BEC of magnons and spin wave interactions in QAF

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Interacting bosons T=0

- Bogoliubov theory (1947)
  euclidean action for interacting Bose gas

$$\begin{aligned}
S[\bar{\Psi},\Psi] &= S_2[\bar{\Psi},\Psi] + S_4[\bar{\Psi},\Psi] \\
S_2[\bar{\Psi},\Psi] &= \int_K (-i\omega + \epsilon_{\mathsf{k}} - \mu)\bar{\Psi}_K \Psi_K \\
S_4[\bar{\Psi},\Psi] &= \frac{1}{(2!)^2}\int_{K_1}\cdots\int_{K_4} \delta_{K_1+K_2,K_3+K_4} V(\{K_i\})\bar{\Psi}_{K_1}\bar{\Psi}_{K_2}\Psi_{K_3}\Psi_{K4}
\end{aligned}$$

- Bogoliubov shift $\Psi_K = \Psi_K^0 + \Delta\Psi_K$ $\quad$ $\Psi_K^0 = \delta_{K,0}\sqrt{\rho_0}$

$$\langle \Delta\Psi_k \rangle = 0$$

$$\Rightarrow \quad \tilde{S}\left[\Delta\bar{\Psi},\Psi\right] = \sum_{i=0}^{4} \tilde{S}_i\left[\Delta\bar{\Psi},\Delta\Psi\right]$$

N. Bogoliubov (1947)

Goldstone bosons

- constraint to chemical potential (minimum)

$$\tilde{S}_1\left[\Delta\bar{\Psi}, \Delta\Psi\right] = 0$$

- diagonalization

$$E_{\mathsf{k}} = \sqrt{\epsilon_{\mathsf{k}}^2 + 2\rho_0 U(\mathsf{k})\epsilon_{\mathsf{k}}} = c_0\left|\mathsf{k}\right| + \alpha\left|\mathsf{k}\right|^3 + \mathcal{O}(\mathsf{k}^5)$$

- condensate density

$$\rho_0 = \frac{\mu}{U(0)}$$

- gapless excitations: velocity

$$c_0 = \sqrt{\frac{\mu}{m}}$$

Spin waves in Heisenberg-magnets

- Heisenberg ferromagnet $\hat{H} = \frac{1}{2}\sum_{ij} J_{ij} \mathsf{S}_i \cdot \mathsf{S}_j \qquad J_{ij} = J < 0$
- Holstein-Primakoff transformation around classical ground state

$$\begin{aligned} S_i^- &= \sqrt{2S} b_i^\dagger \sqrt{1 - \frac{n_i}{2S}} = (S_i^+)^\dagger \qquad n_i = b_i^\dagger b_i \quad , \quad \left[b_i^\dagger , b_j \right] = \delta_{ij} \\ S_i^z &= S - n_i \end{aligned}$$

- bosonic hamiltonian $H = E_{\mathrm{cl}} + H_2 + H_4 + \mathcal{O}(b^6)$

$$\begin{aligned} E_{\mathrm{cl}} &= NS[2DJS - h] \\ H_2 &= -\frac{S}{2} \sum_{ij} J_{ij} \left(n_i + n_j + b_i^\dagger b_j - b_j^\dagger b_i \right) + h \sum_i n_i \\ H_4 &= \frac{1}{4} \sum_{ij} J_{ij} \left(2 n_i n_j - b_i^\dagger n_i b_j - b_j^\dagger n_i b_j \right) \end{aligned}$$

N. Hasselmann, P. Kopietz, Europhys. Lett. 74, 1067 (2006)

BEC of Holstein-Primakoff bosons in ferromagnets?

$$\begin{aligned} E_{\mathsf{k}} &= \tilde{J}_0 S(1-\gamma_{\mathsf{k}} \\ \gamma_{\mathsf{k}} &= \sum_{\delta} e^{i\mathsf{k}\cdot\delta} \\ \tilde{J}_0 &= 2DJ \end{aligned}$$

- FT: diagonalizes quadratic part

$$b_i = \frac{1}{\sqrt{N}} \sum_{\mathsf{k}} e^{-i\mathsf{k}\cdot\mathsf{r}_i} b_{\mathsf{k}}$$

$$H_2 = \sum_{\mathsf{k}} E_{\mathsf{k}} b^{\dagger}_{\mathsf{k}} b_{\mathsf{k}}$$

- interaction vanishes: to weak interaction of ferromagnetic spin waves

$$H_4 = \frac{1}{2V} \sum_{\mathsf{k}_1 \cdots \mathsf{k}_4} \delta_{\mathsf{k}_1+\mathsf{k}_2,\mathsf{k}_3+\mathsf{k}_4} V\left(\{\mathsf{k}_i\}\right) b^{\dagger}_{\mathsf{k}_1} b^{\dagger}_{\mathsf{k}_2} b_{\mathsf{k}_3} b_{\mathsf{k}_4}$$

$$V\left(\{\mathsf{k}_i\}\right) \sim \mathsf{k}_1 \cdot \mathsf{k}_2 + \mathsf{k}_3 \cdot \mathsf{k}_4 + \mathcal{O}(\mathsf{k}^4)$$

- no BEC!

Heisenberg antiferromagnet

$$\hat{H} = \frac{1}{2}\sum_{ij} J_{ij}\mathsf{S}_i \cdot \mathsf{S}_j \quad J_{ij} = J > 0$$

- conventional spin wave expansion
- transformations
  - Holstein-Primakoff

$$\begin{aligned} H &= E_{\mathrm{cl}} + H_2 + H_4 + \mathcal{O}(b^6) \\ E_{\mathrm{cl}} &= -DNJS^2 \\ H_2 &= S\sum_{ij} J_{ij}\left(b_i^\dagger b_i + b_j^\dagger b_j + b_i b_j + b_i^\dagger b_j^\dagger\right) \end{aligned}$$

  - Fourier transformation using reduced BZ

$$b_i = \sqrt{\frac{2}{N}}\sum_{\mathsf{k}} e^{i\mathsf{k}\cdot\mathsf{r}_i} A_{\mathsf{k}} \qquad b_i = \sqrt{\frac{2}{N}}\sum_{\mathsf{k}} e^{i\mathsf{k}\cdot\mathsf{r}_i} B_{\mathsf{k}}$$

  - diagonalization: Bogoliubov transformation

$$\begin{pmatrix} A_{\mathsf{k}} \\ B^\dagger_{-\mathsf{k}} \end{pmatrix} = \begin{pmatrix} u_{\mathsf{k}} & v_{\mathsf{k}} \\ -v_{\mathsf{k}} & u_{\mathsf{k}} \end{pmatrix} \begin{pmatrix} \alpha_{\mathsf{k}} \\ \beta^\dagger_{-\mathsf{k}} \end{pmatrix}$$

$$\begin{aligned} \epsilon_{\mathsf{k}} &= \sqrt{1 - \gamma_{\mathsf{k}} 2} \\ \gamma_{\mathsf{k}} &= \frac{1}{D}\sum_{i=1}^{D} \cos(k_i a) \end{aligned}$$

Anderson (1952), Kubo (1952), Oguchi (1960)

Spin-wave interactions

- 2 degenerate magnon modes

$$H_2 = \sum_{\mathsf{k}} E_{\mathsf{k}} \left(\alpha^{\dagger}_{\mathsf{k}} \alpha_{\mathsf{k}} + \beta^{\dagger}_{\mathsf{k}} \beta_{\mathsf{k}} + 1 \right) + E_0 \qquad E_{\mathsf{k}} = \tilde{J}_0 S \epsilon_{\mathsf{k}}$$

- spin-wave interactions: complicated

$$H_4 = \frac{1}{4V\chi_0} \sum_{\mathsf{k}_1 \cdots \mathsf{k}_4} V^{(1)}_{1234} \left(\beta^{\dagger}_1 \beta^{\dagger}_2 \beta_3 \beta_4 + \alpha^{\dagger}_3 \alpha^{\dagger}_4 \alpha_1 \alpha_2 \right) + \ldots$$

- interaction vertices

$$V^{(1)}_{1234} \sim \frac{1}{2} \sqrt{\frac{|k_1||k_2|}{|k_3||k_4|}} \left(1 + \frac{\mathsf{k}_1 \cdot \mathsf{k}_2}{|k_1||k_2|} \right)$$

infrared singular
in some limits

- but: all divergences cancel in physical quantities

N. Hasselmann, P. Kopietz, Europhys. Lett. 74, 1067 (2006)

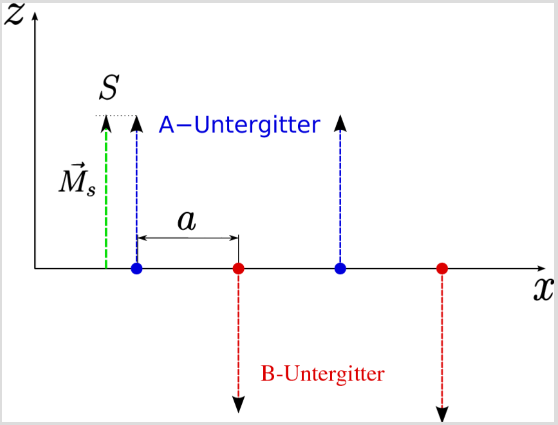
BEC of magnons in QAF

- quantum antiferromagnet in magnetic field $h = g\mu_B B$

$$\hat{H} = \frac{1}{2}\sum_{ij} J_{ij}\mathsf{S}_i \cdot \mathsf{S}_j - h\sum_i S_i^z$$

has $U(1)$ symmetry
rotational symmetry
around magnetic field

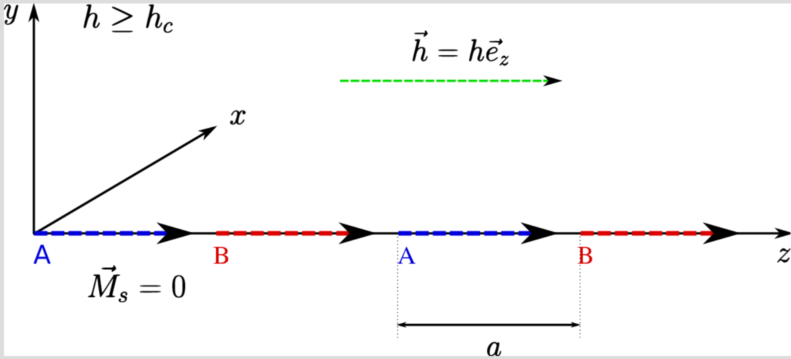
- classical ground states



I. $h = 0$

II. $h < h_c = 2\tilde{J}_0 S$

III. $h > h_c$



A. Kreisel, N. Hasselmann, and P. Kopietz,
Phys. Rev. Lett. **98**, 067230 (2007)

Formulation in terms of magnons

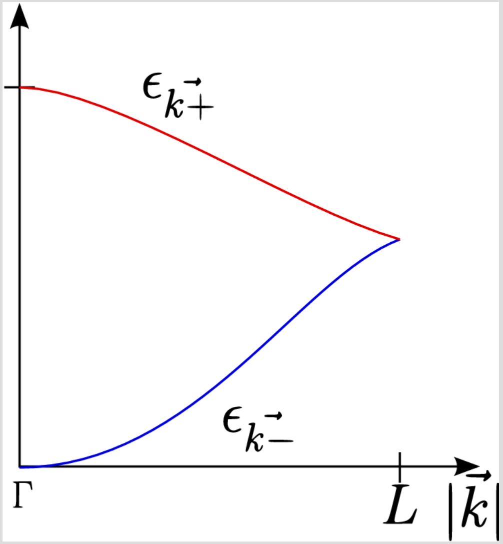
- 2 Transformations
  - Holstein-Primakoff (fluctuations around FM ground state)

$$S_i^- = \sqrt{2S} b_i^\dagger \sqrt{1 - \frac{n_i}{2S}} = (S_i^+)^\dagger \qquad n_i = b_i^\dagger b_i \quad , \quad \left[b_i^\dagger, b_j\right] = \delta_{ij}$$

$$S_i^z = S - n_i$$

$$H = E_{\mathrm{cl}} + H_2 + H_4 + \mathcal{O}(b^6)$$

  - fourier transformation: red. BZ (2 Modes)



$$H_2 = \sum_{\mathsf{k},\sigma=\pm} (\epsilon_{\mathsf{k}\sigma} - \mu)\, b_{\mathsf{k}\sigma}^\dagger b_{\mathsf{k}\sigma}$$

$$\epsilon_{\mathsf{k}-} = \frac{\mathsf{k}^2}{2m} + \mathcal{O}(\mathsf{k}^4) \qquad \epsilon_{\mathsf{k}+} = h_c + \mathcal{O}(\mathsf{k}^2) \quad h_c = 2\tilde{J}_0 S$$

Continuum formulation

- neglect gapped mode for simplicity

$$\Psi_{\mathsf{k}} = \sqrt{Na^D} b_{\mathsf{k}-}$$

- interacting bosonic fields

$$H = \int \frac{d^D \mathsf{k}}{(2\pi)^D} \left(\frac{\mathsf{k}^2}{2m} - \mu \right) \Psi^\dagger_{\mathsf{k}} \Psi_{\mathsf{k}} + \frac{1}{2} \int_{\mathsf{k}} \int_{\mathsf{k}'} \int_{\mathsf{q}} U_{\mathsf{q}} \Psi^\dagger_{\mathsf{k}+\mathsf{q}} \Psi^\dagger_{\mathsf{k}'-\mathsf{q}} \Psi_{\mathsf{k}'} \Psi_{\mathsf{k}}$$

- effective interaction: constant for small momenta

$$U_{\mathsf{q}} = \Theta(\Lambda_0 - |\mathsf{q}|) \chi_0^{-1}$$

$$\chi_0 = \frac{1}{2\tilde{J}_0 a^D} \qquad m = \frac{1}{2JSa^2}$$

Hermitian parametrization

- hermitian operators $\Pi_{\mathsf{k}} = \Pi^{\dagger}_{-\mathsf{k}} \qquad \Phi_{\mathsf{k}} = \Phi^{\dagger}_{-\mathsf{k}}$

$$[\Pi_{\mathsf{k}}, \Phi_{\mathsf{q}}] = i(2\pi)^D \delta(\mathsf{k} - \mathsf{q})$$

- transformation: split up field

$$\Psi_{\mathsf{k}} = \sqrt{\frac{s}{2}}\theta\Pi_{\mathsf{k}} + \frac{i}{\sqrt{2s\theta}}\Phi_{\mathsf{k}} \qquad s = \frac{S}{a^D}$$

transverse

longitudinal

- advantage: physical interpretation of operators

$$\Pi_{\mathsf{k}} \sim \sum_i \zeta_i e^{-i\mathsf{k}\cdot\mathsf{r}_i} S_i^x$$

$$\Phi_{\mathsf{k}} \sim \sum_i \zeta_i e^{-i\mathsf{k}\cdot\mathsf{r}_i} S_i^y$$

$$\mathsf{h} = h\mathsf{e}_z$$

$$\zeta_i = \begin{cases} 1 & i \in A \\ -1 & i \in B \end{cases}$$

P. W. Anderson (1952)

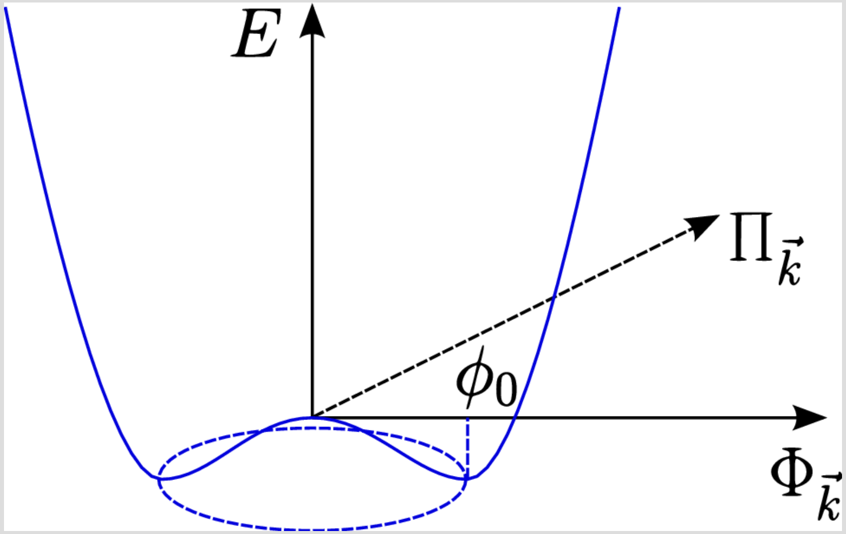
N. Hasselmann, P. Kopietz, Europhys. Lett. **74**, 1067 (2006)

Symmetry breaking

- impose nonzero expectation value

$$\Phi_{\mathsf{k}} = \Delta\Phi_{\mathsf{k}} + (2\pi)^D \, \delta(\mathsf{k})\phi_0$$

$$H = E_0 + H_1 + H_2 + H_3 + H_4$$



- fix chemical potential

$$\mu = \frac{U_0 s \phi_0}{2}$$

- diagonalization: Bogoliubov transformation

$$H_2 = \frac{1}{2} \int \frac{d^D \mathsf{k}}{(2\pi)^D} \epsilon_{\mathsf{k}} \left(\alpha^\dagger_{\mathsf{k}} \alpha_{\mathsf{k}} + \beta^\dagger_{\mathsf{k}} \beta_{\mathsf{k}} \right) + E_0$$

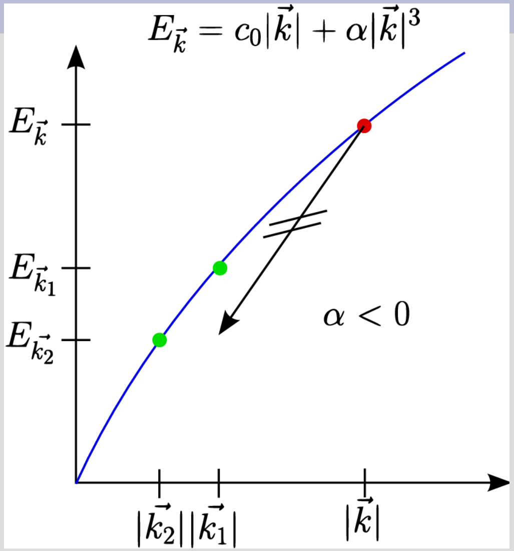
$$\epsilon_{\mathsf{k}} = c_0|k| + \alpha|k|^3 + \mathcal{O}(k^5) \qquad c_0 = \sqrt{\frac{\mu}{m}} \quad \alpha = \frac{1}{8\sqrt{m^3\mu}}$$

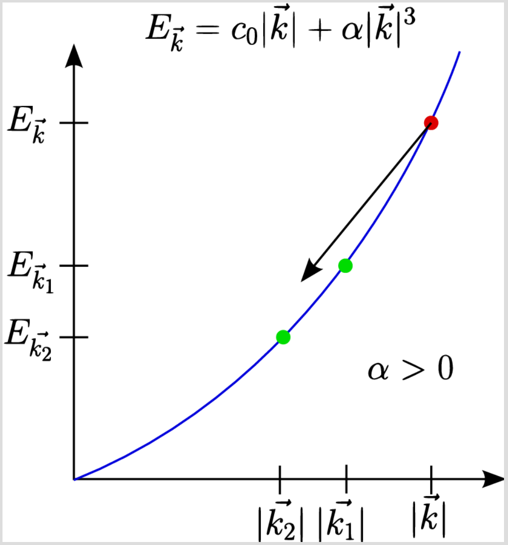
Quasiparticle decay

- decay in two or more quasiparticles
  - energy conservation $E_{\mathsf{k}} = E_{\mathsf{k}_1} + E_{\mathsf{k}_2}$
  - momentum conservation $\mathsf{k} = \mathsf{k}_1 + \mathsf{k}_2$
- consider energy dispersion

$$E_{\mathsf{k}} = c_0|\mathsf{k}| + \alpha|\mathsf{k}^3 + \mathcal{O}(\mathsf{k}^5)$$

  - stable quasiparticles $\alpha > 0$
  - unstable quasiparticles concerning spontaneous decay $\alpha < 0$





M. E. Zhitormirsky, A. L. Chernyshev, Phys. Rev. Lett. 82, 4536 (1999)

Correlation functions

- hermitian fields

$$\Psi_{\mathsf{k}} = \sqrt{\frac{s}{2}}\theta\Pi_{\mathsf{k}} + \frac{i}{\sqrt{2s\theta}}\Phi_{\mathsf{k}}$$

transverse

longitudinal

- correlations in gaussian approximation (linear spin-wave theory)

$$\begin{aligned}\langle\Pi_K\Pi_{K'}\rangle &= \delta_{K,-K'}\frac{\chi_0^{-1}}{\omega^2+c_0^2\mathsf{k}^2}\\ \langle\Pi_K\Phi_{K'}\rangle &= \delta_{K,-K'}\frac{\omega}{\omega^2+c_0^2\mathsf{k}^2}\\ \langle\Phi_K\Phi_{K'}\rangle &= \delta_{K,-K'}\frac{\chi_0 c_0^2\mathsf{k}^2}{\omega^2+c_0\mathsf{k}^2}\end{aligned}$$

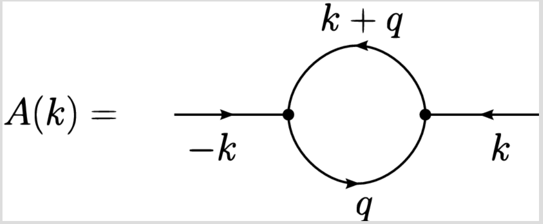
$$K = (\omega, \mathsf{k})$$

quantitatively wrong in gaussian approximation

C. Castellani, C. Di Castro, F. Pistolesi, and G. C. Strinati, Phys. Rev. Lett. 78, 1612 (1997)
F. Pistolesi, C. Castellani, C. Di Castro, and G. C. Strinati, Phys. Rev. B 69, 024513 (2004)

Beyond Bogoliubov

- divergences in pertubation theory (contribution to anomal Propagator)



- cancellations controlled by Ward identities due to $\mathcal{U}(1)$ - symmetry (2 correlations correct)
- critical continuum in longitudinal correlation function

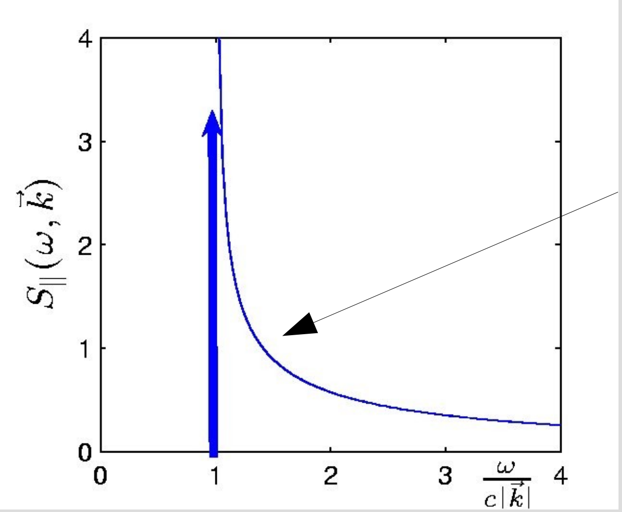
$$\langle \Phi_K \Phi_{-K} \rangle = -Z_\parallel^2 \frac{\omega^2}{\omega^2 + c^2 \mathsf{k}^2} + K_{D+1} \frac{(mc)^3}{Z_\rho^2 \rho_0} \begin{cases} \ln\left[\frac{(mc)^2}{\omega^2/c^2+\mathsf{k}^2}\right] & D=3 \\ \frac{2}{3-D}\left[\frac{\omega^2}{c^2}+\mathsf{k}^2\right]^{\frac{D-3}{2}} & 1<D<3 \end{cases}$$

- contribution cancels in physical quantities of the Bose gas
- observable? Yes, BEC of magnons!
  fields: staggered magnetization

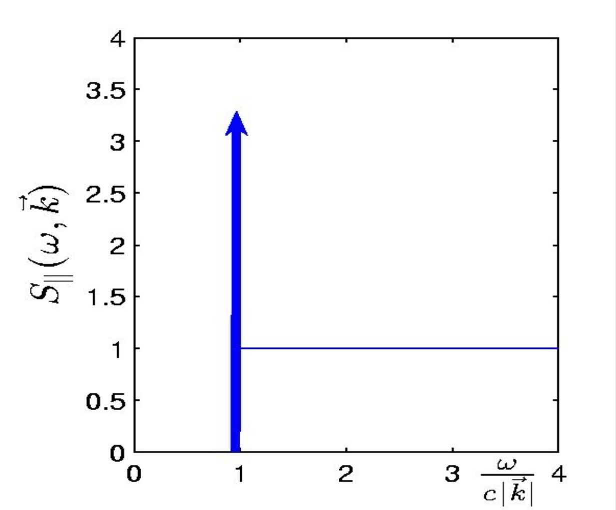
Possible measurement

- neutron scattering might detect anomalous dimension in correlation function
- longitudinal staggered structure factor

$$S_{\parallel}(\mathsf{k},\omega) = \frac{\chi s^2}{M_s^2}\left[\frac{Z_{\parallel}^2}{2} c|\mathsf{k}|\,\delta(\omega - c|\mathsf{k}|) + \frac{C_D (mc)^2 \Theta(\omega - c|\mathsf{k}|)}{Z_{\rho}^3 \rho_0 \left(\frac{\omega^2}{c^2} - \mathsf{k}^2\right)^{\frac{3-D}{2}}}\right]$$

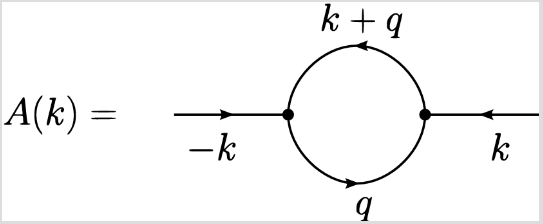


critical continuum only valid at $\frac{\omega}{c} \leq k_G$ (Ginzburg-scale)



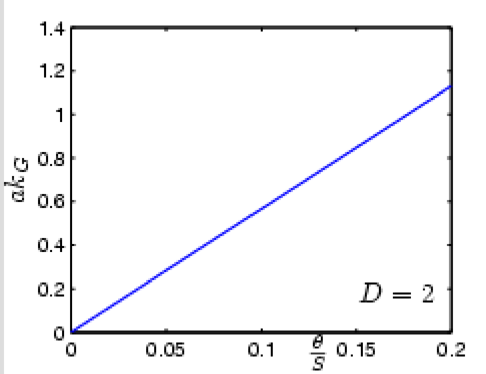
Restrictions

- Ginzburg-scale $\frac{\omega}{c} \leq k_G$
- 2 contributions to abnormal propagator
  - symmetry breaking
  - divergent diagrams



$$k_G \approx \frac{(mc)^3}{\rho_0} \approx \frac{4\sqrt{2}}{Sa}\theta \sim \theta \qquad D = 2$$

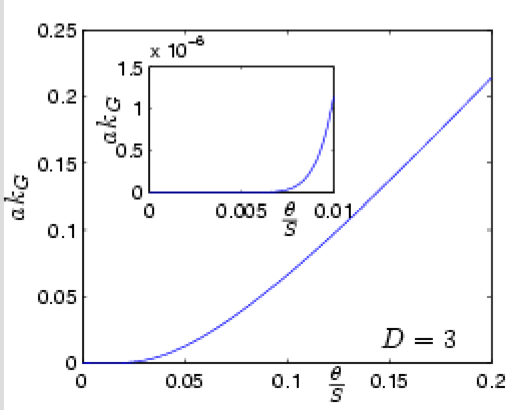
$$k_G \approx mc\, e^{-\frac{\rho_0}{(mc)^3}} \approx \frac{\sqrt{3}}{a}\theta e^{-\frac{S}{6\sqrt{3}\theta}} \qquad D = 3$$



- energy integrated structure factor
  spectral weight
  I_δ : δ -function $\quad I_c$: critical continuum

$$\frac{I_c}{I_\delta} \approx \frac{k_G}{|\mathsf{k}|} \ln\left(\frac{k_G}{|\mathsf{k}|}\right) \qquad D = 2$$

$$\frac{I_c}{I_\delta} \approx \frac{(mc)^2}{\rho_0}\frac{k_G}{|\mathsf{k}|} \qquad D = 3$$



Summary

- not-so-well known facts about the Bose gas
  - large corrections to Bogoliubov approximation in $1 < D \leq 3$
- Spin-wave interactions in QAF
  - interaction vertices: divergent (usual spin-wave expansion)
  - hermitian field parametrization
    - weak interactions between Goldstone modes
    - fields: physical quantities
- QAF in uniform magnetic field: BEC of magnons
  - measurable quantities via
    neutron scattering
    but only at D=2