Introduction

We study the critical behavior of the q-state random-bond Potts model with a bimodal distribution of two positive coupling strengths J_1 and J_2. Using the star-graph expansion technique we calculated high-temperature series expansions for the free energy and susceptibility for general dimensions D and any value of the parameters q, p and $R = J_2/J_1$ to the order of K^{11}. In order to analyse the series we specialize to a symmetric distribution with $D = 2$, $p = 1/2$ and $q = 2, 3, 5,$ and 8, since in this case the exact transition temperature T_c can be computed for any positive value of J_1 and J_2 from a (transcendental) self-duality relation [1].

The random-bond Potts model

- Hamiltonian:

$$\mathcal{H} = -\sum_{<ij>} J_{ij}\delta_{S_i S_j}\,; \quad S_i, S_j \in 1 \,..\, q$$

- Interaction: Nearest-neighbor interactions on a square lattice with a symmetric bimodal distribution of the coupling strengths $J_{ij} > 0$

$$P(J_{ij}) = \frac{1}{2}\delta(J_{ij} - J_1) + \frac{1}{2}\delta(J_{ij} - J_2)$$

- The pure model ($J_1 = J_2$) undergoes a second-order transition for $q \leq 4$ and a first-order transition for $q > 4$ [1].

High-temperature series expansion

- Example: Naive expansion of the partition function:

$$\mathcal{Z} = Tr[1+\beta \sum_{<ij>} J_{ij}\delta_{S_iS_j} + \frac{1}{2!}\beta^2(\sum_{<ij>} J_{ij}\delta_{S_iS_j})^2 + \ldots] \quad (1)$$

  The algebraic expressions can be represented in terms of graphs [2]. Each spin S_i corresponds to a vertex i, and each product $J_{ij}\delta_{S_iS_j}$ corresponds to a bond between the vertices i and j.
  The computation of physical quantities proceeds by considering larger and larger graphs that can be embedded in a given lattice type.

- Star graphs:
  Star graphs are multiply connected graphs without articulation points.
  It can be shown that in the high-temperature series expansion of the free energy and the susceptibility of Potts models only star graphs yield non-vanishing contributions [3].

Analysis of the series

If for a function $F(z)$,

$$F(z) \sim A(1 - z/z_c)^{-\lambda} \doteq \sum a_n z^n \,, \quad (2)$$

then z_c defines the critical (singular) point, λ is the critical exponent, and A is the critical amplitude.

Ratio Method

- The coefficients a_n yield biased extrapolants for the critical exponent (Biased Ratio I):

$$\lambda_n = n\frac{a_n}{a_{n-1}}z_c - n + 1 \tag{3}$$

- A possible variation is to consider the logarithmic derivative of the series (Biased Ratio II)

$$\frac{d}{dz}\ln F(z) = \sum a_n^* z^n \sim \frac{\lambda}{z_c\left(1-\frac{z}{z_c}\right)} \tag{4}$$

- There are also methods to handle multiplicative logarithmic corrections.

Padé-Approximant Methods

- Biased Padé: Consider the function:

$$F^*(z) = (z_c - z)\frac{d}{dz}\ln(F(z)) \sim \lambda \tag{5}$$

  The $[L/M]$ Padé-Approximant to $F^*(z)$ is the rational polynomial expression

$$[L/M] \equiv \frac{P_L(z)}{Q_M(z)} \equiv \frac{p_0 + p_1 z + p_2 z^2 + \ldots + p_L z^L}{q_0 + q_1 z + q_2 z^2 + \ldots + q_M z^M} \tag{6}$$

  where the coefficients are chosen to make the first $L+M+1$ terms of the expansion of (6) agree with the corresponding terms in the Taylor expansion of $F^*(z)$.

- Padé approximant methods are also suitable for multiplicative logarithmic corrections [4].

Results: Ising model ($q = 2$) [5]

There are two theoretical predictions for the critical behavior of the susceptibility:

- Dotsenko and Dotsenko (DD) [6]:

$$\chi \sim t^{-2} \exp\left[-a\left(\ln\ln\frac{1}{t}\right)^2\right] \tag{7}$$

- Shalaev, Shankar, and Ludwig (SSL) [7–9]:

$$\chi \sim t^{-7/4}|\ln t|^{7/8} \tag{8}$$

  t: reduced temperature $t = (T - T_c)/T_c$

- Assuming a pure power-law behavior according to the DD prediction (7) we obtained $\gamma > 2$ for strong disorder (Fig.1).

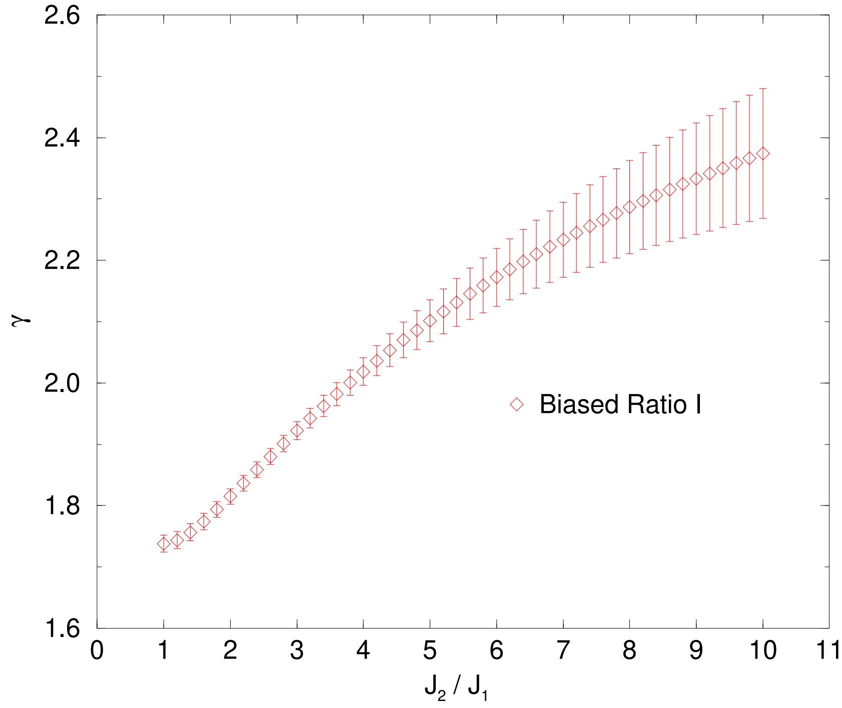

Figure 1: Analysis of the susceptibility ($\chi(q = 2)$, Biased Ratio I)

- Assuming the same leading singularity as in the pure case ($\gamma = 7/4$), but modified by a multiplicative logarithmic correction with exponent p as in (8), we obtained p compatible with $7/8 = 0.875$ (Fig. 2).

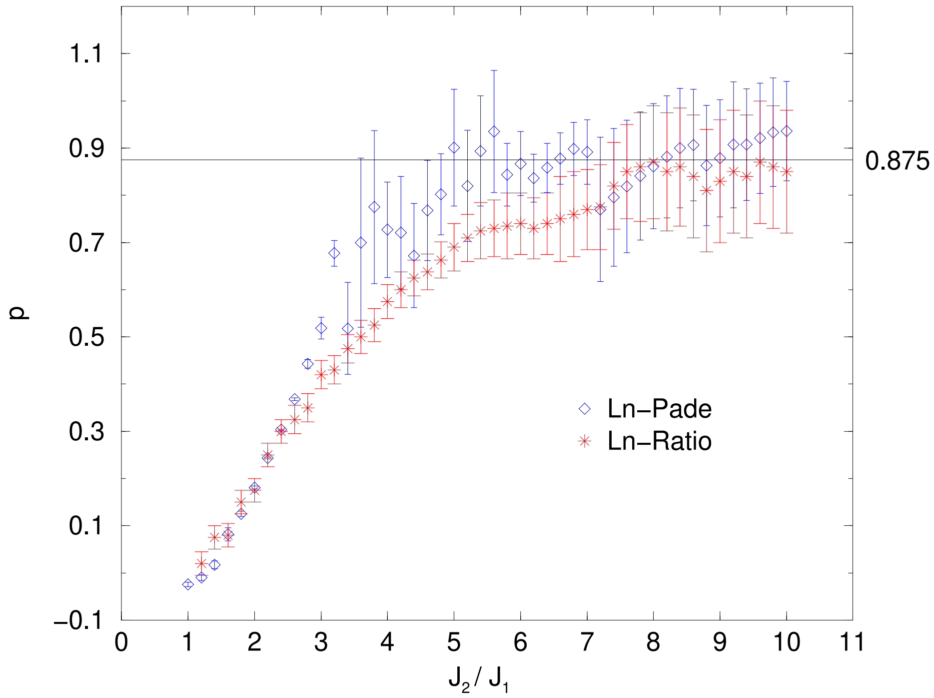

Figure 2:
Analysis of $\chi(q = 2)$ assuming multiplicative logarithmic corrections, using Padé approximants and a ratio method.

- To justify this claim we have investigated a model function (Fig. 3) simulating the "true" susceptibility [10, 11]:

$$\chi \sim t^{-\frac{7}{4}} \left[1 + \frac{4g_0}{\pi} \ln(\frac{1}{t}) \right]^{\frac{7}{8}} \quad ; \quad g_0 = \text{const.} \tag{9}$$

$4g_0/\pi$ is an increasing function of the degree of disorder.

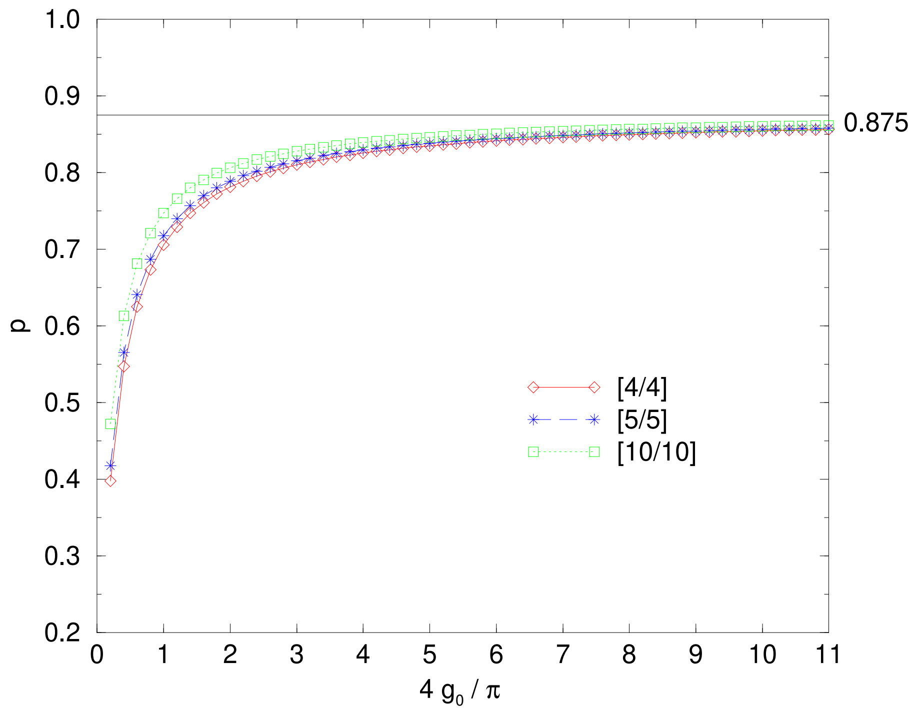

Figure 3:
Analysis of the series expansion of the model function with logarithmic corrections. The legend indicates the three different Padé approximants shown.

Results: $q = 3, q = 5$, and $q = 8$

- We first analysed the susceptibility series for $q = 3$ using the methods Biased Ratio I, Biased Ratio II, and Biased Padé. However, more precise estimates can be obtained by using the so-called methods M1 and M2 [12], respectively, which also take confluent corrections into account (Fig. 4).
- Analysis of the transition point for $q = 8$:
  At a first-order phase transition it is expected that the transition point K_c is systematically over-estimated (contrary to a second-order transition) [13]. The result of such an analysis is shown in Fig. 5.

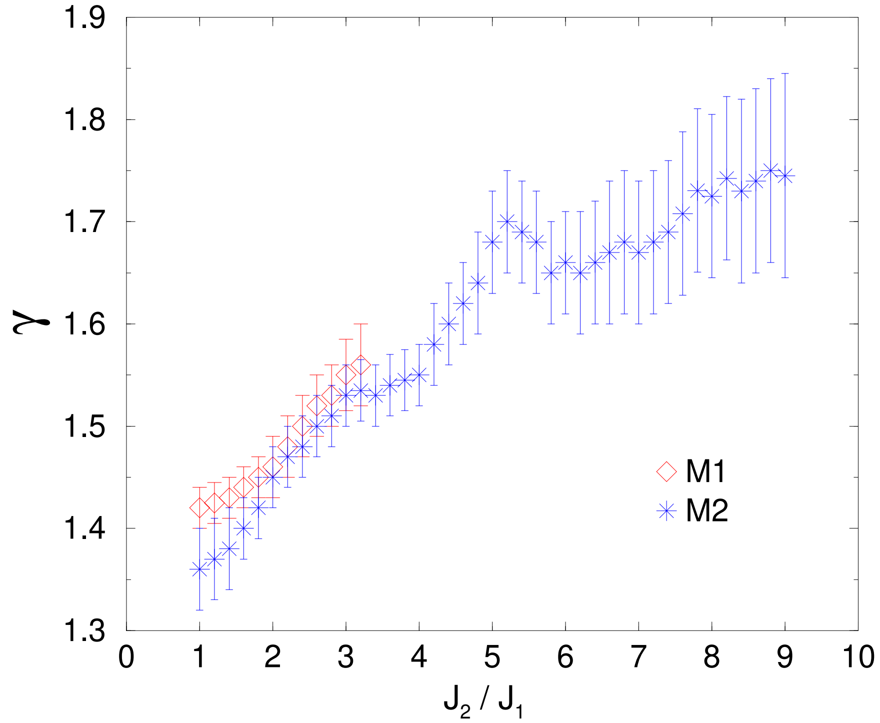

Figure 4: Analysis of the susceptibility series for $q = 3$ using M1/M2

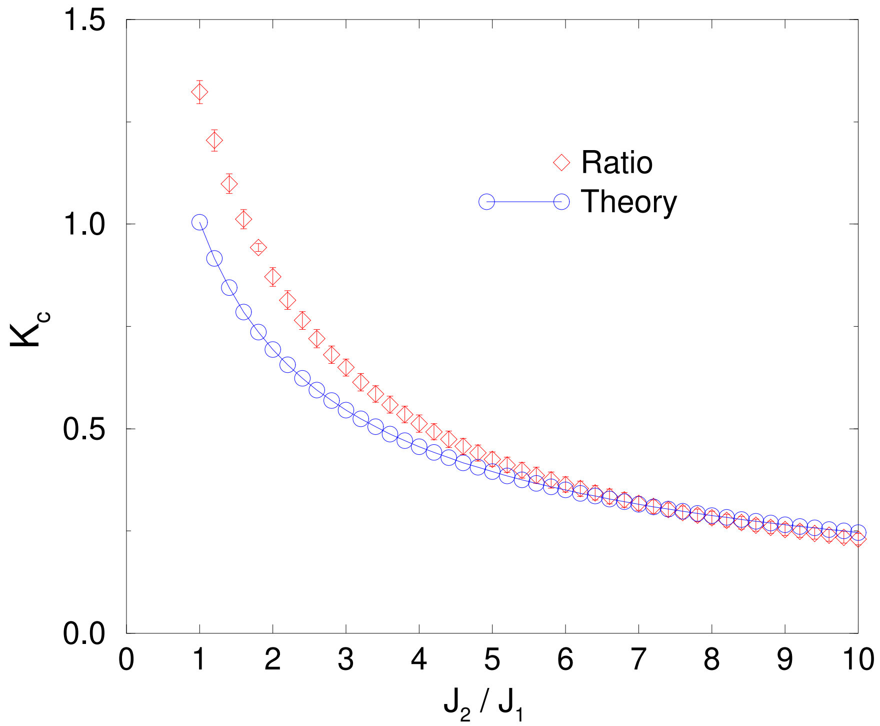

Figure 5: Transition point for $q = 8$

Summary

For the random-bond Ising model ($q = 2$) we obtain for the first time a direct confirmation of the theoretical predictions by Shalaev, Shankar, and Ludwig, $\chi \propto t^{-7/4}|\ln t|^p$, with a critical exponent $p = 7/8$. For $q = 3$ our results indicate that the critical exponent γ takes the Ising value of $\gamma = 7/4$. Also for $q = 5$ and 8, where the pure model exhibits a first-order phase transition, our analyses are compatible with this value. Here, longer series expansions would be necessary for more precise estimates. We obtain, however, clear evidence for the theoretically expected softening of the transition.

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