Information geometry of the ising model on planar random graphs

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It has been suggested that an information geometric view of statistical mechanics in which a metric is introduced onto the space of parameters provides an interesting alternative characterization of the phase structure, particularly in the case where there are two such parameters, such as the Ising model with inverse temperature β and external field h. In various two-parameter calculable models, the scalar curvature $\mathcal{R}$ of the information metric has been found to diverge at the phase transition point β_c and a plausible scaling relation postulated: $\mathcal{R}\sim|\beta-\beta_c|^{\alpha-2}$. For spin models the necessity of calculating in nonzero field has limited analytic consideration to one-dimensional, mean-field and Bethe lattice Ising models. In this paper we use the solution in field of the Ising model on an ensemble of planar random graphs (where $\alpha=-1$, $\beta=1/2$, $\gamma=2$) to evaluate the scaling behavior of the scalar curvature, and find $\mathcal{R}\sim|\beta-\beta_c|^{-2}$. The apparent discrepancy is traced back to the effect of a negative α.

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I. GENERALITIES: THE INFORMATION GEOMETRY

Various authors, motivated by ideas in parametric statistics [1], have discussed the advantages of taking a geometrical perspective on statistical mechanics [2–8]. The "distance" between two probability distributions in parametric statistics can be measured using a geodesic distance that is calculated from the Fisher information matrix for the system. In a statistical mechanical context the probability distributions of interest are Gibbs measures

$$p(x|\theta)=\exp\left(-\sum_{i=1}^{r}\theta^i H_i(x)-\ln Z(\theta)\right), \qquad (1)$$

where x characterizes the state of the system (e.g., spins), $H_i(x)$ are the various terms in the Hamiltonian, $Z(\theta)$ is the normalizing partition function, and θ^i are the various parameters such as the inverse temperature β, the external field h, etc.

The manifold $\mathcal{M}$ of parameters is endowed with a natural Riemannian metric, the Fisher-Rao metric [1], which measures the distance between different configurations. For a spin model in field $\mathcal{M}$ is a two-dimensional manifold parametrized by $(\theta^1,\theta^2)=(\beta,h)$. The components of the Fisher-Rao metric take the simple form $G_{ij}=\partial_i\partial_j f$ in this case, where f is the reduced free energy per site and $\partial_i=\partial/\partial\theta^i$. A natural object to consider in any geometrical approach is the scalar or Gaussian curvature which may be calculated as

$$\mathcal{R}=-\frac{1}{2G^2}\begin{vmatrix}\partial_\beta^2 f & \partial_\beta\partial_h f & \partial_h^2 f\\ \partial_\beta^3 f & \partial_\beta^2\partial_h f & \partial_\beta\partial_h^2 f\\ \partial_\beta^2\partial_h f & \partial_\beta\partial_h^2 f & \partial_h^3 f\end{vmatrix}, \qquad (2)$$

where $G=\det(G_{ij})$.

It is worth remarking that, unlike most standard statistical mechanical observables, the curvature $\mathcal{R}$ depends on third order derivatives. Nonetheless, a plausible scaling relation has been advanced for $\mathcal{R}$ in the critical region. The hypothesis, on dimensional grounds, is that the curvature depends on the correlation volume for a second-order transition $\mathcal{R}\sim\xi^d$, where ξ is the correlation length and d is the dimension of the system. This is reasonable since ξ is the only physical scale in the system near criticality.

Combined with hyperscaling, $\nu d=2-\alpha$, and standard scaling assumptions, this leads to

$$\mathcal{R}\sim|\beta-\beta_c|^{\alpha-2}. \qquad (3)$$

In the above, α is the standard exponent characterizing the scaling of the specific heat, so consideration of $\mathcal{R}$ clearly offers a way of determining critical exponents in a nonstandard manner.

Analytic determination of $\mathcal{R}$ in spin models has been limited by the necessity of carrying out calculations in field. One case where this is possible is the one-dimensional (1D) Ising model [3], where the curvature was calculated to be

$$\mathcal{R}=1+\eta^{-1}\cosh h \qquad (4)$$

with $\eta=\sqrt{\sinh^2 h+e^{-4\beta}}$. The 1D Ising model can be thought of as having a zero-temperature transition, so looking at $h=0$, $\beta\to\infty$ we see that $\mathcal{R}\sim e^{2\beta}$, corresponding to the expected $\alpha=1$. Similarly, it is possible to obtain an expression for the scalar curvature for the Ising model on a Bethe lattice [6], where the scaling behavior is also verified. Both these examples have unsatisfactory aspects—the 1D Ising model has no real transition and the Bethe lattice Ising model is mean field in nature.

Given the relative paucity of models which are soluble in field, any further explicit calculations would be welcome, particularly in a non-mean-field model with a genuine finite-temperature phase transition. In the sequel we discuss one such case, the Ising model on dynamical planar random graphs.

II. PARTICULARS: THE ISING MODEL ON PLANAR GRAPHS

The solution of the Ising model on an ensemble of Φ^4 (4-regular) or Φ^3 (3-regular) planar random graphs was first presented by Boulatov and Kazakov [9,10], who were motivated by string-theoretic considerations, since the continuum limit of the theory represents matter coupled to 2D quantum gravity. They considered the partition function for the Ising model on a single n vertex planar graph with connectivity matrix G^n_{ij}

$$Z_{\text{single}}(G^n,\beta,h)=\sum_{\{\sigma\}}\exp\left(\beta\sum_{\langle i,j\rangle}G^n_{ij}\sigma_i\sigma_j+h\sum_i\sigma_i\right), \tag{5}$$

then summed it over all n vertex graphs $\{G^n\}$ resulting in

$$Z_n=\sum_{\{G^n\}}Z_{\text{single}}(G^n,\beta,h), \tag{6}$$

before finally forming the grand-canonical sum over graphs with different numbers n of vertices

$$\mathcal{Z}=\sum_{n=1}^{\infty}\left(\frac{-4gc}{(1-c^2)^2}\right)^n Z_n, \tag{7}$$

where $c=\exp(-2\beta)$. This last expression could be calculated exactly as matrix integral over $N\times N$ Hermitian matrices,

$$\mathcal{Z}=-\ln\int\mathcal{D}\phi_1\ \mathcal{D}\phi_2\ \exp\left(-\text{Tr}\left[\frac{1}{2}(\phi_1^2+\phi_2^2)-c\,\phi_1\phi_2-\frac{g}{4}(e^h\phi_1^4+e^{-h}\phi_2^4)\right]\right), \tag{8}$$

where the $N\to\infty$ limit is to be taken to pick out the planar diagrams and the potential appropriate for Φ^4 (4-regular) random graphs has been shown.

When the matrix integral is carried out the solution is given in parametric form by

$$\mathcal{Z}=\frac{1}{2}\ln\frac{z}{g}-\frac{1}{g}\int_0^z\frac{dt}{t}g(t)+\frac{1}{2g^2}\int_0^z\frac{dt}{t}g(t)^2, \tag{9}$$

where the function $g(z)$ is

$$g(z)=\frac{1}{9}c^2z^3+\frac{z}{3}\left[\frac{1}{(1-z)^2}-c^2+\frac{zB}{(1-z^2)^2}\right] \tag{10}$$

and $B=2[\cosh(h)-1]$.

In the thermodynamic limit the reduced free energy per site is given by

$$f=\ln\left(\frac{-4cg(z_0)}{(1-c^2)^2}\right), \tag{11}$$

where $z_0=z_0(\beta,h)$ is the appropriate low- or high-temperature solution of $g'(z)=0$. When $h=0$ this may be solved in closed form, and although the solution is not available explicitly for nonzero h it can still be developed as a power series in h around the zero-field solutions in order to obtain expressions for quantities such as the energy, specific heat, magnetization and susceptibility. It was found that the critical exponents were given by $\alpha=-1$, $\beta=1/2$, $\gamma=2$, so the transition was *third* order with, intriguingly, the same exponents as the 3D spherical model on a regular lattice [11].

If we carry out a perturbative expansion for the high-temperature solution, which is symmetric in h and hence a series in even powers, we find

$$z_0=1-\frac{1}{u}-\frac{(u-1)(2u^2-2u+1)}{(2u-1)^4}h^2+\frac{(u-1)(2u^2-2u+1)(4u^5-10u^4+10u^3-5u^2+5u+1)}{24(2u-1)^9}h^4+\dots, \tag{12}$$

where the coefficients in the series are most naturally expressed in terms of $u=\exp(-\beta)=\sqrt{c}$, as above.

III. GENERALITIES: SCALING OF THE SCALAR CURVATURE

The expected scaling form of the various components of $\mathcal{R}$ for a generic spin model in field is discussed at some length in Ref. [4], and we now recapitulate these results briefly for comparison with the specific results for the Ising model on planar random graphs in the following section. The starting point is the scaling form of the free energy per spin near the critical point,

$$f(\epsilon,h)=\lambda^{-1}f(\epsilon\lambda^{a_\epsilon},h\lambda^{a_h}), \tag{13}$$

where $\epsilon\equiv\beta_c-\beta$ and a_ϵ,a_h are the scaling dimensions for the energy and spin operators. For $\epsilon>0$, i.e., in the unbroken high-temperature phase, we can use standard scaling assumptions to write this as

$$f(\epsilon,h)=\epsilon^{1/a_\epsilon}\psi_+(h\epsilon^{-a_h/a_\epsilon}), \tag{14}$$

where ψ_+ is a scaling function and we also define $A=1/a_\epsilon$ and $C=-a_h/a_\epsilon$ for later convenience. In terms of the standard critical exponents $A=2-\alpha$ and $A+C=\beta$.

This generic scaling form can now be substituted into Eq. (2) to find the scaling behavior of the various components and the scalar curvature (2) itself near criticality (i.e. $h=0$, $\epsilon\to0$),

$$\mathcal{R} = -\frac{1}{2G^2}\begin{vmatrix} A(A-1)\epsilon^{A-2}\psi_+(0) & 0 & \epsilon^{A+2C}\psi_+''(0) \\ -A(A-1)(A-2)\epsilon^{A-3}\psi_+(0) & 0 & -(A+2C)\epsilon^{A+2C-1}\psi_+''(0) \\ 0 & -(A+2C)\epsilon^{A+2C-1}\psi_+''(0) & \epsilon^{A+3C}\psi_+'''(0) \end{vmatrix}, \tag{15}$$

where the scaling of the metric determinant is

$$G = A(A-1)\epsilon^{2A+2C-2}\psi_+(0)\psi_+''(0). \tag{16}$$

Expanding the determinant one finds two terms of similar order contributing to give

$$\mathcal{R} = \frac{(A+2C)[(A+2C)-(A-2)]}{2A(A-1)\psi_+(0)}\epsilon^{-A} \tag{17}$$

or, translating back to the standard critical exponents,

$$\mathcal{R} = \frac{\gamma(\gamma-\alpha)}{2(2-\alpha)(1-\alpha)\psi_+(0)}\epsilon^{\alpha-2}. \tag{18}$$

The discussion in Ref. [4] was intended to be as general as possible, one should note that for Ising-like models with a $\pm h$ symmetry all odd derivatives of the scaling function with respect to h will vanish so $\partial_h^3 f = 0$ rather than $\epsilon^{A+3C}\psi_+'''(0)$. This does not affect the stated scaling relations.

However, one feature of these scaling relations does have an impact on the observed scaling for the Ising model. Generically one expects that $\partial_\beta^2 f = A(A-1)\epsilon^{A-2}\psi_+(0)$, which contributes to both the metric and the determinant involved in calculating $\mathcal{R}$. If $A>2$, i.e., $\alpha<0$, this naively suggests a vanishing $\partial_\beta^2 f$ at criticality, which will in general *not* be the case. There would instead be a contribution from a regular term, which would give a constant at the critical point. Having such a constant term modifies the scaling form of $\mathcal{R}$ in the case $\alpha<0$, $A>2$ to

$$\mathcal{R} = -\frac{1}{2G^2}\begin{vmatrix} A(A-1)\phi(0) & 0 & \epsilon^{A+2C}\psi_+''(0) \\ -A(A-1)(A-2)\epsilon^{A-3}\psi_+(0) & 0 & -(A+2C)\epsilon^{A+2C-1}\psi_+''(0) \\ 0 & -(A+2C)\epsilon^{A+2C-1}\psi_+''(0) & \epsilon^{A+3C}\psi_+'''(0) \end{vmatrix}, \tag{19}$$

where we have denoted the constant by $A(A-1)\phi(0)$. The scaling for G is also modified to

$$G = A(A-1)\epsilon^{A+2C}\phi(0)\psi_+''(0). \tag{20}$$

When expanded, the expression for $\mathcal{R}$ contains two terms that now have differing orders in ϵ. The leading term for $A>2$, the case which we are interested in, is

$$\mathcal{R} = \frac{(A+2C)^2}{2A(A-1)\phi(0)}\epsilon^{-2} \tag{21}$$

or

$$\mathcal{R} = \frac{\gamma^2}{2(2-\alpha)(1-\alpha)\phi(0)}\epsilon^{-2}, \tag{22}$$

so the critical exponent α no longer appears in the scaling exponent.

By virtue of the Boulatov and Kazakov solution, the Ising model on planar random graphs allows us to explicitly confirm these observations, as we see in the following section. Since $\alpha=-1$, $\beta=1/2$, $\gamma=2$, we have $A=3$, $C=-5/2$ and the modified discussion of scaling should apply.

IV. PARTICULARS: THE SCALAR CURVATURE FOR ISING

We can now take the series expansion for z_0 from Eq. (12), insert this into $g(z)$ and use the resulting expression for f in Eq. (11) to calculate the various terms that appear in the scalar curvature $\mathcal{R}$ as a power series in h^2. We find that the leading terms at $h=0$, with $\epsilon_u \equiv u - u_{cr} = \epsilon/2 + \dots$ and $u_{cr}=1/2$, and using β, h as co-ordinates are

$$\mathcal{R} = -\frac{1}{2G^2}\begin{vmatrix} \frac{352}{225} & 0 & \frac{3}{20}\epsilon_u^{-2} \\ -\frac{1072}{675} & 0 & \frac{3}{20}\epsilon_u^{-3} \\ 0 & \frac{3}{20}\epsilon_u^{-3} & 0 \end{vmatrix}. \tag{23}$$

The determinant of the metric scales as $G = \frac{88}{375}\epsilon_u^{-2} + \dots$, so the final scaling expression for the scalar curvature is

$$\mathcal{R} \sim \frac{225}{704}\epsilon_u^{-2} + \dots = \frac{225}{176}\epsilon^{-2} + \dots. \tag{24}$$

A glance back at Eqs. (19)–(22) shows that the modified scaling for $A>2$ that these incorporate is, indeed, followed for the individual components in Eq. (19), the metric in Eq. (20) and the scalar curvature itself in Eqs. (21) and (22).

It is an easy matter to calculate $\mathcal{R}$ for any u when h is small, using the expansion for z_0 in Eq. (12). Writing

$$\mathcal{R}=\mathcal{R}_0+\mathcal{R}_2h^2+\ \dots, \tag{25}$$

the first two coefficients are given by

$$\mathcal{R}_0=\frac{(6u^8+43u^7+357u^6+1265u^5+2123u^4+1841u^3+783u^2+75u+3)}{2(6u^5+27u^4+56u^3+54u^2+18u+3)^2(2u-1)^2}\times(2u^2+2u+1)(u+1)(u^2+1) \tag{26}$$

and

$$\mathcal{R}_2=\tfrac{1}{2}(u+1)(u^2+1)u(u+2)[144u^{18}+1008u^{17}-3276u^{16}-31\,180u^{15}-79\,106u^{14}-129\,786u^{13}-135\,424u^{12}-92\,093u^{11} -78\,645u^{10}-37\,499u^9+54\,941u^8+245\,658u^7+410\,788u^6+328\,760u^5+139\,986u^4+33\,183u^3+5331u^2+765u +45]/[(6u^5+27u^4+56u^3+54u^2+18u+3)^3(2u-1)^7], \tag{27}$$

which give the scaling of Eq. (24) when u is set equal to $1/2+\epsilon_u$.

In Fig. 1 we have plotted $\mathcal{R}$ close to $u_{cr}=1/2$ using a series correct up to $O(h^6)$ terms. The scaling region in h is very narrow, with the approximation to $\mathcal{R}$ rapidly giving large negative values outside this region due to the increasingly strong divergences in the series coefficients as $u\to u_{cr}$ for increasing order in h. This turnover is just visible on the edges of the plotted surface. The sensitivity to h would have to be carefully handled in any numerical investigations of $\mathcal{R}$. Within the domain of validity of the expansion in h it appears that $\mathcal{R}$ is positive. It has been remarked [3,5] that $\mathcal{R}$ is positive in the thermodynamic limit for Ising models when the parameters take physical values, with the only divergence being at the critical point, and the Ising model here provides another example. This feature is apparently not universal, calculations of $\mathcal{R}$ for the one-dimensional Potts model [12] and field theories [8] do not give positive curvatures throughout the physical parameter space.

It has been observed that the line $h=0$ is a geodesic of the metric for the one-dimensional Ising and Potts models [12]. The geodesic equations using co-ordinates β,h are given in general by

$$\frac{dV^\beta}{ds}+\Gamma^\beta_{\beta\beta}V^\beta V^\beta+2\Gamma^\beta_{\beta h}V^\beta V^h+\Gamma^\beta_{hh}V^hV^h=\lambda(s)V^\beta, \tag{28}$$

$$\frac{dV^h}{ds}+\Gamma^h_{\beta\beta}V^\beta V^\beta+2\Gamma^h_{\beta h}V^\beta V^h+\Gamma^h_{hh}V^hV^h=\lambda(s)V^h, \tag{29}$$

where s parametrizes the flow lines, $V^\beta=d\beta/ds$, $V^h=dh/ds$, the Γ are the Christoffel symbols and $\lambda(s)$ allows for the possibility of a nonaffine parameter choice.

A vector field with a flow line along $h=0$ has $V^h=0$, so in this case Eqs. (28) and (29) reduce to

$$\frac{dV^\beta}{ds}+(\Gamma^\beta_{\beta\beta})|_{h=0}V^\beta V^\beta=\lambda(s)V^\beta, \tag{30}$$

$$(\Gamma^h_{\beta\beta})|_{h=0}V^\beta V^\beta=0. \tag{31}$$

The first of these equations for $\beta(s)$ always has a solution and the second requires $(\Gamma^h_{\beta\beta})|_{h=0}=0$. This is satisfied for the Ising model on planar random graphs because the Christoffel symbol $\Gamma^h_{\beta\beta}$ vanishes at $h=0$ for the same reason as in

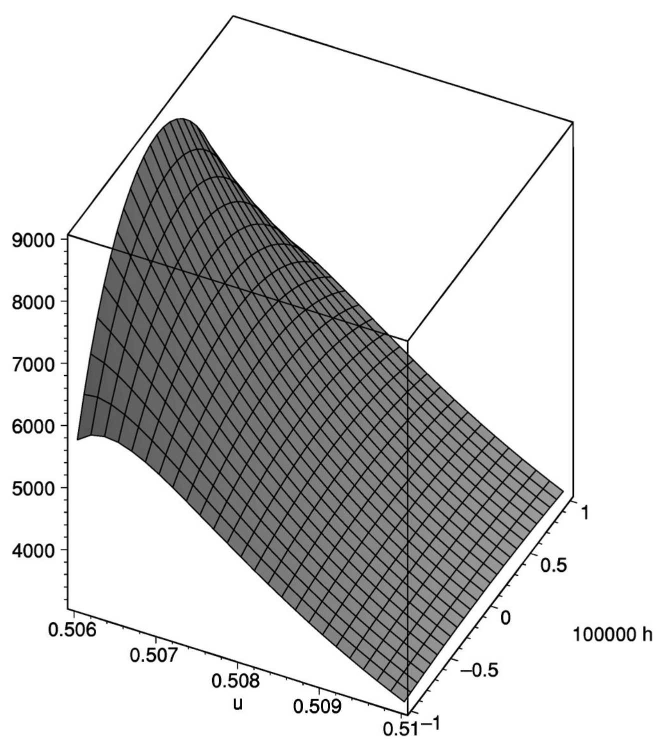

FIG. 1. A plot of $\mathcal{R}$ close to $u_{cr}=1/2$. Note that the external field h is scaled by a factor of 10^5 and so covers a very narrow range.

the one-dimensional models—the off-diagonal components of the metric $\partial_\beta \partial_h f$ are $O(h)$ in all cases. We therefore find that for the Ising model on planar random graphs $h=0$ is also a geodesic line in the β, h plane.

We close with a remark on the infinite temperature ($T \to \infty$, $\beta \to 0$, $u \to 1$) limit of $\mathcal{R}$ when $h=0$. In this limit $\mathcal{R}$ was found to be 2 for the one-dimensional Ising model [3] and $z/2$ for the Ising model on a z co-ordinated Bethe lattice [6]. Here we find that $\mathcal{R}(T=\infty)=4060/1681=2.415\ldots$, so if we accept the suggestion in Ref. [3] that $\mathcal{R}-\mathcal{R}(T=\infty)$ should be taken as a measure of fluctuations caused by the spin interactions the correct measure of the deviation from ideal paramagnetism for the Ising model on planar random graphs is $\mathcal{R}-4060/1681$.

V. CONCLUSIONS

We have calculated the scaling behavior of the scalar curvature of the Fisher-Rao metric for the Ising model on planar random graphs using the exact solution of [9,10] combined with a perturbative expansion in the external field h. Although $\alpha=-1$ for this model, $\mathcal{R} \sim \epsilon^{-2}$ rather than the $\mathcal{R} \sim \epsilon^{-3}$ one might have expected naively from general scaling arguments. This discrepancy was traced back to the effect that a negative value of α had on the scaling of the various components of the metric and the terms that contributed to $\mathcal{R}$.

Various qualitative features of the calculated $\mathcal{R}$ tally with earlier observations of one-dimensional and mean-field Ising models. It is positive (within the domain of the applicability of our semiperturbative calculation) and diverges only at the critical point. The zero-field line is seen to be a geodesic, just as for the one-dimensional Ising and Potts models.

It would be an interesting exercise to calculate $\mathcal{R}$ for other models where some form of solution in field was accessible.

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