Spin Wave Theory for Magnetic Insulators

From textbook knowledge towards BEC of magnons

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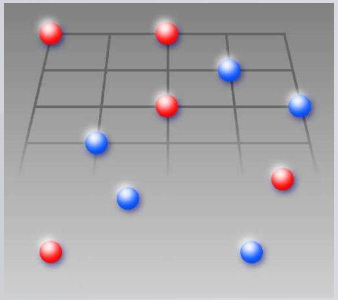
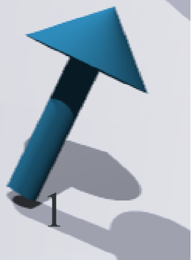
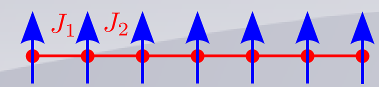




Outline

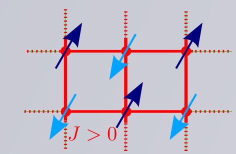
1. Introduction to magnetism



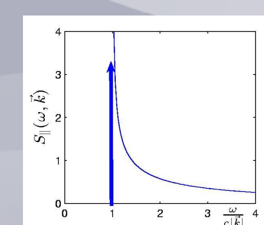
$$\chi = \frac{\partial \vec{M}}{\partial \vec{H}}$$

2. Introduction to spin wave theory

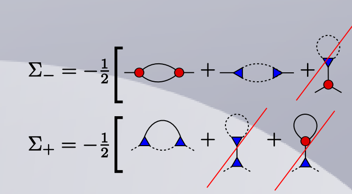
$$H = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$$



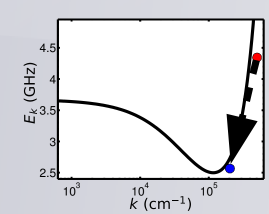
3. Applications
   1. BEC of magnons in an Antiferromagnet anomalous longitudinal fluctuations



   2. Antiferromagnet in a magnetic field: non analytic properties of the magnon spectrum



   3. Spin waves in thin film ferromagnets with dipole-dipole interactions

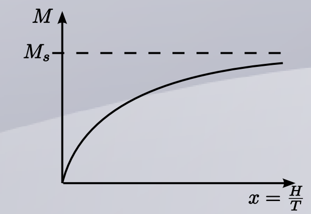


1. Magnetism

- general types of magnetism in condensed matter

$$\chi = \frac{\partial \vec{M}}{\partial \vec{H}}$$

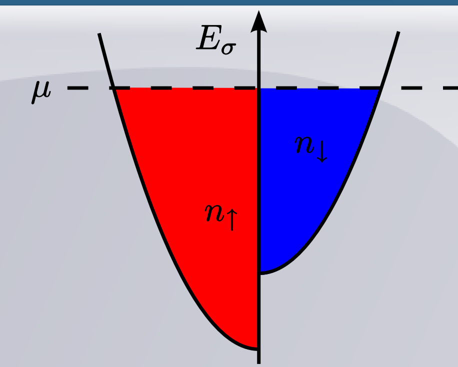
  - diamagnetism
    response to magnetic field due to Lenz' rule
    $\vec{L}\,|0\rangle = 0 \quad \vec{S}\,|0\rangle = 0$ $\qquad \chi = -\frac{e^2}{6m}\mu_0 n r_a^2 Z_a \approx -10^{-4} < 0$
  - paramagnetism
    alignment of magnetic moment in a magnetic field
    $$\chi \sim \frac{1}{T} > 0$$



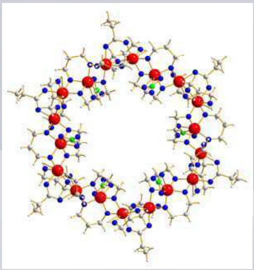
  - **collective magnetism**
    correlation effects of spin degrees of freedom
    $\chi \to \infty$ (phase transitions)

1. Collective magnetism

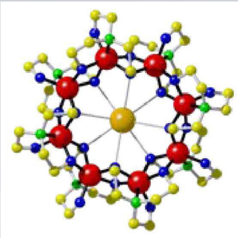
- itinerant magnetism in metals
  magnetization from occupation of states
  $M \sim n_\uparrow - n_\downarrow$



- magnetism on mesoscopic scales

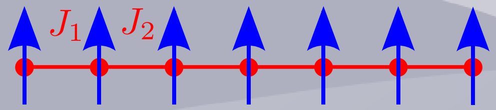


Fe_{18}



$CsFe_8$

finite numbers of spins:
nevertheless collective
behaviour



model: finite string
even-odd effects

method: exact diagonalisation
O. Waldmann *et al.* PRL ('06)
V. Paschenko *et al.* ('07)
K. Removic-Langer *et al.* ('09)

- magnetism of localized spins in insulators

$$H = J\vec{S}_1 \cdot \vec{S}_2 \quad \rightarrow \quad H = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

Heisenberg model

W. Heisenberg Z. Phys. **49**, 619 (1928)

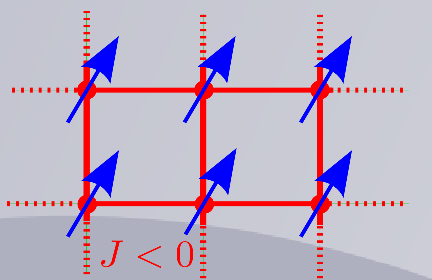
1. Heisenberg Model

$$H = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

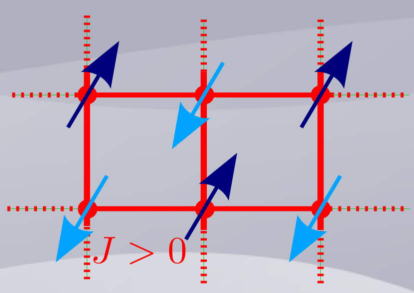
- dimensionality
  - Mermin Wagner theorem: no spontaneous ordering in isotropic Heisenberg models at T>0 in D<3, (PRL **17**, 1133 (1966))
  - different methods e.g. Bethe-Ansatz (D=1), spin-wave theory (D>1), quantum field theory (NLSM)
- sign of couplings
  - ferromagnetism J<0
  - antiferromagnetism J>0
- lattice
  - hypercubic lattice
  - triangular, honeycomb lattice (frustration)

2. Spin wave theory

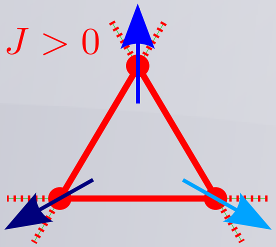
- determine ordered classical groundstate
  - ferromagnet
    classical groundstate=quantum groundstate



  - anti-ferromagnet
    (2 sublattices, Néel groundstate)



  - triangular anti-ferromagnet
    (3 sublattices, frustration)



Chernychev, Zhitomirsky ('09)
Veillette *et al.* ('05)

2. Spin wave theory

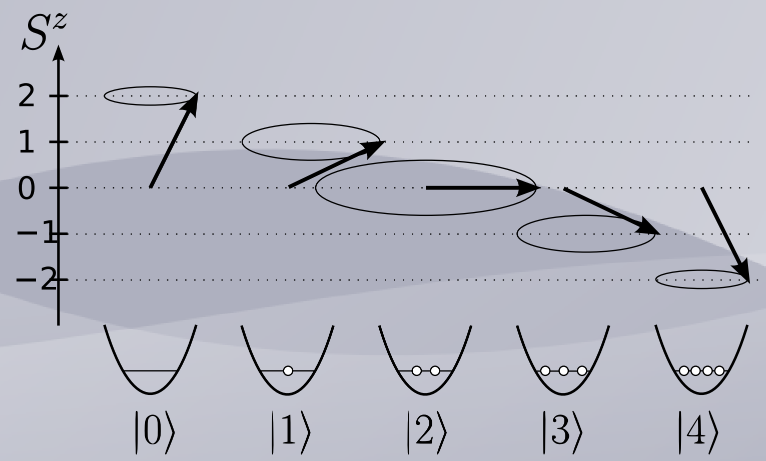
- expand in terms of bosons (1/S expansion), Holstein-Primakoff transformation

$$H = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

$$\hat{S}^z = S - \hat{n} \qquad \hat{n} = \hat{b}^\dagger \hat{b} \qquad \left[\hat{b}, \hat{b}^\dagger\right] = 1$$

$$\hat{S}^+ = \sqrt{2S}\sqrt{1 - \frac{\hat{n}}{2S}}\,\hat{b}$$

$$\hat{S}^- = \sqrt{2S}\,\hat{b}^\dagger\sqrt{1 - \frac{\hat{n}}{2S}} \qquad \sqrt{1 - \frac{\hat{n}}{2S}} = 1 - \frac{\hat{n}}{4S} + \mathcal{O}(\frac{1}{S^2})$$



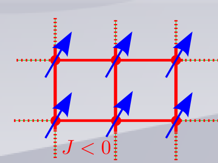
Holstein, Primakoff, Phys. Rev. **58**, 1098 (1940)

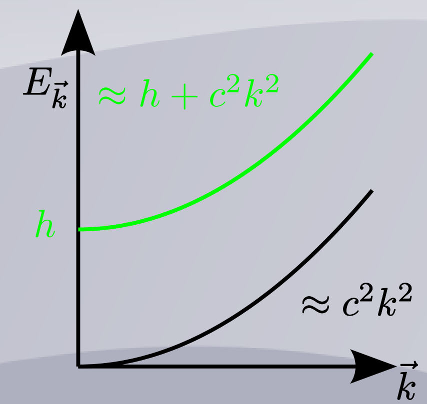
- determine properties of resulting interacting theory of bosons

$$H = \sum_{\vec{k}} E_{\vec{k}} b^\dagger_{\vec{k}} b_{\vec{k}} + \sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \Gamma^3(\vec{k}_1, \vec{k}_2, \vec{k}_3) b^\dagger_{\vec{k}_1} b_{\vec{k}_2} b_{\vec{k}_3} + \sum_{1,2,3,4} \Gamma^4(1,2;3,4) b^\dagger_1 b^\dagger_2 b_3 b_4 + \ldots$$

2. Spin wave theory: General results

- ferromagnet

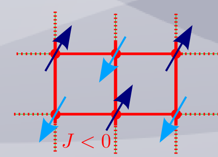


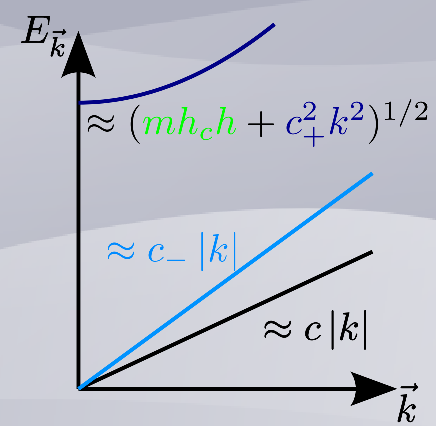


  - quadratic excitation spectrum
  - vanishing interaction vertices

$$\Gamma^4 \sim -(\vec{k}_1 \cdot \vec{k}_2 + \vec{k}_3 \cdot \vec{k}_4)$$

- antiferromagnet





  - linear spectrum (Goldstone mode)
  - two modes in magnetic field (2 sublattices)
  - divergent interaction vertices

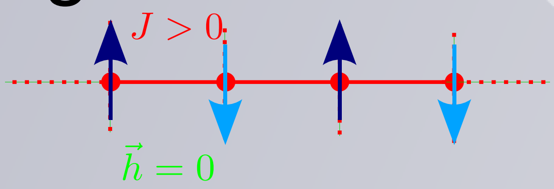
$$\Gamma^4 \sim \sqrt{\frac{|\vec{k}_1||\vec{k}_2|}{|\vec{k}_3||\vec{k}_4|}} \left(1 \pm \frac{\vec{k}_1 \cdot \vec{k}_2}{|\vec{k}_1||\vec{k}_2|} \right)$$

Hasselmann, Kopietz ('06)

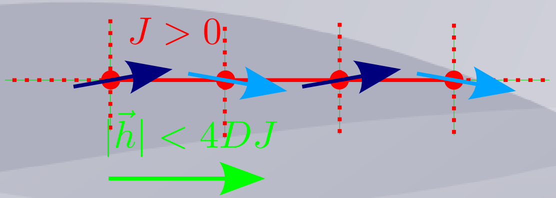
3.1 Bose Einstein Condensation of magnons in Quantum AF

- quantum Antiferromagnet in a magnetic field

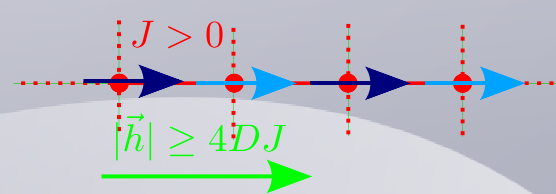
$$\hat{H} = \frac{1}{2}\sum_{ij} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j - h\sum_i S_i^z$$



- rotational symmetry around magnetic field
  U(1) symmetry of Hamiltonian



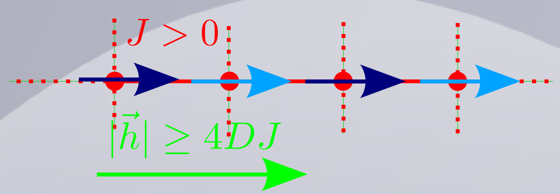
$$\hat{b} \rightarrow \hat{b}' = e^{i\phi} b \quad H \rightarrow H' = H$$



AK, Hasselmann, Kopietz ('07)

3.1 Spin Wave Theory

- expansion around ferromagnetic state for strong field

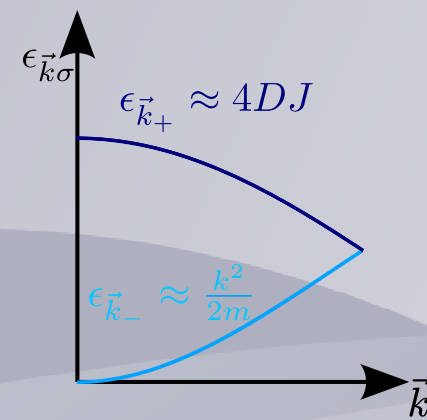


$$S_i^- = \sqrt{2S} b_i^\dagger \sqrt{1 - \frac{n_i}{2S}} = (S_i^+)^\dagger$$

$$S_i^z = S - n_i$$

$$H_2 = \sum_{\vec{k},\sigma=\pm} \left(\epsilon_{\vec{k}\sigma} - \mu\right) b_{\vec{k}\sigma}^\dagger b_{\vec{k}\sigma}$$

Batyev, Braginskii ('84)



- describe antiferromagnetic phase using condensation of magnons $\rho_0 = \frac{\mu}{U(0)}$ condensate density
- important quantities

$$E_{\vec{k}} = \sqrt{\epsilon_{\vec{k}}^2 + 2\rho_0 U(\vec{k})\epsilon_{\vec{k}}} = c_0 \left|\vec{k}\right| + \alpha \left|\vec{k}\right|^3 + \mathcal{O}(\vec{k}^5)$$ dispersion of Goldstone mode (gapless, linear)

$$c_0 = \sqrt{\frac{\mu}{m}}$$ spin wave velocity

Chernychev *et al.* ('98)

3.1 Theory of interacting Bose Gas in symmetry broken phase

- infra-red divergences in perturbation theory
- technically:

$$\left[\hat{\Pi}_{\vec{k}}, \hat{\Phi}_{\vec{q}}\right] = i(2\pi)^D \delta(\vec{k} - \vec{q})$$

  - split field

$$\Psi_{\vec{k}} = \sqrt{\frac{s}{2}}\theta\Pi_{\vec{k}} + \frac{i}{\sqrt{2s\theta}}\Phi_{\vec{k}} \qquad s = \frac{S}{a^D}$$

transverse

longitudinal

  - physical interpretation in the case of the magnetic system: staggered magnetization

$$\Pi_{\vec{k}} \sim \sum_i \zeta_i e^{-i\vec{k}\cdot\vec{r}_i} S_i^x \qquad \zeta_i = \begin{cases} 1 & i \in A \\ -1 & i \in B \end{cases}$$

$$\Phi_{\vec{k}} \sim \sum_i \zeta_i e^{-i\vec{k}\cdot\vec{r}_i} S_i^y$$

3.1 Correlations

- correlators in gaussian approximation

$$\langle \Pi_K \Pi_{K'} \rangle = \delta_{K,-K'} \frac{\chi_0^{-1}}{\omega^2 + c_0^2 \vec{k}^2} \qquad K = (\omega, \vec{k})$$

$$\langle \Pi_K \Phi_{K'} \rangle = \delta_{K,-K'} \frac{\omega}{\omega^2 + c_0^2 \vec{k}^2}$$

$$\langle \Phi_K \Phi_{K'} \rangle = \delta_{K,-K'} \frac{\chi_0 c_0^2 \vec{k}^2}{\omega^2 + c_0 \vec{k}^2}$$

quantitatively wrong in Gaussian approximation
mean field theory wrong in D=3 and D=2
Castellani. *et al.* ('00)
Pistolesi *et al.* ('04)

- beyond Bogoliubov (RG approach for Bose gas)

renormalization

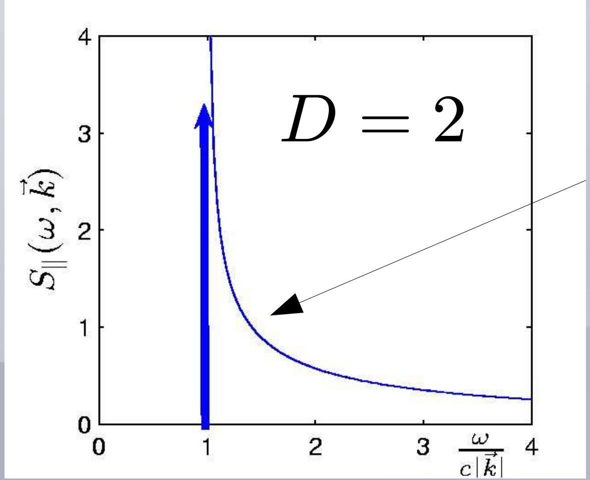
$$\langle \Phi_K \Phi_{-K} \rangle = -Z_\parallel^2 \frac{\omega^2}{\omega^2 + c^2 \vec{k}^2} + K_{D+1} \frac{(mc)^3}{Z_\rho^2 \rho_0} \begin{cases} \ln\left[\frac{(mc)^2}{\omega^2/c^2 + \vec{k}^2}\right] & D = 3 \\ \frac{2}{3-D}\left[\frac{\omega^2}{c^2} + \vec{k}^2\right]^{\frac{D-3}{2}} & 1 < D < 3 \end{cases}$$

extra term

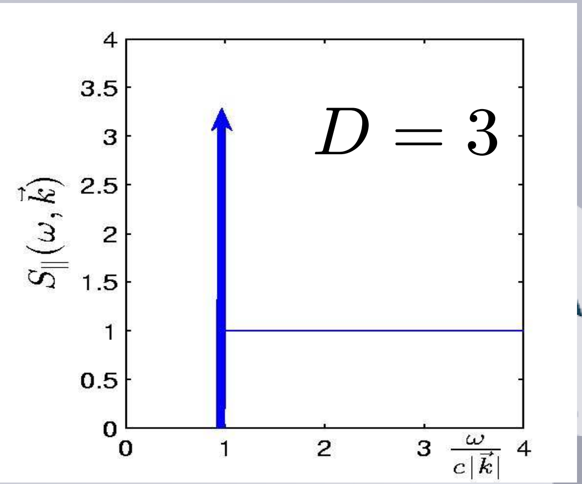
3.1 Possible measurement in magnetic systems

- physical interpretation of longitudinal fluctuations
- scattering experiment: longitudinal staggered structure factor

$$S_{\|}\left(\vec{k},\omega\right)=\frac{\chi s^2}{M_s^2}\left[\frac{Z_{\|}^2}{2}c|\vec{k}|\,\delta\left(\omega-c|\vec{k}|\right)+\frac{C_D(mc)^2\Theta\left(\omega-c|\vec{k}|\right)}{Z_\rho^3\rho_0\left(\frac{\omega^2}{c^2}-\vec{k}^2\right)^{\frac{3-D}{2}}}\right]$$

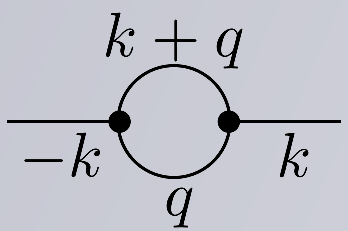


critical continuum only valid at $\omega/c \leq k_G$ (Ginzburg-scale)



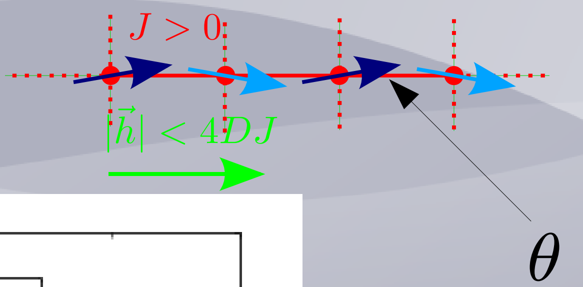
3.1 Restrictions: Ginzburg scale

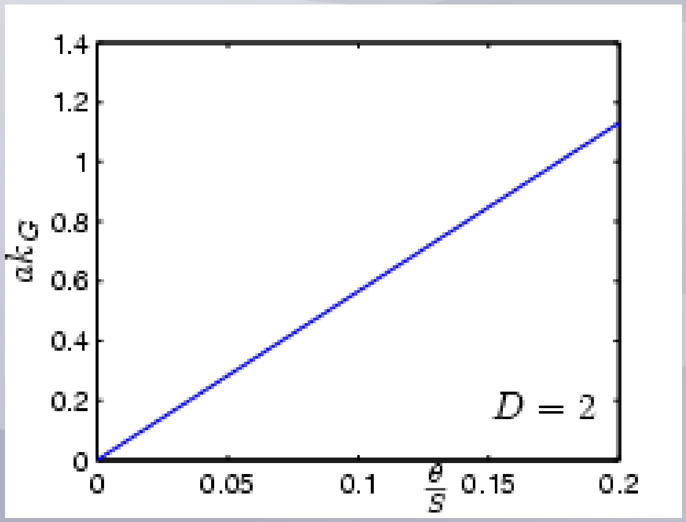
- 2 contributions to abnormal propagator
  - symmetry breaking
  - divergent diagrams

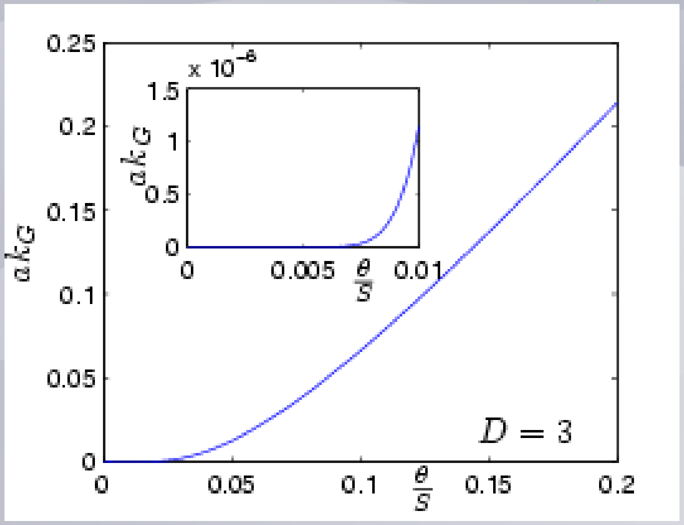
$A(k) =$ 

$$k_G \approx \frac{(mc)^3}{\rho_0} \approx \frac{4\sqrt{2}}{Sa}\theta \sim \theta \qquad D = 2$$

$$k_G \approx mc\, e^{-\frac{\rho_0}{(mc)^3}} \approx \frac{\sqrt{3}}{a}\theta e^{-\frac{S}{6\sqrt{3}\theta}} \qquad D = 3$$







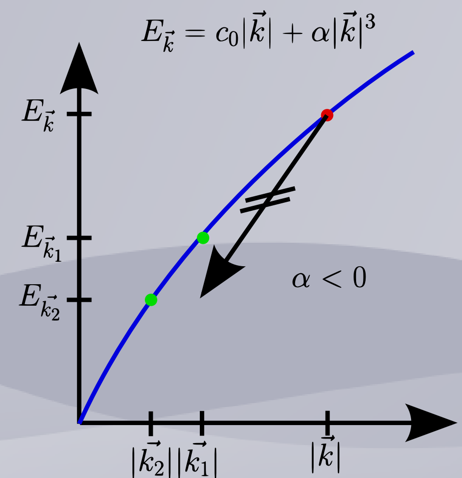
3.1 Spin wave damping (Quasiparticle decay)

- spontaneous decay (3-point vertex)
  - energy conservation

$$E_{\vec{k}} = E_{\vec{k}_1} + E_{\vec{k}_2}$$

  - momentum conservation

$$\vec{k} = \vec{k}_1 + \vec{k}_2$$

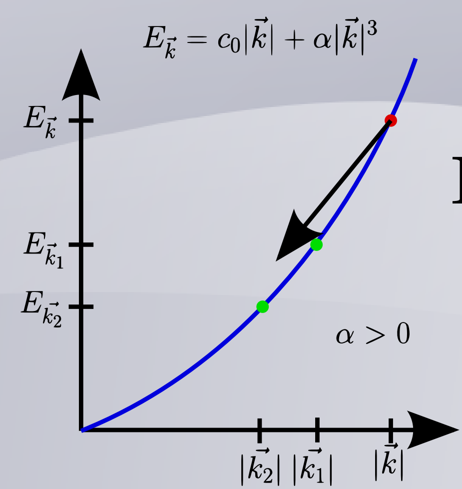


$$\Gamma_{\vec{k}} \approx 0$$

- energy dispersion

$$E_{\vec{k}} = c_0|\vec{k}| + \alpha|\vec{k}|^3 + \mathcal{O}(\vec{k}^5)$$

$$c_0 = \sqrt{\frac{\mu}{m}} \quad \alpha = \frac{1}{8\sqrt{m^3\mu}}$$



$$\Gamma_{\vec{k}} \sim \frac{1}{S}|\vec{k}|^3$$

3.1 Recapitulation

- description of the antiferromagnetic phase below the critical magnetic field as BEC of magnons
- non analytic behaviour of the interacting bose gas in the symmetry broken phase
- possible measurement via neutron scattering experiment
- restrictions:
  - material with reachable critical field
  - D=2

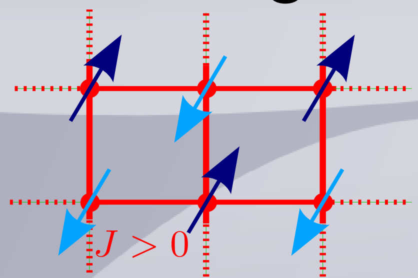
3.2 Quantum Antiferromagnet in a magnetic field

- **Non Linear Sigma Model**
  [Chakravarty, Halperin, Nelson, PRB **39**, 2344 (1989)]
  - effective continuum theory for staggered spin fluctuations
  - continuum theory yields long wavelength results
  - simple interaction vertices without singularities
  - regularization using arbitrary ultraviolett cutoff: method can only be used to obtain universal quantities

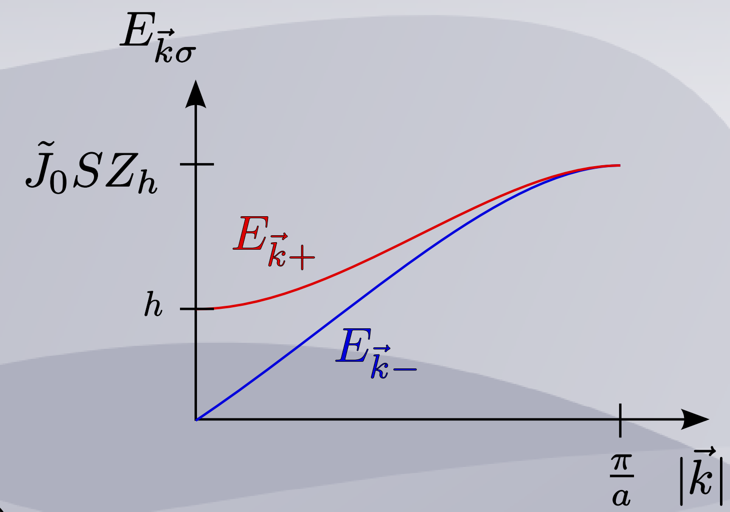
- **1/S-expansion**
  [Oguchi, Phys. Rev. **117**, 117 (1960), Harris *et al.* PRB **3**, 961 (1971)]
  - perturbative expansion in powers of 1/S
  - lattice theory to describe short wavelength results
  - tedious interaction vertices with infrared divergences which anyway cancel in physical quantities
  - method can be used to obtain all physical quantities

3.2 Energy dispersion in leading order spin-wave theory

- Long-wavelength limit



$$E^2_{\vec{k}+} = h^2 + c_+^2 \vec{k}^2,$$
$$E^2_{\vec{k}-} = c_-^2 \vec{k}^2,$$



with the spin-wave velocities

$$c_+^2 = c_0^2 (1 - \frac{3}{\Delta_0^2} h^2),$$
$$c_-^2 = c_0^2 (1 - \frac{1}{\Delta_0^2} h^2),$$

AK, Sauli, Hasselmann, Kopietz ('08)

where $\Delta_0 = 4DJS$ and c_0 is the leading large-S result for spin-wave velocity for $h = 0$.

3.2 Euclidean action of NLSM with uniform magnetic field

- $$\begin{aligned} S_{\text{NLSM}}[\vec{\Omega}] &= \frac{\rho_s}{2}\int_0^\beta d\tau \int d^D r \Big[\sum_{\mu=1}^{D}(\partial_\mu \vec{\Omega})^2 \\ &\quad + c^{-2}(\partial_\tau \vec{\Omega} - i\vec{h}\times\vec{\Omega})^2\Big], \end{aligned}$$

  spin field

- spin stiffness and spin wave velocity at T=0

$$\begin{aligned} \rho_s &= JS^2 a^{2-D} \\ c &= 2JSa\sqrt{D} \end{aligned}$$

- effect of the magnetic field $\partial_\tau \to \partial_\tau - i\vec{h}\times$
- but the magnon dispersions can not be characterized by a single $c(h)$!

3.2 Spin wave theory with Hermitian field operators

- $$\hat{\Psi}_{\vec{k}\sigma} = p_\sigma \left[\sqrt{\frac{\nu_{\vec{k}\sigma}}{2}} \hat{X}_{\vec{k}\sigma} + \frac{i}{\sqrt{2\nu_{\vec{k}\sigma}}} \hat{P}_{\vec{k}\sigma} \right]$$
- commutation relations: $[\hat{X}_{\vec{k}\sigma}, \hat{P}_{\vec{k}'\sigma'}] = i\delta_{\vec{k},-\vec{k}'}\delta_{\sigma,\sigma'}$
- physical meaning for $k \ll 1$

  $\hat{P}_{\vec{k}\sigma}$: uniform spin fluctuations

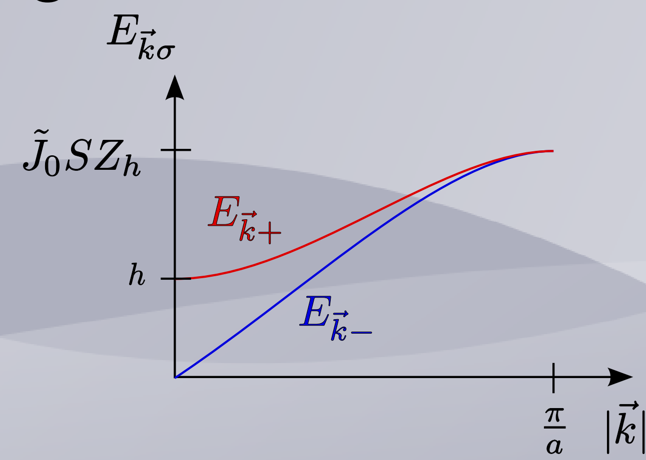
  $\hat{X}_{\vec{k}\sigma}$: staggered spin fluctuations
- those operators yield regular vertices!
- construct effective action for staggered fluctuations

P. W. Anderson, Phys. Rev. 86, 694 (1952),
N. Hasselmann and P. Kopietz, Europhys. Lett. 74, 1067 (2006)

3.2 Effective Model

- eliminate degrees of freedom associated with the operator of the ferromagnetic fluctuations

$$e^{-S_{\text{eff}}[X_\sigma]} = \int \mathcal{D}[P_\sigma] e^{-S[P_\sigma, X_\sigma]}$$



$$\begin{aligned} S_{\text{eff}}[X_\sigma] &= S_0 + \frac{\beta}{2} \sum_{K\sigma} \frac{E^2_{\mathsf{k}\sigma} + \omega^2}{\Delta_{\mathsf{k}\sigma}} X_{-K\sigma} X_{K\sigma} \\ &+ \beta \sqrt{\frac{2}{N}} \sum_{K_1 K_2 K_3} \delta_{K_1+K_2+K_3,0} \Big[\frac{1}{3!} \Gamma^{(3)}_{---}(K_1, K_2, K_3) X_{K_1-} X_{K_2-} X_{K_3-} \\ &+ \frac{1}{2!} \Gamma^{(3)}_{-++}(K_1; K_2, K_3) X_{K_1-} X_{K_2+} X_{K_3+} \Big] \end{aligned}$$

$$K = (\mathsf{k}, i\omega_n)$$

3.2 Hybrid approach: Comparison

- generalization of Non-Linear-Sigma-Model for QAF subject to magnetic field

$$S_{\mathrm{NLSM}}[\Omega] \approx -\beta V \frac{\chi}{2} h^2 + \frac{\chi}{2} \int_K \sum_\sigma (\omega^2 + c^2 \mathsf{k}^2 + m_\sigma^2) \Pi_{-K\sigma} \Pi_{K\sigma} \qquad \chi = \frac{\rho}{c^2}$$

$$-i\chi h \int_0^\beta d\tau \int d^D r \Pi_+^2 \partial_\tau \Pi_- + \mathcal{O}(\Pi_\sigma^4)$$

$$\partial_\tau \Omega \to (\partial_\tau - i\mathsf{h}\times)\Omega$$

$$X_\sigma \hat{=} \Pi_\sigma$$

$$E_{\mathsf{k}+}^2 \approx m\Delta_0 h + c_+^2 \mathsf{k}^2$$
$$E_{\mathsf{k}-}^2 \approx c_-^2 \mathsf{k}^2$$

$$S_{\mathrm{eff}}[X_\sigma] = S_0 + \frac{\beta}{2} \sum_{K\sigma} \frac{E_{\mathsf{k}\sigma}^2 + \omega^2}{\Delta_{\mathsf{k}\sigma}} X_{-K\sigma} X_{K\sigma} +$$

$$+\beta\sqrt{\frac{2}{N}} \sum \left[\frac{1}{3!} \Gamma_{---}^{(3)} X_- X_- X_- + \frac{1}{2!} \Gamma_{-++}^{(3)} X_- X_+ X_+ \right]$$

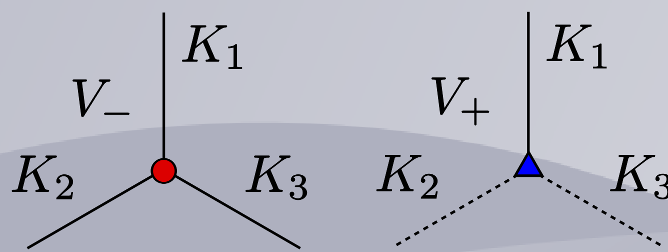
$$\Gamma_{---}^{(3)} \approx 0 \qquad \Gamma_{-++}^{(3)} \approx -2 \frac{\lambda}{\sqrt{8S}} \omega_1$$

3.2 1/S Corrections

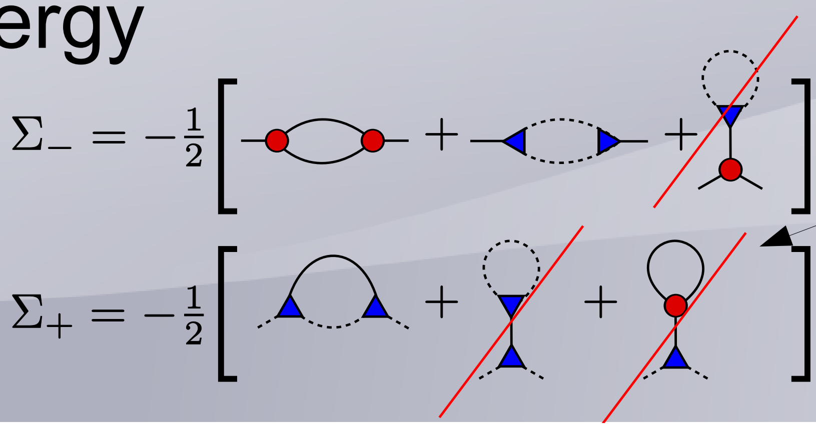
- diagrammatic perturbation theory

$$S^{\text{int}}_{\text{eff}}[X_\sigma] \quad = \beta\sqrt{\frac{2}{N}}\sum\left[\frac{1}{3!}V_-^{(3)}X_-X_-X_- + \frac{1}{2!}V_+^{(3)}X_-X_+X_+\right]$$

$$G_\sigma(K) = \frac{\Delta_{\mathsf{k}\sigma}}{E^2_{\mathsf{k}\sigma}+\omega^2}$$



- perturbation theory: 1/S corrections to self energy



no frequency dependence, negligible

3.2 Results

- leading order expansion of self energy

$$\Sigma_-(K) = C^\omega \omega^2 + C^k c_0^2 \mathsf{k}^2 + \mathcal{O}(\omega^4, k^4)$$

Zhitomirsky, Chernychev, '98

- full propagator

$$G_-(K) = \frac{\Delta_{\mathsf{k}-}}{\omega^2 + E_{\mathsf{k}-}^2 + \Delta_{\mathsf{k}-}\Sigma_-(K)} \approx \frac{Z_-\Delta_0 n^2}{\omega^2 + c_-(h)^2\mathsf{k}^2}$$

- spin wave velocity of gapless mode

non analytic in h^2

$$\frac{c_-^2}{c_0^2} \approx 1 - \Delta_0 n^2 C^\omega \approx 1 - \frac{6\sqrt{3}\,\tilde{h}^2}{\pi^2 S} \ln\left(\frac{2}{\tilde{h}}\right) \qquad D = 3$$

$$\frac{c_-^2}{c_0^2} \approx 1 - \Delta_0 n^2 C^\omega \approx 1 - \frac{2\tilde{h}}{\pi S} \qquad D = 2$$

$$\tilde{h} = \frac{h}{\Delta_0}$$

3.2 Recapitulation

- new formulation for the quantum antiferromagnet in a magnetic field: Combine NLSM with 1/S expansion (spin wave theory)
- advantage: physical interpretation of field operators
- results: non analytic dependences of the spin wave velocity

3.3 Spin wave theory for thin film ferromagnets

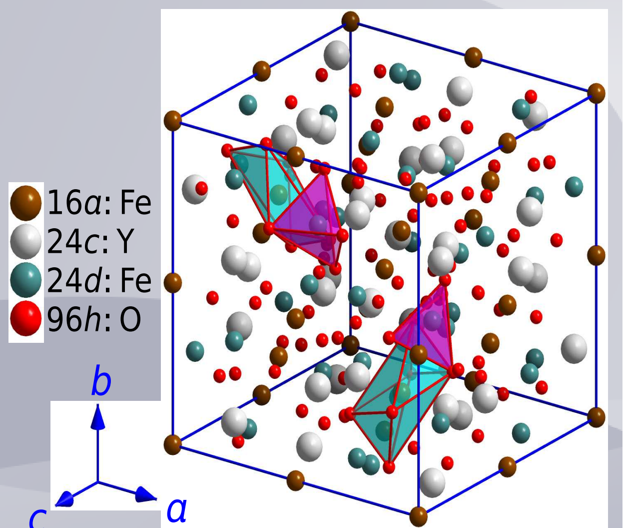
- Motivation: Experiments on YIG
  - Crystal structure:

space group: **Ia3d**
Y: 24(c) white
Fe: 24(d) green
Fe: 16(a) brown
O: 96(h) red

Gilleo *et al.* '58

Magnetic system:
40 magnetic ions in elementary cell
40 magnetic bands

Elastic system:
160 atoms in
elementary cell
3x160 phonon bands

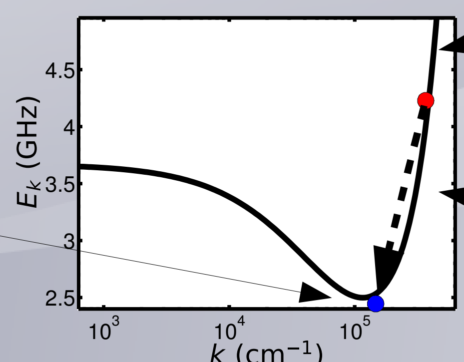


  - low spin wave damping
  - good experimental control

Observation of the occupation number using microwave antennas or Brillouin Light Scattering (BLS)

Bose-Einstein Condensation of magnons at room temperature!

Demokritov *et al.* Nature **443**, 430 (2006)

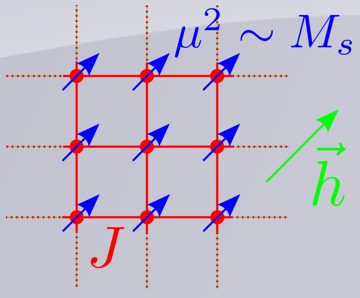


Parametric pumping of magnons at high k-vectors creates magnetic excitations

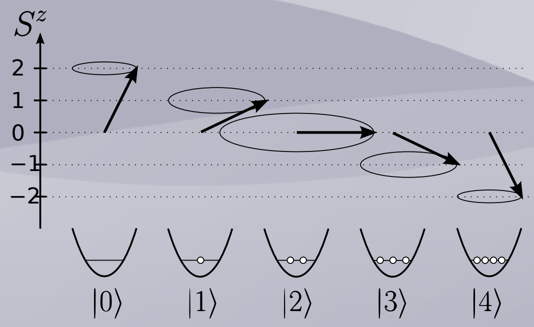
Question:
Time evolution of magnons:
Non-equilibrium physics of interacting quasiparticles

3.3 Procedure

- Microscopic Hamiltonian



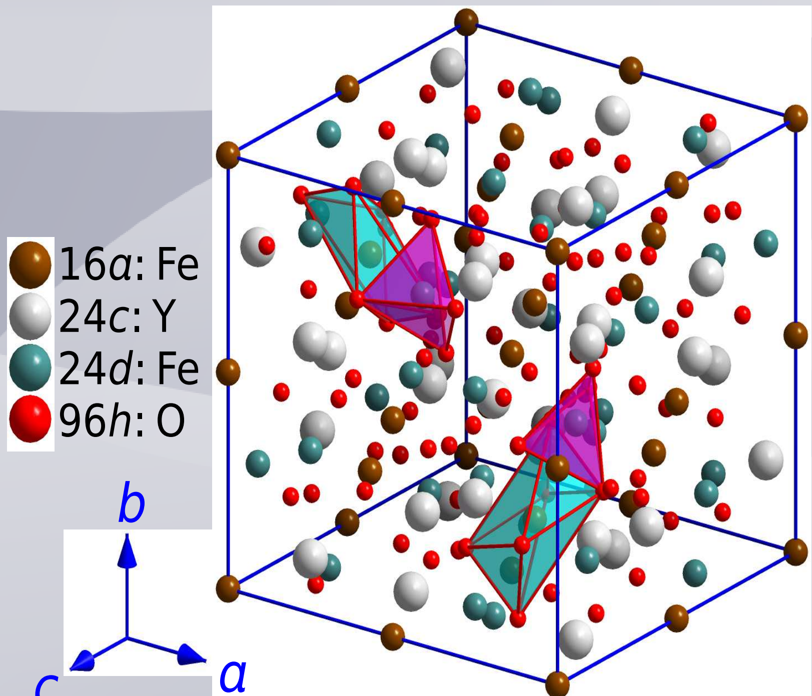
- Quantum Theory of Magnons
  - Linear Spinwave theory: spectrum



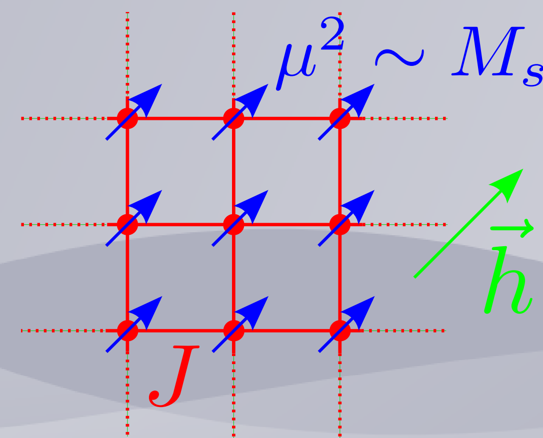
  - Interactions: damping, energy shift

3.3 Simplifications to relevant physical properties

crystal structure of YIG



microscopic Hamiltonian



quantum spin S ferromagnet

Zeeman term

dipole-dipole interactions

$$\hat{H}_{\text{mag}} = -\frac{1}{2}\sum_{ij} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j - h\sum_i S_i^z$$

$$-\frac{1}{2}\sum_i \sum_{j \neq i} \frac{\mu^2}{|\mathbf{r}_{ij}|^3}\left[3(\mathbf{S}_i \cdot \hat{\mathbf{r}}_{ij})(\mathbf{S}_j \cdot \hat{\mathbf{r}}_{ij}) - \mathbf{S}_i \cdot \mathbf{S}_j\right]$$

3.3 Microscopic Hamiltonian: Heisenberg model (lowest band)

quantum spin S ferromagnet

Zeeman term

dipole-dipole interactions

$$\hat{H}_{\mathrm{mag}} = -\frac{1}{2}\sum_{ij} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j - h\sum_i S_i^z$$

$$- \quad \frac{1}{2}\sum_i \sum_{j\neq i} \frac{\mu^2}{|\mathbf{r}_{ij}|^3}\left[3(\mathbf{S}_i \cdot \hat{\mathbf{r}}_{ij})(\mathbf{S}_j \cdot \hat{\mathbf{r}}_{ij}) - \mathbf{S}_i \cdot \mathbf{S}_j\right]$$

$$= \quad -\frac{1}{2}\sum_{ij}\sum_{\alpha\beta}\left[J_{ij}\delta^{\alpha\beta} + D_{ij}^{\alpha\beta}\right] S_i^\alpha S_j^\beta - h\sum_i S_i^z$$

- Dipolar tensor

$$D_{ij}^{\alpha\beta} = (1-\delta_{ij})\frac{\mu^2}{|\mathbf{r}_{ij}|^3}\left[3\hat{r}_{ij}^\alpha \hat{r}_{ij}^\beta - \delta^{\alpha\beta}\right]$$

material parameters

$a = 12.376$ Å

Gilleo *et al.* '58

$4\pi M_s = 1750$ G

Tittmann '73

$\frac{\rho_{\mathrm{ex}}}{\mu} = 5.17 \cdot 10^{-13}$ Oe m^2

Cherepanov *et al.* '93

alternatively

$J = 1.29$ K

$S = 14.2 \qquad \mu = 2\mu_B$

Tupitsyn *et al.* '08

3.3 Linear Spin Wave Theory

- classical groundstate for stripe geometry
- Holstein Primakoff transformation (bosons)

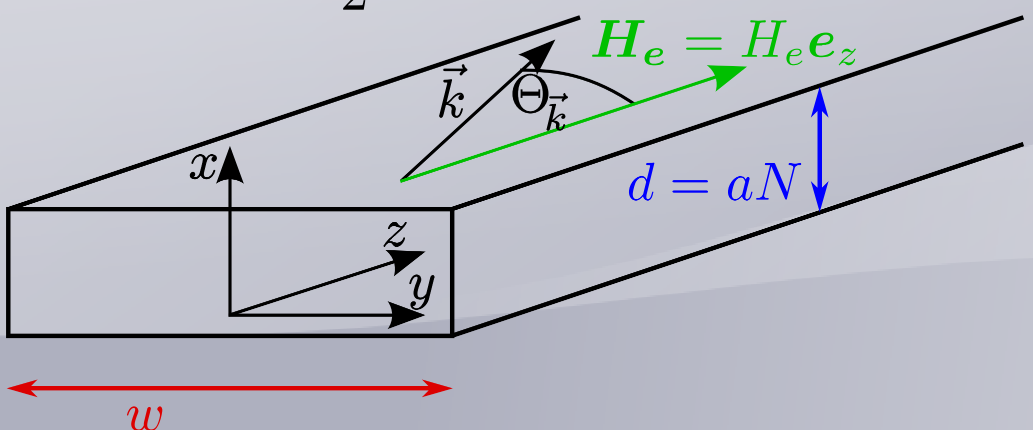
$$\hat{H}_2 = \sum_{ij} \left[A_{ij} b_i^\dagger b_j + \frac{B_{ij}}{2} \left(b_i b_j + b_i^\dagger b_j^\dagger \right) \right]$$

Filho Costa *et al.* Sol. State Comm. **108**, 439 (1998)

$$A_{ij} = \delta_{ij} h + S(\delta_{ij} \sum_n J_{in} - J_{ij}) + S \left[\delta_{ij} \sum_n D_{in}^{zz} - \frac{D_{ij}^{xx} + D_{ij}^{yy}}{2} \right],$$

$$B_{ij} = -\frac{S}{2} [D_{ij}^{xx} - 2iD_{ij}^{xy} - D_{ij}^{yy}]$$

dipolar tensor



3.3 Stripe geometry

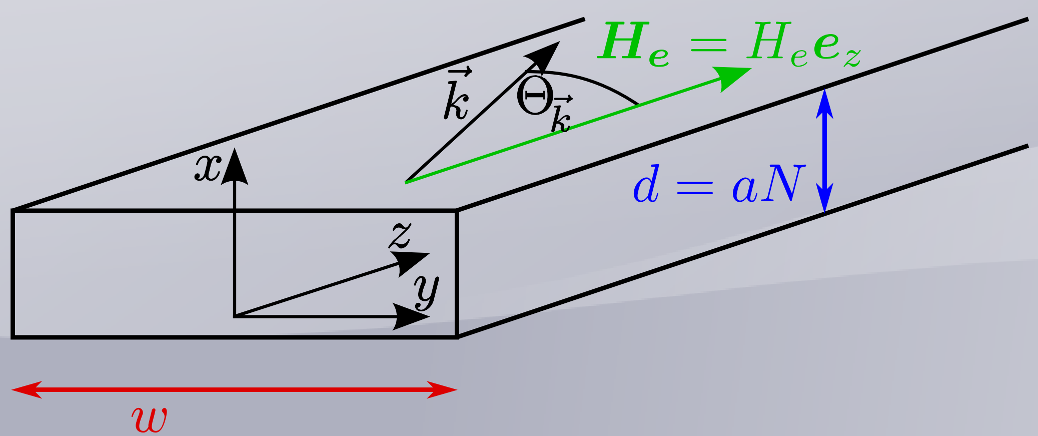
- partial Fourier transformation

$w \to \infty$

$$b_i = \frac{1}{\sqrt{N_y N_z}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_i} b_{\vec{k}}(x_i)$$

- find all branches

$$\det \begin{pmatrix} E_{\vec{k}}\mathsf{I} - \mathsf{A}_{\vec{k}} & -\mathsf{B}_{\vec{k}} \\ -\mathsf{B}^*_{\vec{k}} & -E_{\vec{k}}\mathsf{I} - \mathsf{A}_{\vec{k}} \end{pmatrix} = 0$$



Problems:
1) dipolar sums
2) large matrices

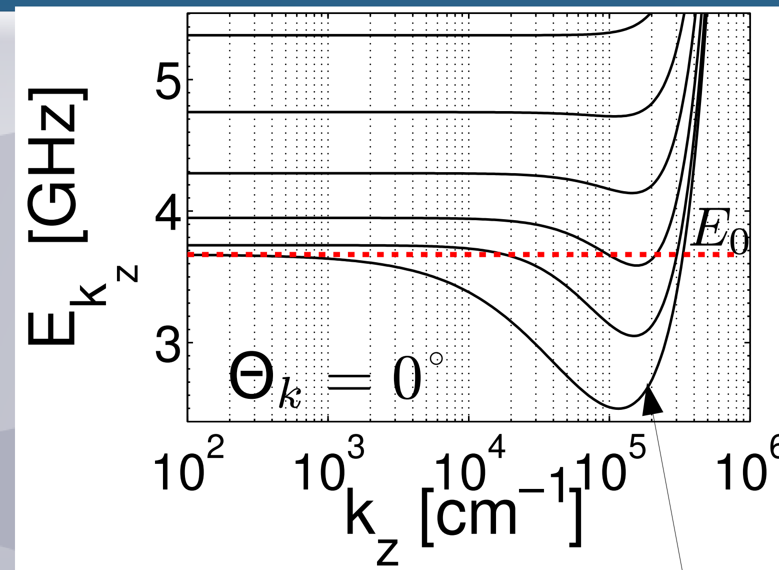
3.3 Numerical approach

1) numerical diagonalization of 2Nx2N matrix

$$H_2 = \begin{pmatrix} A_{\vec{k}} & B_{\vec{k}} \\ -B^T_{\vec{k}} & -A_{\vec{k}} \end{pmatrix}$$

2) evaluation of dipol sums (Ewald summation technique)

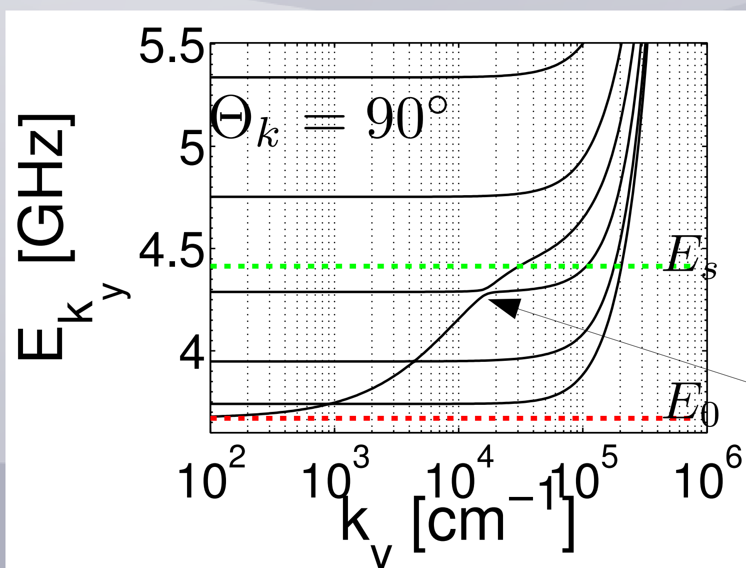
Bartosch *et al.* '06



$d = 400a \approx 0.5\mu m \qquad N = 400 \quad H_e = 700$ Oe

minimum for BEC

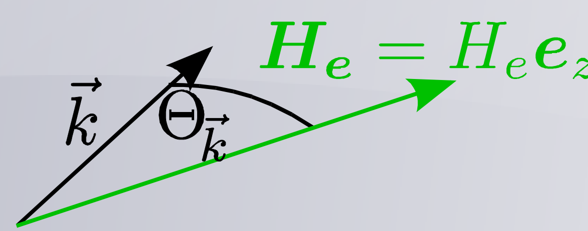
Demokritov *et al.* '06



$$E_0 = \sqrt{h(h + 4\pi\mu M_s)}$$

$$E_s = h + 2\pi\mu M_s$$

hybridization: surface mode



3.3 Analytical approach

$$\hat{H} = \sum_{\vec{k}} \sum_{x_i x_j} \Big[A_{\vec{k}}(x_{ij}) b^\dagger_{\vec{k}}(x_i) b_{\vec{k}}(x_j) + \frac{B_{\vec{k}}(x_{ij})}{2} b_{\vec{k}}(x_i) b_{\vec{k}}(x_j) + \frac{B^*_{\vec{k}}(x_{ij})}{2} b^\dagger_{\vec{k}}(x_i) b^\dagger_{\vec{k}}(x_j) \Big]$$

- uniform mode approximation

$$b_{\vec{k}}(x_i) = \frac{1}{\sqrt{N}} b_{\vec{k}}$$

- lowest eigenmode approximation

$$b_{\vec{k}}(x_i) = \sqrt{\frac{2}{N}} \cos\!\left(\frac{\pi x_i}{d}\right) b_{\vec{k}}$$

3.3 Analytical results with approximation

$$\hat{H} = \sum_{\vec{k}} \left[A_{\vec{k}} b^\dagger_{\vec{k}} b_{\vec{k}} + \frac{B_{\vec{k}}}{2} b_{\vec{k}} b_{\vec{k}} + \frac{B^*_{\vec{k}}}{2} b^\dagger_{\vec{k}} b^\dagger_{\vec{k}} \right]$$

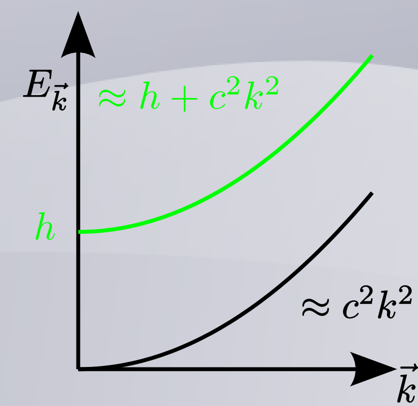
- dispersion via Bogoliubov transformation

$$E_{\vec{k}} = \sqrt{[h + \rho_{\mathrm{ex}} \vec{k}^2 + \Delta(1 - f_{\vec{k}}) \sin^2 \Theta_{\vec{k}}][h + \rho_{\mathrm{ex}} \vec{k}^2 + \Delta f_{\vec{k}}]}$$

$\Delta = 4\pi\mu M_S$

- no dipolar interaction: $\Delta = 0$

$$E_{\vec{k}} = h + \rho_{\mathrm{ex}} \vec{k}^2$$



3.3 Analytical results with approximation

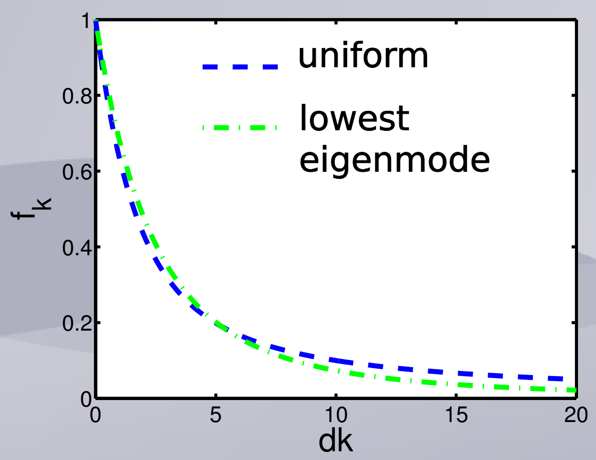
$$E_{\vec{k}} = \sqrt{[h + \rho_{\mathrm{ex}}\vec{k}^2 + \Delta(1 - f_{\vec{k}})\sin^2\Theta_{\vec{k}}][h + \rho_{\mathrm{ex}}\vec{k}^2 + \Delta f_{\vec{k}}]}$$

$$\Delta = 4\pi\mu M_S$$

- uniform mode approximation
  $\Rightarrow$ form factor

  $$f_{\vec{k}} = \frac{1 - e^{-|\vec{k}|d}}{|\vec{k}|d}$$

  compare: Kalinikos *et al.* ‘86
  Tupitsyn *et al.* ‘08

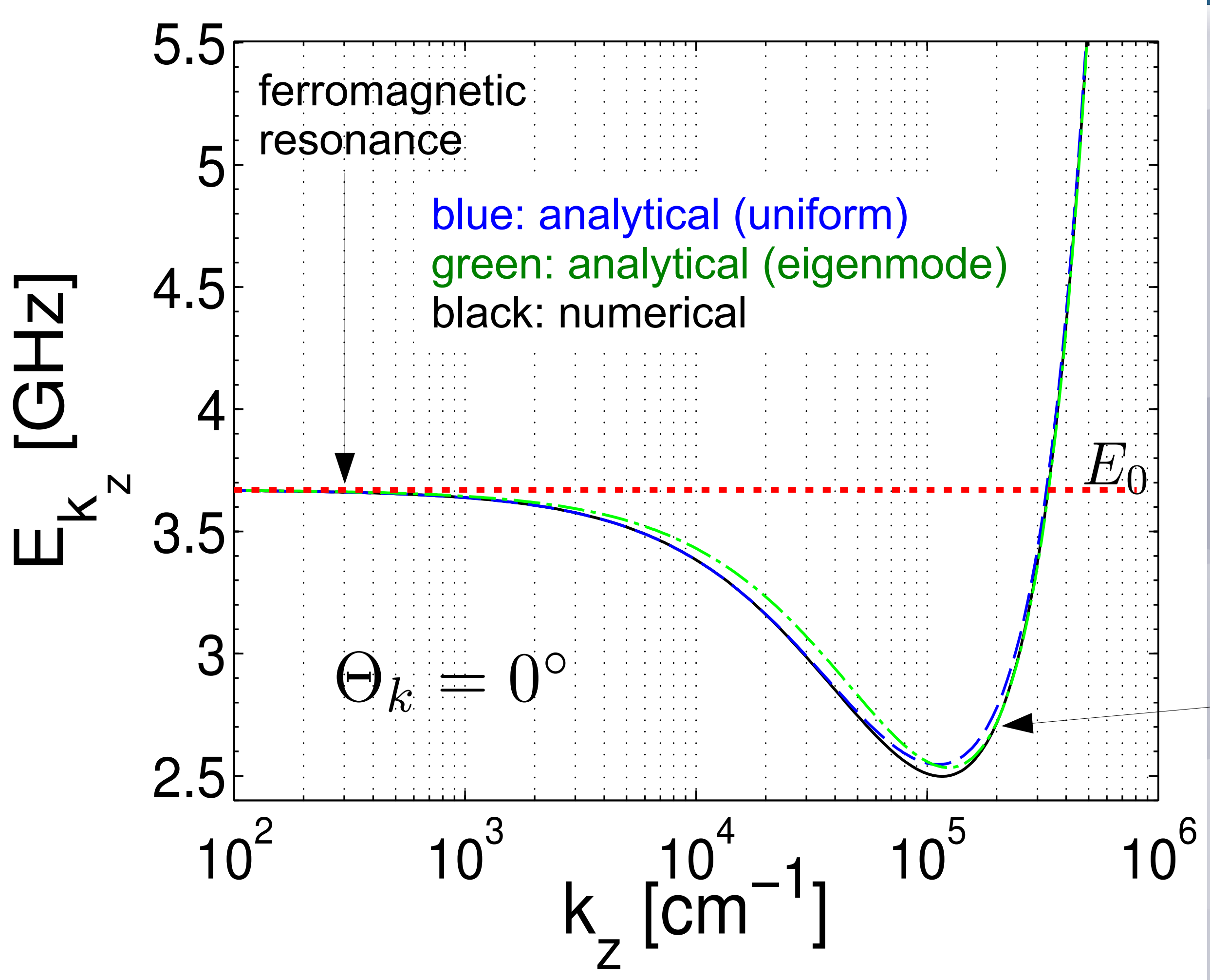


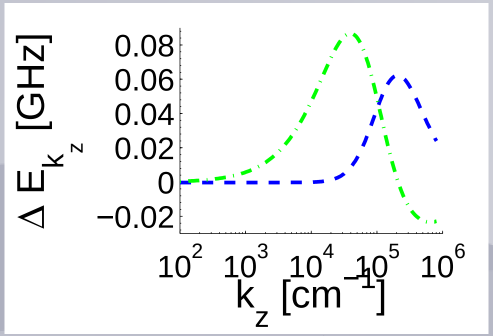
- eigenmode approximation
  $\Rightarrow$ different form factor:

$$f_{\vec{k}} = 1 - |\vec{k}d| \frac{|\vec{k}d|^3 + |\vec{k}d|\pi^2 + 2\pi^2(1 + e^{-|\vec{k}d|})}{(\vec{k}^2 d^2 + \pi^2)^2}$$

AK, Sauli, Bartosch, Kopietz ('09)

3.3 Comparison: lowest mode





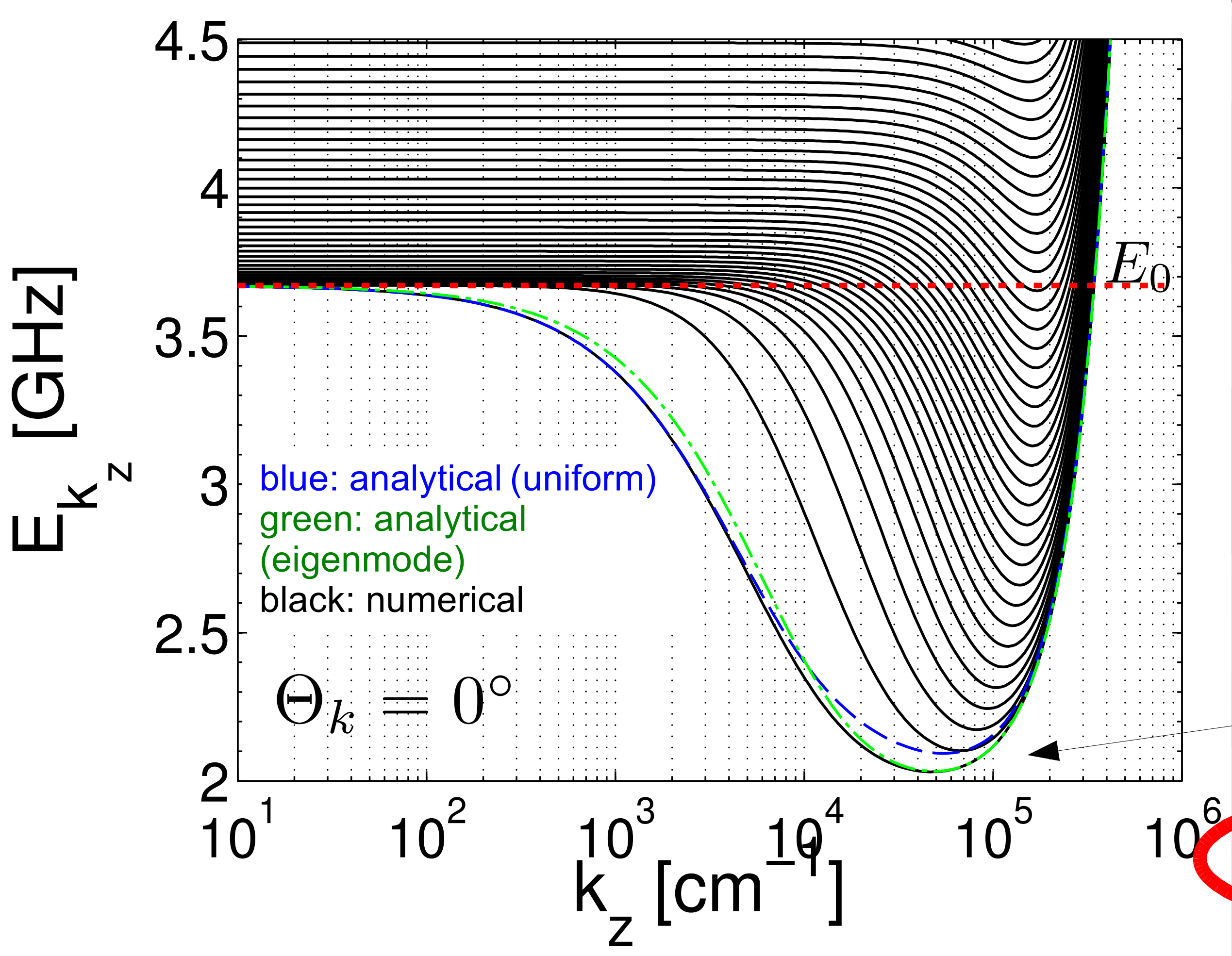
small deviations

$d = 400a \approx 0.5\mu\mathrm{m}$

$H_e = 700$ Oe

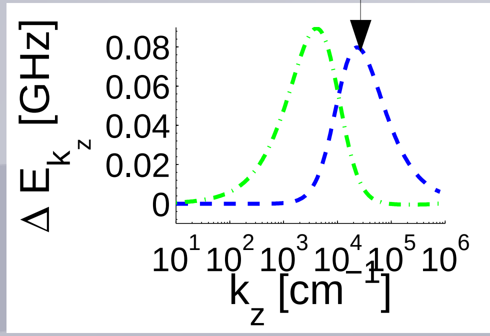
$$E_{\vec{k}} = \sqrt{[h + \rho_{\mathrm{ex}}\vec{k}^2 + \Delta(1 - f_{\vec{k}})\sin^2\Theta_{\vec{k}}][h + \rho_{\mathrm{ex}}\vec{k}^2 + \Delta f_{\vec{k}}]}$$

3.3 Real system: all modes



eigenmode approximation

$$f_{\vec{k}} = 1 - |\vec{k}d| \frac{|\vec{k}d|^3 + |\vec{k}d|\pi^2 + 2\pi^2(1 + e^{-|\vec{k}d|})}{(\vec{k}^2 d^2 + \pi^2)^2}$$



minimum for BEC
Demokritov *et al.* '06

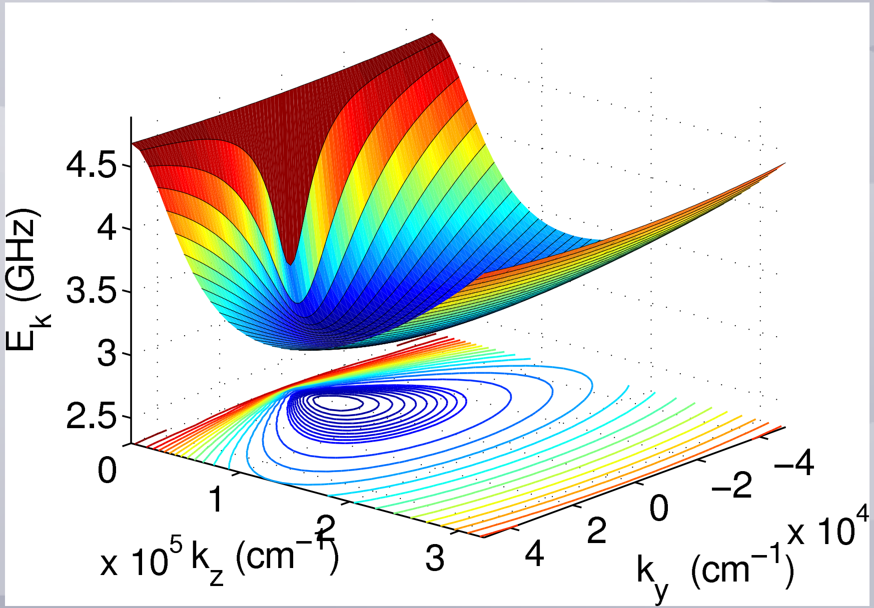
$d = 4040a \approx 5\mu\mathrm{m}$

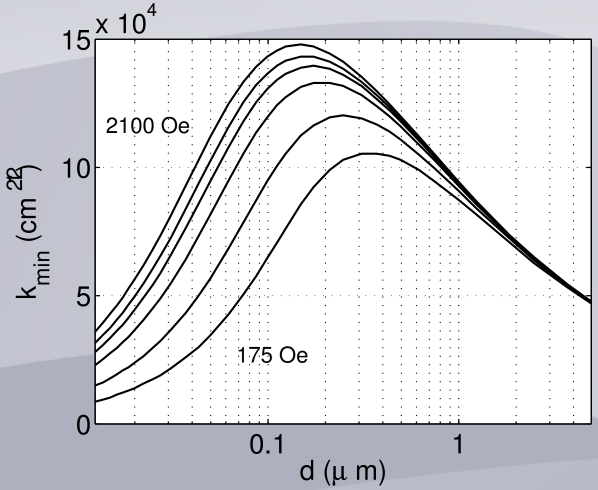
$H_e = 700$ Oe

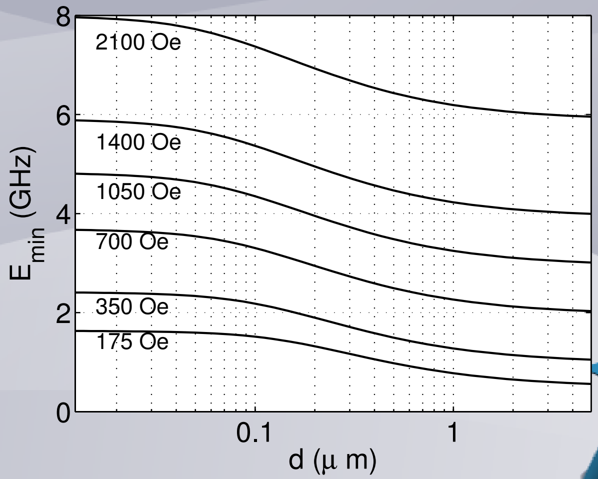
$$E_{\vec{k}} = \sqrt{[h + \rho_{\mathrm{ex}}\vec{k}^2 + \Delta(1 - f_{\vec{k}})\sin^2\Theta_{\vec{k}}][h + \rho_{\mathrm{ex}}\vec{k}^2 + \Delta f_{\vec{k}}]}$$

3.3 Spectrum: Summary

- Interplay
  - exchange interaction
  - dipolar interaction
  - finite thickness







3.4 Vertices in diagonal basis

F. Sauli (in preparation)

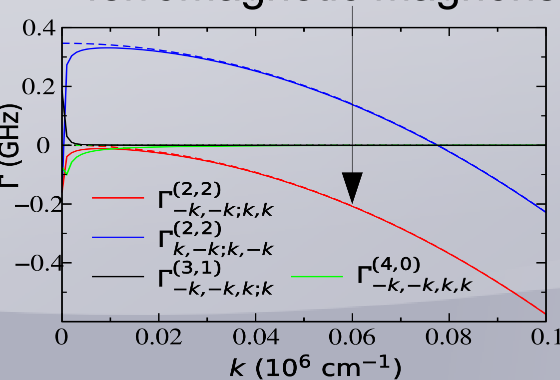
$$\hat{H}_2 = \sum_{\vec{k}} E_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}}$$

$$H_4 = \frac{1}{N} \sum_{\vec{k}_1 \cdots \vec{k}_4} \delta_{\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4,0} \left(\frac{1}{(2!)^2} \Gamma^{(2,2)}_{\vec{k}_1,\vec{k}_2;\vec{k}_3,\vec{k}_4} b^\dagger_{\vec{k}_1} b^\dagger_{\vec{k}_2} b_{\vec{k}_3} b_{\vec{k}_4} \right.$$

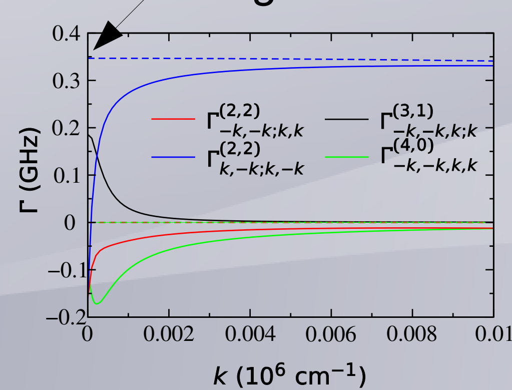
$$\left. + \frac{1}{3!} \left\{ \Gamma^{(3,1)}_{\vec{k}_1,\vec{k}_2,\vec{k}_3;\vec{k}_4} b^\dagger_{\vec{k}_1} b_{\vec{k}_2} b_{\vec{k}_3} b_{\vec{k}_4} + \text{h.c.} \right\} + \frac{1}{4!} \left\{ \Gamma^{(4,0)}_{\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4} b_{\vec{k}_1} b_{\vec{k}_2} b_{\vec{k}_3} b_{\vec{k}_4} + \text{h.c.} \right\} \right)$$

- properties of vertex functions

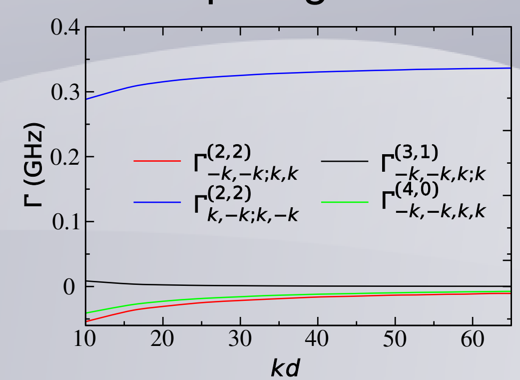
$\Gamma \sim -Jk^2$
ferromagnetic magnons



$\Gamma \sim M_s$
Sergio Rezende '09



$k = k_{\min}$ finite size effects
vertices stay finite at minimum,
but competing interactions



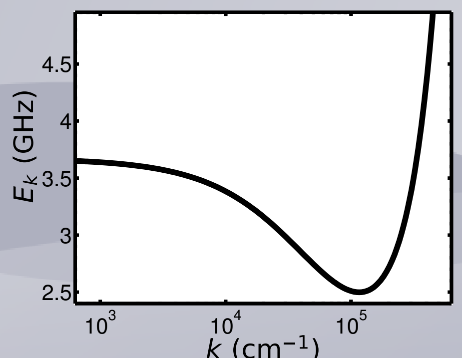
3.4 Spontaneous symmetry breaking

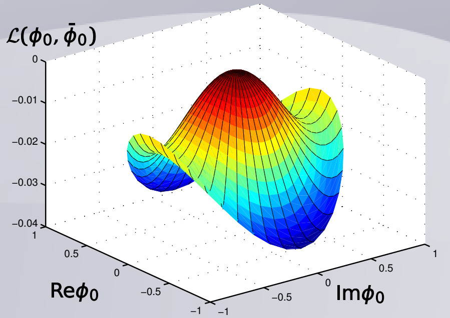
- Bogoliubov shift $\phi_{\vec{k}} = \langle \phi_{\vec{k}} \rangle + \delta\phi_{\vec{k}}$
- calculation of Landau function

$$Z \approx e^{-\beta \mathcal{L}(\bar{\phi}_0, \phi_0)}$$

- new features
  - condensate at finite wave-vectors
    $\langle \phi_k \rangle = \delta_{k,k_{\mathrm{min}}} \phi_0$
  - possible 2 condensates $\epsilon_{\vec{k}} = \epsilon_{-\vec{k}}$
    $\langle \phi_k \rangle = \delta_{k,k_{\mathrm{min}}} \phi_0^+ + \delta_{k,-k_{\mathrm{min}}} \phi_0^-$
  - explicitly symmetry breaking term

$$\begin{aligned} H &= \sum_{\vec{k}} \epsilon_{\vec{k}} b^\dagger_{\vec{k}} b_{\vec{k}} + \frac{1}{2} \sum \left(\gamma b^\dagger b^\dagger + \gamma^* bb \right) \\ &+ \frac{1}{N} \sum \Gamma^{(2,2)} b^\dagger b^\dagger bb \end{aligned}$$

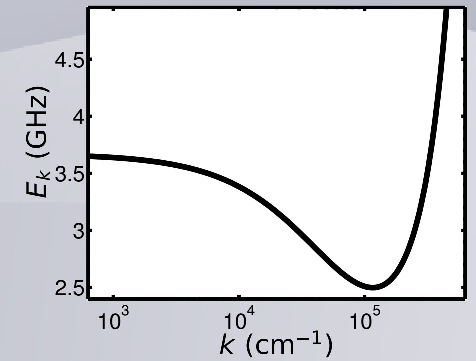




Napoleons hat potential

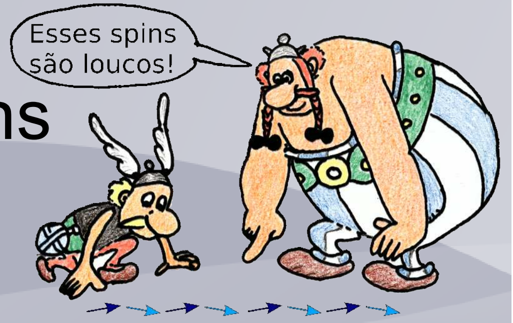
3.3 Recapitulation

- development of **interacting spin-wave theory** with dipole dipole interactions (straightforward)
- interesting properties of the energy dispersion
- interactions: possible **condensation** of bosons at finite wave-vectors



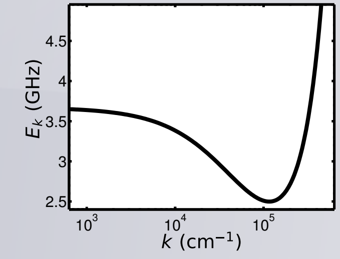
4 Summary

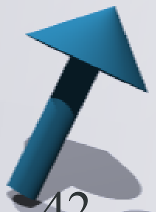
- Description of magnetic insulators: Spin wave theory
- Applications
  - Bose Einstein Condensation of magnons in Quantum Antiferromagnets



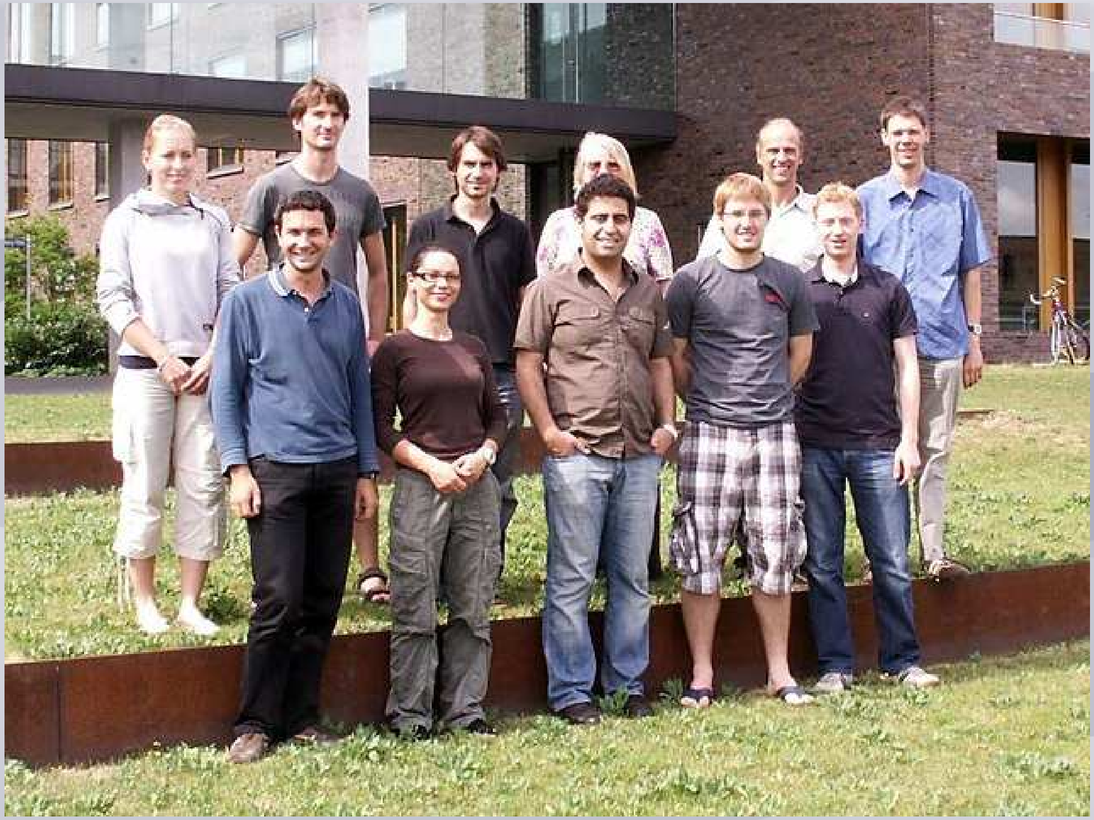
  - Hybrid approach for Quantum Antiferromagnets in a magnetic field

  - Spin wave theory for thin film ferromagnets





5 Acknowledgement



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