TP4 – Statistical Physics I
Homework 10

Due: Friday, 18 June 2010

28. **Cumulant expansion**

The *characteristic function* of a probability density $W(y)$ is the Fourier transform (see problems 24, respectively, 27e)

$$\tilde{W}(k) \equiv \langle e^{iky} \rangle = \int_{-\infty}^{\infty} \mathrm{d}y \, \exp(iky) \, W(y). \tag{1}$$

If all moments $\langle y^n \rangle$ exist, then $\tilde{W}(k)$ can be expanded into a power series:

$$\tilde{W}(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle y^n \rangle. \tag{2}$$

Express the moments as derivatives of $\tilde{W}(k)$.

The expansion of $\ln \tilde{W}(k)$ in k yields a power series which defines the so-called *cumulants* κ_n:

$$\ln \tilde{W}(k) = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \kappa_n. \tag{3}$$

Express the first four cumulants $\kappa_1, \ldots, \kappa_4$ in terms of the moments, and then rewrite these formulas in terms of the so-called *central moments*,

$$\mu_n = \langle (y - \langle y \rangle)^n \rangle. \tag{4}$$

Determine $\tilde{W}(k)$ for a Gauß distribution and calculate in this case *all* cumulants.

29. **One-dimensional percolation**

a) Use the exact solution of the one-dimensional percolation problem to show that the kth moment of the cluster-size distribution, $M_k = \sum_s s^k n_s$, diverges as $\Gamma_k(1-p)^{1-k}$, and calculate explicitly the amplitudes Γ_k.

b) For one dimension, calculate the (site-) cluster numbers for the case that the lattice sites are occupied with probability p and the lattice bonds are occupied with probability x. Repeat part a) for this distribution.

c) Calculate the ratios $\Gamma_k\Gamma_l/\Gamma_m\Gamma_{k+l-m}$ for both cases. What do you conclude about these ratios?

30. **Maxwell distribution**

a) Assuming a *Maxwell distribution* $f(\vec{v})$ resp. $\omega(\epsilon)$ with $\epsilon = m\,\vec{v}^2/2$, show that the mean squared fluctuations of the kinetic energy of a single particle are given by $\langle(\epsilon - \langle\epsilon\rangle)^2\rangle = (3/2)(k_BT)^2$. What is the general formula for $\langle\epsilon^k\rangle \equiv \int_0^\infty \mathrm{d}\epsilon\, \epsilon^k\omega(\epsilon)$?

b) Consider now the total energy for N non-interacting particles, $E = \sum_{n=1}^N \epsilon_n$, and calculate the expectation value $\langle E\rangle$ and the variance $\langle(E - \langle E\rangle)^2\rangle$. Here the ϵ_n, $n = 1, \ldots, N$ are assumed to be uncorrelated. What follows in each case for the total energy per particle, $e = E/N$?

c) How does the distribution $W(e)$ of $e = E/N = \frac{1}{N}\sum_{n=1}^N \epsilon_n \equiv \overline{\epsilon}$ look like for large N? Compare this distribution in a graphical representation with the distribution $\omega(\epsilon_n)$ of the single-particle energies ϵ_n.

Hint: See problem 27.