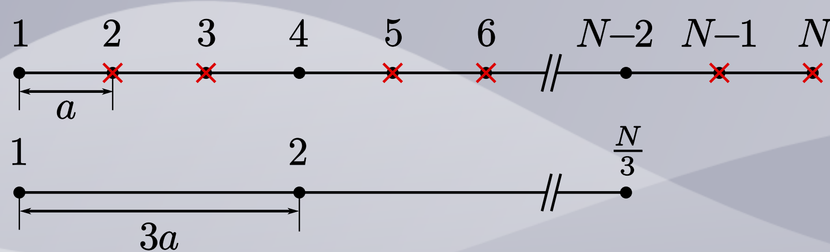
$$Z = \mathrm{Tr}[T^N] \qquad T = \begin{pmatrix} e^{g} & e^{-g} \\ e^{-g} & e^{g} \end{pmatrix} \qquad g = \beta J$$

- calculate trace in diagonal basis

$$T = U^\dagger \tilde{T} U \qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Exercise 2: Real-space RG

- keep only every b's spin and derive effective model with new coupling



$$Z = \mathrm{Tr}[T^N] = \mathrm{Tr}[T^b]^{\frac{N}{b}}$$

$$T^b = T' = \begin{pmatrix} e^{g'} & e^{-g'} \\ e^{-g'} & e^{g'} \end{pmatrix}$$

- derive recursion relation (RG transformation)

$$g'(g) = \mathrm{Artanh}(\tanh^b(g))$$

Exercise 2: Real-space RG

- variable transformation

$$y = e^{-2g} \quad y' = e^{-2g} \qquad\qquad y'(y) = \frac{(1+y)^b - (1-y)^b}{(1+y)^b + (1-y)^b}$$

- infinitesimal transformation (differential equation)

$$b = e^{\delta l} \approx 1 + \delta l \qquad\qquad \frac{dy}{dl} = \frac{1-y^2}{2} \ln(\frac{1+y}{1-y})$$

- fixed points and flow

$$\frac{dy}{dl} = 0$$

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