NUCLEAR PHYSICS B
PROCEEDINGS
SUPPLEMENTS

Nuclear Physics B (Proc. Suppl.) 34 (1994) 698–701
North-Holland

Ising model on 2D random lattices*

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We report single-cluster Monte Carlo simulations of the Ising model on two-dimensional Poissonian random lattices constructed according to the Voronoi/Delaunay prescription. One set of simulations is performed near criticality on lattices with up to 80 000 sites. Here we apply reweighting techniques to obtain the critical exponents from a finite-size scaling analysis. The other set of simulations is performed in the disordered phase and the critical parameters are extracted from fits to power-law divergencies as the critical point is approached. From both sets we obtain unambiguous support for lattice universality, i.e., that the critical exponents of the Ising model on a two-dimensional random lattice agree with the exactly known values for regular lattices.

1. INTRODUCTION

Random lattices [1,2] are a useful tool to discretize space without introducing any kind of anisotropy. Recent applications can be found in such diverse fields as quantum field theory or quantum gravity [1,2], the statistical mechanics of membranes [3], diffusion limited aggregation [4], or growth models of sandpiles [5].

Here we consider the Ising model defined by the partition function

$$Z = \sum_{\{s_i\}} e^{-KE}; \quad E = -\sum_{\langle ij \rangle} s_i s_j; \quad s_i = \pm 1, \qquad (1)$$

where $K = J/k_B T > 0$ is the inverse temperature in natural units and $\langle ij \rangle$ denote nearest-neighbour links of two-dimensional Poissonian random lattices constructed according to the Voronoi/Delaunay prescription [1,2]. We thus take the relative weights of the links to be constant as in previous work by Espriu *et al.* [6], who studied this model using standard Metropolis Monte Carlo (MC) simulations in the low- and high-temperature phase on lattices with $N = 10\,000$ $(10k)$ sites. Focussing mainly on the question of lattice universality, we report high-statistics simulations in the very vicinity of the phase transition, using considerably larger lattices of size up to $N = 80k$, as well as further runs in the disordered phase. To achieve the desired accuracy of the data we made extensively use of recently developed greatly refined MC simulation techniques, such as the single-cluster update algorithm [7] and reweighting methods [8]. As a result of finite-size scaling (FSS) analyses of our data at criticality and power-law fits in the disordered phase we obtain very strong support for (lattice) universality in this model [9].

2. SIMULATION

The lattice sizes investigated in the FSS study are $N = 5k$, $10k$, $20k$, $40k$, and $80k$, with three replicas for each of the two smallest lattices, and two replicas for $N = 20k$. We always employed periodic boundary conditions, i.e., the topology of a torus with an average coordination number $\overline{q} = 6$ (Euler's theorem). Locally the coordination numbers q vary for Poissonian random lattices (see Fig.1) between 3 and ∞ with an exactly known [10] distribution $P(q)$. All our lattices satisfy these constraints, with $q = 13$ being the highest coordination number actually observed in the $N = 80k$ lattice simulation.

To update the spins s_i we employed the single-

*Work supported in part by The Florida High Technology and Industry Council under Contract FHTIC-15423ERAU.

SSDI 0920-5632(94)00378-9

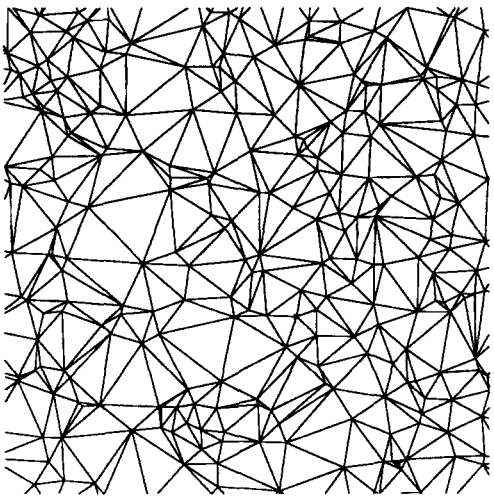

Figure 1. Section of a Poissonian random lattice.

cluster update algorithm [7]. All runs were performed at $K = 0.263$, the estimate of the critical coupling K_c in Ref.[6]. After discarding from 50 000 to 150 000 clusters to reach equilibrium from an initially completely disordered state, we generated a further 4×10^6 clusters and recorded every $10th$ cluster measurements of the energy per spin, $e = E/N$, and the magnetization per spin, $m = \sum_i s_i/N$ in a time-series file. At the scale of our measurements the corresponding integrated autocorrelation times are $\hat{\tau}_e \approx 0.8 - 1.3$ and $\hat{\tau}_{m^2} \approx 0.7 - 0.9$, respectively. The statistical errors are estimated by dividing the time series into 20 blocks, which are jack-knived to avoid bias problems in reweighted data.

3. RESULTS

3.1. Finite-size scaling region

To determine the transition point K_c and the correlation length exponent ν we first considered the Binder parameter $U_L(K) = 1 - \langle m^4 \rangle / 3 \langle m^2 \rangle^2$, where $L \equiv \sqrt{N}$ is defined as the linear length of the lattice in natural units. The intersection points $(K^\times(L, L'), U^\times(L, L'))$ of these curves approach (K_c, U^*) for large L,L'. In Table 1 we quote as estimate for K_c the average of the $K^\times(L, L')$ for the three largest lattices, with the (rough) error estimate reflecting also the fluctuations between different replicas, and from an average over all lattice sizes at K_c we obtain

$$U^* = 0.6123 \pm 0.0025. \tag{2}$$

The very good agreement with MC estimates for the square (sq) lattice of $U^* = 0.615(10)$ and $U^* = 0.611(1)$ [11] may be taken as a first indication of lattice universality.

Recalling that $U'_L \equiv dU_L/dK \propto L^{1/\nu}$ at $K^\times$, the exponent ν can be estimated from the effective exponents

$$\nu_{\text{eff}} = \frac{\ln(L'/L)}{\ln\left(U'_{L'}(K^\times)/U'_L(K^\times)\right)}. \tag{3}$$

Averages over various combinations of L and L' give

$$\begin{aligned} \nu &= 1.008 \pm 0.022 \quad \text{(all crossings)}, \\ \nu &= 1.0043 \pm 0.0036 \ (N = 80k \text{ crossings}), \end{aligned} \tag{4}$$

in very good agreement with the exact regular lattice value of $\nu = 1$.

Another possibility to extract the exponent ν is to analyze the scaling of $d \ln\langle |m|^p \rangle / dK$ at their maxima, $(d \ln\langle |m|^p \rangle / dK)|_{\max} \propto L^{1/\nu}$. The least-square fits in Fig.2 give

$$\begin{aligned} \nu &= 1.037 \pm 0.031 \qquad (p = 1), \\ \nu &= 1.042 \pm 0.030 \qquad (p = 2), \end{aligned} \tag{5}$$

again consistent with $\nu = 1$.

The ratio of exponents γ/ν follows from the scaling of the maxima of the (finite lattice) susceptibility $\chi'(K) = K\,N(\langle m^2 \rangle - \langle |m| \rangle^2)$. From a straight line fit through all data points in a log-log plot of $\chi'_{\max}(L) \propto L^{\gamma/\nu}$ vs L in Fig.3 we obtain

$$\gamma/\nu = 1.7503 \pm 0.0059. \tag{6}$$

This is again in perfect agreement with the exact value for regular lattices, $\gamma/\nu = 1.75$.

All other exponents can in principle be calculated by scaling or hyperscaling relations, e.g., $2\beta/\nu = d - \gamma/\nu$, where d is the dimension. An independent estimate can be obtained from the FSS behaviour of the magnetization $\langle |m| \rangle$ at its point of inflection, $\langle |m| \rangle|_{\text{inf}}(L) \propto L^{-\beta/\nu}$. A linear fit through all data points gives

$$\beta/\nu = 0.1208 \pm 0.0092, \tag{7}$$

Table 1
Estimates of the critical coupling K_c from extrapolations of various quantities.

HTS (Ref.[6])	MC (Ref.[6])	$K^{\times}(L,L')$	$K^{C}_{\max}$	$K^{\chi'}_{\max}$	$K^{(|m|)}_{\inf}$
≈ 0.26303	0.2631(3)	0.2630(2)	0.26295(33)	0.262947(77)	0.26304(14)

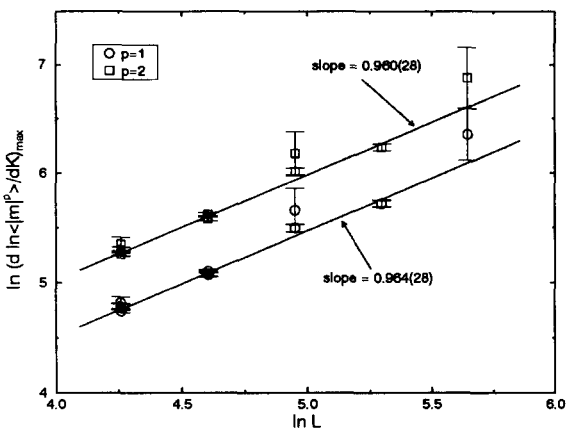

Figure 2. FSS of the maxima of $d\ln\langle|m|^p\rangle/dK$. The slope of the linear fits is an estimate for $1/\nu$.

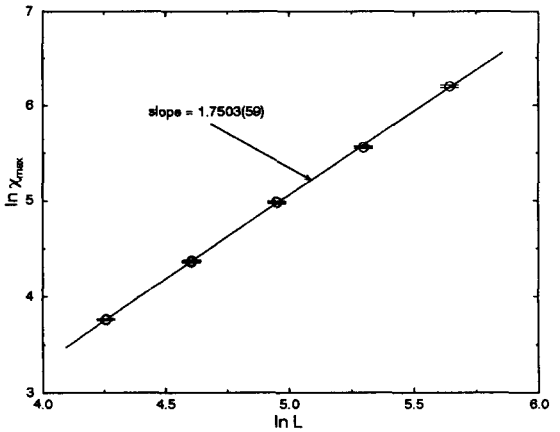

Figure 3. FSS of the susceptibility maxima. The slope of the linear fit is an estimate for γ/ν.

in agreement with the scaling expectations.

Let us finally consider the specific heat $C = K^2N(\langle e^2\rangle - \langle e\rangle^2)$. Assuming hyperscaling, $\alpha = 2 - d\nu$, and $\nu = 1$, the maxima of C should scale as $C_{\max}(L) = B_0 + B_1 \ln L$, with non-universal constants B_0 and B_1. As is demonstrated in the semi-log plot in Fig.4 our data is consistent with this prediction. We cannot claim, however, unambiguous support for logarithmic scaling since fits to a power-law Ansatz, $C_{\max} = b_0 + b_1 L^{\alpha/\nu}$ with $\alpha/\nu = 0.17(16)$ are equally acceptable. In fact, even when imposing $b_0 = 0$, we obtain fits of very high quality. These problems are, however, not special to random lattices but also occur for the *sq* lattice.

Finally, to have collected in Table 1 further estimates of the critical coupling K_c obtained from fits to the asymptotic FSS behaviour of the pseudo transition points, e.g., $K^{C}_{\max}(L) = K_c + aL^{-1/\nu}$ (assuming $\nu = 1$).

3.2. Disordered phase

In the disordered phase we have concentrated on how the susceptibility and specific heat approach K_c. Most data is obtained from one random lattice with $N = 40k$ sites in the inverse temperature range $K = 0.22 \ldots 0.26$. For χ we have assumed a leading singularity of the form $\chi = A(K_c - K)^{-\gamma}$. Empirically it turned out that in the range $K \in (0.22, 0.25)$ correction terms can only be omitted if χ/K is fitted instead of χ. Using the data from the improved estimator $\chi/K = \langle|C|\rangle$, where $\langle|C|\rangle$ is the average cluster size, the fit shown in Fig.5 yields $K_c = 0.26281(10)$ and $\gamma = 1.7725(76)$.

For the specific heat we find that a logarithmic behaviour, $C = A_0 - A_1 \ln(K_c - K)$, is slightly favored over a power-law Ansatz, but similar to the FSS region also here the numerical data for this quantity is not really conclusive.

4. CONCLUSIONS

As usual the specific-heat peaks are difficult to analyze, since the asymptotic behaviour sets in only for extremely large lattice sizes in FSS or very close to K_c in the disordered phase. While our data is consistent with a logarithmic scaling,

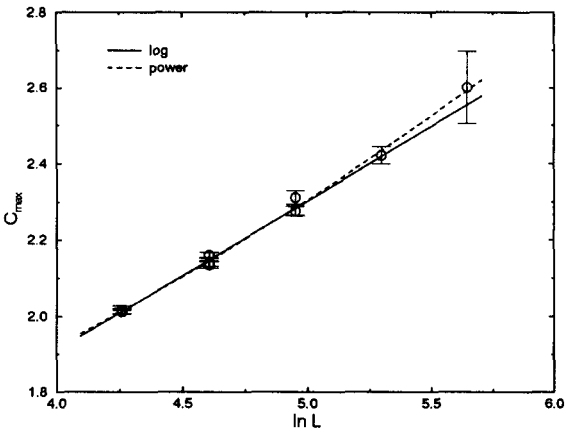

Figure 4. FSS of the specific-heat maxima together with logarithmic and power-law fits.

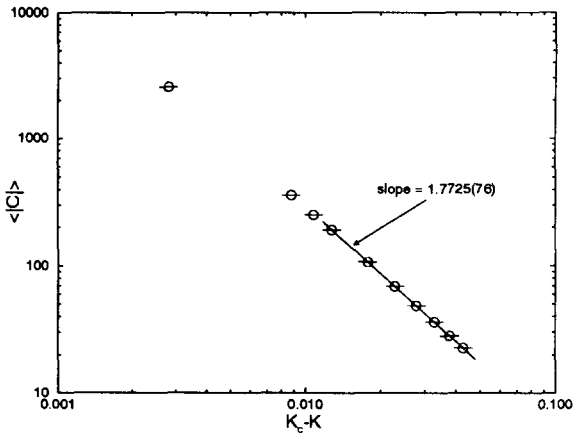

Figure 5. Scaling of the susceptibility in the disordered phase.

i.e., with a critical exponent $\alpha = 0$, it is not yet sufficient to exclude a power-law scaling with $\alpha \neq 0$ on a statistically firm basis.

Our results for the critical exponents ν, γ and β, on the other hand, clearly agree with the regular lattice values and thus provide strong support for lattice universality in the two-dimensional Ising model.

As a future project it would be interesting to perform a similar study for *dynamical* random lattices satisfying the Voronoi/Delaunay construction at all times [12] and to investigate whether the critical behaviour is still governed by the critical exponents of the *static* random (or regular) lattice or by the critical exponents predicted by matrix model theory [13].

ACKNOWLEDGEMENTS

R.V. is supported by a fellowship from the "Centre de Supercomputació de Catalunya", and W.J. thanks the DFG for a Heisenberg fellowship. Some of the simulations were performed on the SCRI cluster of fast RISC workstations.

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