Solution 4-1

1)

Either look up or derive the rotation matrices and Taylor-expand up to first order around $\alpha_i = 0$ w. $sin(\alpha) = \alpha + 0(\alpha^2)$ and $cos(\alpha) = 1 + 0(\alpha^2)$ and get

$$R_1(\alpha_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_1) & -sin(\alpha_1) \\ 0 & \sin(\alpha_1) & \cos(\alpha_1) \end{pmatrix} = \mathbb{1} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha_1 \\ 0 & \alpha_1 & 0 \end{pmatrix} + 0(\alpha_1^2),$$
$$R_2(\alpha_2) = \begin{pmatrix} \cos(\alpha_2) & 0 & \sin(\alpha_2) \\ 0 & 1 & 0 \\ -sin(\alpha_2) & 0 & cos(\alpha_2) \end{pmatrix} = \mathbb{1} + \begin{pmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & 0 \\ -\alpha_2 & 0 & 0 \end{pmatrix} + 0(\alpha_2^2),$$
$$R_3(\alpha_3) = \begin{pmatrix} \cos(\alpha_3) & -\sin(\alpha_3) & 0 \\ sin(\alpha_3) & \cos(\alpha_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1} + \begin{pmatrix} 0 & -\alpha_3 & 0 \\ \alpha_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 0(\alpha_3^2)$$

$\Rightarrow$ read off $\hat{t}_i$'s, i.e.

$$\hat{t}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \hat{t}_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \hat{t}_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using this calculate $[\hat{t}_k, \hat{t}_l] = -i\varepsilon_{klm}\hat{t}_m$ and by
$[\sigma_k, \sigma_l] = 2i\varepsilon_{klm}\sigma_m \Rightarrow$ want $[\hat{t}'_k, \hat{t}'_l] = \frac{-c_m}{c_k c_l} i\varepsilon_{klm}\hat{t}_m = 2i\varepsilon_{klm}\hat{t}'_m$
$\Rightarrow c_j = \frac{\pm 1}{2} \Rightarrow \{c_j\}$ is either $\{\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\}$ or any permutation of $\{\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}\}$
Note that if you defined the $\hat{t}'_i$ as the generators the resulting constants are the reciprocals of the ones stated above.

2) a)
Use the Einstein convention:
$[l_k, x_m] = [\varepsilon_{kab}x_a p_b, x_m] = \varepsilon_{kab}(x_a p_b x_m - x_m x_a p_b + x_a x_m p_b - x_a x_m p_b)$
$=^{(*)} \varepsilon_{kab}(x_a p_b x_m - x_a x_m p_b) = -i\hbar\varepsilon_{kab}x_a\delta_{bm} = i\hbar\varepsilon_{akm}x_a$

$[l_k, p_m] = [\varepsilon_{kab}x_a p_b, p_m] = \varepsilon_{kab}(x_a p_b p_m - p_m x_a p_b + x_a p_m p_b - x_a p_m p_b)$
$=^{(*)} \varepsilon_{kab}(x_a p_m p_b - p_m x_a p_b) = i\hbar\varepsilon_{kab}\delta_{am}p_b = i\hbar\varepsilon_{bkm}p_b$

$[l_k, l_m] = [\varepsilon_{kab}x_a p_b, \varepsilon_{mcd}x_c p_d] = \varepsilon_{kab}\varepsilon_{mcd}(x_a p_b a_c p_d - x_c p_d x_a p_b)$
$= \varepsilon_{kab}\varepsilon_{mcd}(x_a p_b x_c p_d - x_a x_c p_b p_d - x_c p_d x_a p_b + x_c x_a p_d p_b + x_a x_c p_b p_d - x_c x_a p_d p_b) =^{(*)}$
$\varepsilon_{kab}\varepsilon_{mcd}(x_a[p_b, x_c]p_d - x_c[p_d, x_a]p_b)$
$= i\hbar(\varepsilon_{kab}\varepsilon_{mdb}x_a p_d - \varepsilon_{kba}\varepsilon_{mca}x_c p_d) = i\hbar((\delta_{km}\delta_{ad} - \delta_{kd}\delta_{am})x_a p_d - (\delta_{km}\delta_{bc} - \delta_{kc}\delta_{bm})x_c p_b) =$
$i\hbar(-x_m p_k + x_k p_m) = i\hbar\varepsilon_{kmj}l_j$
(*) Since $[x_i, x_j] = [p_i, p_j] = 0$ the middle terms in the first two expressions and the last two terms in the last one cancel.

b)
Use [A,BC] = [A,B]C + B[A,C] to find:
$[l_j, l^2] = [l_j, l_k l_k] = l_k[l_j, l_k] - [l_j, l_k]l_k = i\hbar(\varepsilon_{jkm}l_k l_m + \varepsilon_{jkm}l_m l_k)$
$= [l_k, l_m]_+ = 0$
$[l_j, x^2] = [l_j, x_k x_k] = x_k[l_j, x_k] + [l_j, x_k]x_k = i\hbar(\varepsilon_{jkm} + \varepsilon_{kjm})x_k x_m = 0$
$[l_j, p^2] = [l_j, p_k p_k] = p_k[l_j, p_k] + [l_j, p_k]p_k = i\hbar(\varepsilon_{jkm} + \varepsilon_{kjm})p_k p_m = 0$