TP4 – Statistical Physics I
Homework 10
Due: Friday, 18 June 2010

28. Cumulant expansion

The characteristic function of a probability density \( W(y) \) is the Fourier transform (see problems 24, respectively, 27e)

\[
\tilde{W}(k) \equiv \langle e^{iky} \rangle = \int_{-\infty}^{\infty} dy \exp(iky) W(y) .
\] (1)

If all moments \( \langle y^n \rangle \) exist, then \( \tilde{W}(k) \) can be expanded into a power series:

\[
\tilde{W}(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle y^n \rangle .
\] (2)

Express the moments as derivatives of \( \tilde{W}(k) \).

The expansion of \( \ln \tilde{W}(k) \) in \( k \) yields a power series which defines the so-called cumulants \( \kappa_n \):

\[
\ln \tilde{W}(k) = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \kappa_n .
\] (3)

Express the first four cumulants \( \kappa_1, \ldots, \kappa_4 \) in terms of the moments, and then rewrite these formulas in terms of the so-called central moments,

\[
\mu_n = \langle (y - \langle y \rangle)^n \rangle .
\] (4)

Determine \( \tilde{W}(k) \) for a Gaß distribution and calculate in this case all cumulants.
29. **One-dimensional percolation**

a) Use the exact solution of the one-dimensional percolation problem to show that the $k$th moment of the cluster-size distribution, $M_k = \sum s^k n_s$, diverges as $\Gamma_k (1 - p)^{1-k}$, and calculate explicitly the amplitudes $\Gamma_k$.

b) For one dimension, calculate the (site-) cluster numbers for the case that the lattice sites are occupied with probability $p$ and the lattice bonds are occupied with probability $x$. Repeat part a) for this distribution.

c) Calculate the ratios $\Gamma_k \Gamma_l / \Gamma_m \Gamma_{k+l-m}$ for both cases. What do you conclude about these ratios?

30. **Maxwell distribution**

a) Assuming a Maxwell distribution $f(\vec{v})$ resp. $\omega(\epsilon)$ with $\epsilon = m \vec{v}^2 / 2$, show that the mean squared fluctuations of the kinetic energy of a single particle are given by $\langle (\epsilon - \langle \epsilon \rangle)^2 \rangle = (3/2)(k_B T)^2$. What is the general formula for $\langle \epsilon^k \rangle \equiv \int_0^\infty \epsilon^k \omega(\epsilon) \, d\epsilon$?

b) Consider now the total energy for $N$ non-interacting particles, $E = \sum_{n=1}^{N} \epsilon_n$, and calculate the expectation value $\langle E \rangle$ and the variance $\langle (E - \langle E \rangle)^2 \rangle$. Here the $\epsilon_n$, $n = 1, \ldots, N$ are assumed to be uncorrelated. What follows in each case for the total energy per particle, $e = E/N$?

c) How does the distribution $W(e)$ of $e = E/N = \frac{1}{N} \sum_{n=1}^{N} \epsilon_n \equiv \epsilon$ look like for large $N$? Compare this distribution in a graphical representation with the distribution $\omega(\epsilon_n)$ of the single-particle energies $\epsilon_n$.

*Hint:* See problem 27.