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Extreme order statistics*

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Extreme order statistics has recently been conjectured to be of relevance for a large class of correlated systems, including critical phenomena, turbulent flow problems, some self-organized systems, percolation and other models of lattice field theory. For certain probability densities the theory predicts the characteristic large x fall-off behavior $f(x) \propto \exp(-a\,e^x)$, $a > 0$, usually called Gumbel's first asymptote. Using the multi-overlap algorithm we have tested this prediction over many decades for the overlap distribution $P(q)$ of (i) the Edwards-Anderson Ising spin glass and (ii) the standard Ising model in three dimensions.

1. INTRODUCTION

Inspired by studies of the 2D XY model in the low-temperature phase, Bramwell *et al.* [1] have recently conjectured that a variant of extreme order statistics describes the asymptotic behavior of certain probability densities for a large class of correlated systems. Besides the XY model their class includes turbulent flow problems, percolation models and some self-organized critical phenomena. For large system sizes the asymptotic behavior is claimed to be described by a system size-independent variant of Gumbel's first asymptote,

$$P'(x') = C\exp\left[a\left(x' - x'_{\max} - e^{b(x'-x'_{\max})}\right)\right] \quad ,(1)$$

where C, a, and b are constants, and $x'_{\max} = x_{\max}/\sigma_L$ is the position of the maximum of the scaled probability density $P'_L(x') = \sigma_L P_L(x)$ with σ_L denoting the standard deviation. In its classical form due to Fisher and Tippett, Kawata and Smirnov the exponent a takes the values $a = 1, 2, 3, \ldots$, corresponding, respectively, to the distribution of the first, second, third, ... smallest number of a set of N random numbers, $N \to \infty$ (under certain mild conditions). For reviews on extreme order statistics, see e.g. Ref. [2].

2. 3D EAI SPIN GLASS

Let us start with the three-dimensional (3D) Edwards-Anderson Ising (EAI) [3] spin-glass,

$$H = -\sum_{\langle ik\rangle} J_{ik}\, s_i s_k \ , \qquad s_i = \pm 1 \ , \qquad (2)$$

where $J_{ik} = \pm 1$ are quenched, random coupling constants. The overlap of the spins $s_i^{(1)}$ and $s_i^{(2)}$ of two copies (replica) of the realization $\mathcal{J}$,

$$q = \frac{1}{N}\sum_{i=1}^{N} s_i^{(1)} s_i^{(2)} \ , \qquad N = L^3 \ , \qquad (3)$$

serves as an order parameter. Its probability density $P_L(q)$ is, therefore, a quantity of central physical interest. At the freezing temperature [4] $T = 1.14$ we generated 8192 realizations for $L = 4$, 6 and 8, 1024 realizations for $L = 12$ and 256 realizations for $L = 16$, using the multi-overlap algorithm [5] which simulates a statistical ensemble for which the distribution of q-values is approximately flat. As a consequence the tails of the distributions are (for $L = 16$) accurate down to 10^{-160} (for $|q|$ towards 1) [6]. Alongside with our data at the critical point, we analyzed our

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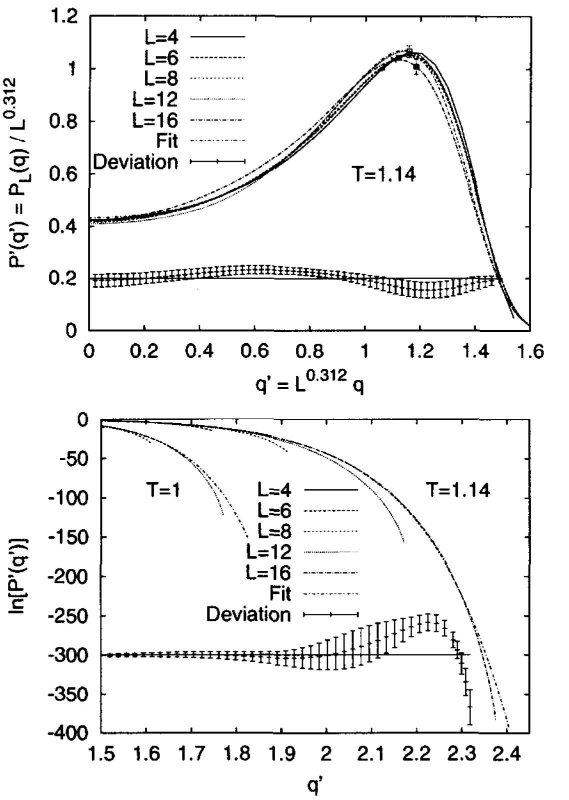

Figure 1. Rescaled overlap probability densities $P'(q') = \sigma_L P_L(q)$ versus $q' = q/\sigma_l$ for the 3D EAI spin-glass model.

data [7] in the spin-glass phase at $T = 1$, based on 8192 realizations for $L = 4$, 6 and 8, and 640 realizations for $L = 12$. In the tails the data (for $L = 12$) are accurate down to 10^{-53}.

A finite-size scaling (FSS) plot [8] of the probability densities at $T = 1.14$ is depicted in Fig. 1 where $P'(q') = \sigma_L P_L(q)$ versus $q' = q/\sigma_L$ with $\sigma_L = c_1 L^{-\beta/\nu}$, $\beta/\nu = 0.312(4)$, is shown. At $T = 1$ we obtained $\beta/\nu = 0.230(4)$. A major focus of our investigation is on the tails of the $P_L(q)$ distribution, which are shown in the lower part of Fig. 1 on a logarithmic scale.

When fitting the density to the Gumbel form (1) we introduced two slight modifications [6]: First, while (1) predicts, on a logarithmic scale, a constant slope a with decreasing $x' \leq x'_{\max}$, for the data of Fig. 1 the slope levels off and at $x' = 0$ the derivative of $P'(x')$ becomes zero. To incorporate this property we replaced the first x' on the r.h.s. of (1) by $c\tanh(x'/c)$, where $c > 0$ is a constant. Second, to take into account the $q' \leftrightarrow -q'$ invariance, we constructed a symmetric expression by multiplying the above construction with its reflection about the $q' = 0$ axis. Of course, the important large x' behavior of (1) is not at all affected by our manipulations.

The results of this fit (with $a = 0.446(37)$ for $T = 1$ and $a = 0.448(40)$ for $T = 1.14$) are shown in Fig. 1 where for $T = 1.14$ also the deviation of the fit from (a subset of) the $L = 16$ data is high-lighted. We observe a very good agreement which extends over the remarkable range of $200/\ln(19) \approx 87$ orders of magnitude.

3. 3D ISING MODEL

By simply setting all coupling constants J_{ik} to one, we have used exactly the same simulation set-up for studying the 3D Ising model at its critical point [9] $\beta_c = 0.221\,654$. Here we performed 32 independent runs (with different pseudo random number sequences) for lattices up to size $L = 30$ and 16 independent runs for $L = 36$. After calculating the multi-overlap parameters [5] the following numbers of sweeps were performed per repetition (i.e. independent run): $2^{19}, 2^{21}, 2^{22}, 2^{23}, 2^{23}, 2^{24}, 2^{25}$, and 2^{24} for $L = 4, 6, 8, 12, 16, 24, 30$, and 36, respectively.

We find the maximum of the $P_L(q)$ densities at $q_{\max} = 0$ [10]. This is in contrast to the well known double-peak of the magnetization probability density. In the tails the $L = 36$ density continues to exhibit accurate results down to -1200, thus the data from this system cover $1200/\ln(10) = 521$ orders of magnitude.

The collapse of the $P_L(q)$ functions on one universal curve $P'(q')$ is depicted in Fig. 2. The figure shows some scaling violations, which become rather small from $L \geq 24$ onwards. The standard deviation σ_L behaves with L according to $\sigma_L \propto L^{-2\beta/\nu}(1 + c_2 L^{-\omega} + \ldots)$, and from fits to our data we obtained $2\beta/\nu = d-2+\eta = 1.030(5)$, in good agreement with FSS estimates for the magnetization which cluster around $\eta = 0.036$.

We compared fits of the data with the Gumbel form (1) and the standard large-deviation behavior, based on the proportionality of the entropy with the volume [11],

$$P_L(q) \propto \exp[-Nf(q)] , \quad (4)$$

where, for large N, $f(q)$ does not depend on N. As is demonstrated by the plot of $f(q)$ in Fig. 2 our data clearly support the prediction (4). Also shown is the scaling form $f(q) \propto q^{d\nu/2\beta}$ with $\beta/\nu = 1.030$. We see excellent convergence towards an L-independent function, but the scaling behavior only holds in the vicinity of $q = 0$.

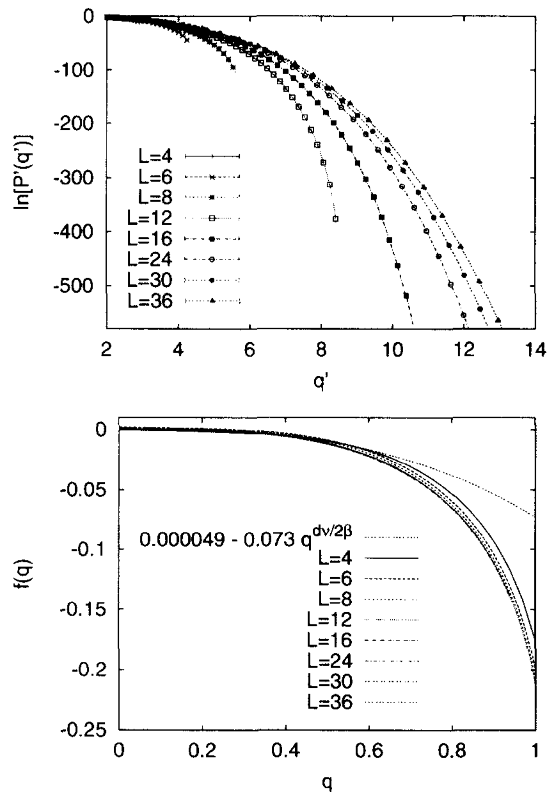

Figure 2. Rescaled overlap probability densities $P'(q') = \sigma_L P_L(q)$ versus $q' = q/\sigma_l$ for the 3D Ising model at the critical point, and the function $f(q)$ extracted from the large-deviation behavior (4) for various lattice sizes. Also shown is a FSS fit valid for small q.

4. CONCLUSIONS

For the 3D EAI spin-glass model we have found numerical evidence that the Parisi overlap distribution at $T = 1.14 \approx T_c$ and $T = 1$ can be described by (a slight modification of) the Gumbel form (1). The detailed relationship between this model and extreme order statistics remains to be investigated and it is certainly a challenge to extend previous work [12] in this direction to more involved scenarios. For the 3D Ising model at T_c, on the other hand, we find support for the standard scaling picture derived from large deviation theory, instead of the Gumbel form.

REFERENCES

1. S.T. Bramwell *et al.*, Phys. Rev. Lett. 84 (2000) 3744; Phys. Rev. E 63 (2001) 041106.
2. E.J. Gumbel, *Statistics of Extremes* (Columbia University Press, New York, 1958); J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, 2nd ed. (Krieger Publishing, Malibar, Florida, 1987).
3. S.J. Edwards and P.M. Anderson, J. Phys. F 5 (1975) 965.
4. M. Palassini and S. Caracciolo, Phys. Rev. Lett. 82 (1999) 5128; H.G. Ballesteros *et al.*, Phys. Rev. B 62 (2000) 14237.
5. B.A. Berg and W. Janke, Phys. Rev. Lett. 80 (1998) 4771.
6. B.A. Berg, A. Billoire, and W. Janke, Phys. Rev. E 65 (2002) 045102(R).
7. B.A. Berg, A. Billoire, and W. Janke, Phys. Rev. B 61 (2000) 12143.
8. M.E. Fisher, in: *Critical Phenomena*, Proc. 1979 E. Fermi Int. School of Physics, Vol. 51, ed. M.S. Green (Academic Press, New York, 1971), p. 1.
9. G.S. Pawley, R.H. Swendsen, D.J. Wallace, and K.G. Wilson, Phys. Rev. B 29 (1984) 4030.
10. B.A. Berg, A. Billoire, and W. Janke, cond-mat/0205377, to appear in Phys. Rev. E.
11. L. Boltzmann, Wiener Berichte 2, 76 (1877) 373.
12. J.-P. Bouchaud and M. Mézard, J. Phys. A 30 (1997) 7997.