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Abstract. We study statistical properties of a zero-range process (ZRP) on random networks. We derive an analytic expression for the distribution of particles (also called node occupation distribution) in the steady state of the ZRP in the ensemble of uncorrelated random graphs. We analyze the dependence of this distribution on the node-degree distribution. In particular, we show that when the degree distribution is tuned properly, one can obtain scale-free fluctuations in the distribution of particles. Such fluctuations lead to a power law in the distribution of particles, just like in the ZRP with the hopping rate $u(m) = 1 + b/m$ on homogeneous graphs.

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1 Introduction

Many statistical systems are defined on random networks or random lattices. Usually, this means that one considers a system on a lattice with some irregularities, or even on a purely random network, but instead of looking at what happens on a single network, one performs an annealed average over a statistical ensemble of networks that is over a collection of random graphs with some statistical weights.

The question of how averaging over the disorder influences the statistical properties of the system has been previously addressed in the context of two-dimensional statistical models. It was shown that additional degrees of freedom related to fluctuations of the geometry can lead to quite distinct behavior, in comparison to analogous systems defined on a fixed lattice. For example, critical properties of the two-dimensional Ising model on a fixed lattice are described by Onsager exponents [1] while on a random lattice, represented as a sum over planar networks, by “dressed” KPZ-DDK ones [2,3]. A similar change of critical exponents is observed for other models.

In this paper we address the same problem but for a statistical system on complex random networks. As a particularly simple but interesting example we consider a zero-range process (ZRP), which has been thoroughly studied on fixed networks [4–10]. The particular feature of this out-of-equilibrium model is that under certain conditions particles tend to condense on a single node. Here

we analyze the influence of the averaging over random networks on the distribution of particles in the steady state. We shall discuss an ensemble of networks with given probability distribution of degrees. To make things as simple as possible we shall restrict the discussion to uncorrelated networks with independent node degrees.

The paper is organized as follows. In the next section we shall recall the definition of the ZRP on a network and introduce quantities describing the steady state of this model. Then we shall consider the free ZRP process being a particular class of ZRP, in which the hopping rates do not depend on the distribution of particles but only on the connectivity of the network. For this case we will present in Sections 3 and 4 an exact solution to the problem of how the distribution of particles averaged over nodes depends on the node-degree distribution and how to choose the latter one to obtain a scale-free distribution of particles. In the following Section 5 we will discuss finite-size effects observed in the distribution of particles for some models of random networks like Erdős-Rényi graphs. We will finish in Section 6 with some concluding remarks.

2 Model description

Consider the ZRP on a connected simple graph with N nodes and a sequence $\mathbf{k} = \{k_1, \dots, k_N\}$ of node degrees. The state of the ZRP is given by the distribution of particles $\mathbf{m} = \{m_1, \dots, m_N\}$ on nodes of the network, where $m_i \geq 0$ is the number of particles at node i . The total

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number of particles $M = m_1 + \dots + m_N$ is conserved during the process. A particle can hop from node i to one of its neighbors with rate $u(m_i)/k_i$. The function $u(m)$, called hopping rate, depends only on the number of particles at the departure node i . The factor $1/k_i$ takes care of distributing the outflow of particles equally between the neighbors. The hopping rates $u(m)$ are non-negative and identical for all nodes.

The ZRP is known to have a unique steady state [11]. Static properties of this state are described by a partition function $Z(N, M, \mathbf{k})$ depending only on the degree sequence \mathbf{k} :

$$Z(N, M, \mathbf{k}) = \sum_{m_1, \dots, m_N=0}^M \delta_{m_1 + \dots + m_N, M} \prod_{i=1}^N p(m_i) k_i^{m_i}, \quad (1)$$

with statistical weights $p(m)$ defined as

$$p(m) = \prod_{n=1}^m \frac{1}{u(n)}, \quad p(0) = 1. \quad (2)$$

We will refer to $Z(N, M, \mathbf{k})$ as to a microcanonical partition function. The main quantity describing the system is the distribution of particles $\pi(m)$, averaged over all configurations \mathbf{m} with the weight given by the partition function (1) and over all nodes:

$$\pi(m, \mathbf{k}) = \frac{1}{N} \sum_i \langle \delta_{m, m_i} \rangle, \quad (3)$$

where the argument \mathbf{k} means that it is calculated for a single network with a given sequence of degrees. It is also called node occupation distribution. It can be calculated [10] as follows:

$$\pi(m, \mathbf{k}) = \frac{1}{N} \sum_{i=1}^N \frac{Z_i(N-1, M-m, \mathbf{k}')}{Z(N, M, \mathbf{k})} k_i^m p(m), \quad (4)$$

where Z_i is the partition function for a graph with $N-1$ nodes and degrees $\mathbf{k}' = \{k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_N\}$. Equation (4) holds for any connected graph with a given degree sequence.

Suppose now that we are interested in the behavior of the ZRP on a random network. In this case we have to take the average over networks. Denote by $P(\mathbf{k}) \equiv P(k_1, \dots, k_N)$ the probability of choosing a network with the degree sequence k_1, \dots, k_N . We can now define a canonical partition function as the average over all degree sequences:

$$Z(N, M) = \sum_{k_1, \dots, k_N} P(\mathbf{k}) Z(N, M, \mathbf{k}). \quad (5)$$

In general, $P(\mathbf{k})$ may have a complicated form. We shall restrict our attention to uncorrelated networks [12,13] for which $P(\mathbf{k})$ is a product measure: $P(\mathbf{k}) = \Pi(k_1) \cdots \Pi(k_N)$. This means that node degrees are independent of each other and that the observed degree distribution is

$\Pi(k)$.¹ The canonical partition function assumes then a simple, symmetric form:

$$Z(N, M) = \sum_{m_1, \dots, m_N} \delta_{m_1 + \dots + m_N, M} \prod_{i=1}^N \hat{p}(m_i), \quad (6)$$

where

$$\hat{p}(m) = p(m) \sum_{k=1}^{\infty} \Pi(k) k^m \quad (7)$$

is an effective weight for a node occupied by m particles. As we see, the effective weight $\hat{p}(m)$ is calculated from the node degree distribution $\Pi(k)$ and the occupation weight $p(m)$. The effective partition function $Z(N, M)$ in equation (6) has the form of a partition function of the balls-in-boxes model with identical weights, which has been thoroughly studied [14–16]. $Z(N, M)$ is invariant with respect to any permutation σ of node occupation numbers $m_i \rightarrow m_{\sigma(i)}$.

3 Free ZRP

In this section we shall consider a particular example of a ZRP for which the hopping rate $u(m) = 1$ is independent of m . We shall call it free ZRP (FZRP). In this case, also the occupation weight (2) is constant, $p(m) = 1$, and the canonical partition function (6) reduces to

$$Z(N, M) = \sum_{m_1, \dots, m_N} \delta_{m_1 + \dots + m_N, M} \prod_{i=1}^N \mu(m_i), \quad (8)$$

where

$$\mu(m) = \sum_{k=1}^{\infty} \Pi(k) k^m \quad (9)$$

is the m th moment of the node-degree distribution. The probability that a node is occupied by m particles is now:

$$\pi(m) = \frac{Z(N-1, M-m)}{Z(N, M)} \mu(m), \quad (10)$$

just as it was in the balls-in-boxes model [14].

For further convenience, let us introduce a generating function for the moments $\mu(m)$:

$$M(z) = \sum_m \mu(m) \frac{z^m}{m!}, \quad (11)$$

which encodes the same information as $\Pi(k)$. Indeed, inserting equation (9) into the last equation we see that it can be interpreted as a Fourier series with coefficients given by the node-degree distribution,

$$M(-iz) = \sum_k \Pi(k) e^{-ikz}. \quad (12)$$

¹ We neglect the fact that the total number of links is often fixed, which leads to an additional Kronecker delta constraint $\delta_{2L, k_1 + \dots + k_N}$. This constraint can be usually neglected in the thermodynamic limit.

From the generating function one can formally reconstruct the moments,

$$\mu(m) = \frac{m!}{2\pi i} \oint M(z)z^{-m-1}dz, \quad (13)$$

as well as the degree distribution,

$$\Pi(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dz e^{izk} M(-iz). \quad (14)$$

The partition function (8) is well defined if all moments of the distribution $\Pi(k)$ are finite. Usually, we are interested in the behavior of the system in the thermodynamic limit $N \rightarrow \infty$. We can distinguish two cases: (a) the limiting distribution $\Pi(k)$ for $N \rightarrow \infty$ has all moments finite, as for instance for Erdős-Rényi (ER) graphs, where it is Poissonian, (b) higher moments of the limiting distribution $\Pi(k)$ diverge for $N \rightarrow \infty$ as it happens for scale-free graphs [17]. For (a), the large- N limit presents no difficulty, while for (b) it has to be taken very carefully since it depends on the details of how the ensemble is defined. Moreover, in case (a) one can show that for random graphs² the probability of any sequence of degrees \mathbf{k} factorizes in the limit $N \rightarrow \infty$ [17]. This factorization often breaks down for (b). One observes particularly strong deviations from the factorization for $\Pi(k) \sim k^{-\gamma}$ with $2 < \gamma \leq 3$ where finite-size effects are especially strong [12,18,19]. Below we shall discuss only the case (a) which is free of these problems.

4 Power-law distribution of particles

We have shown in the previous section that averaging over fluctuating geometries leads to an effective model with the partition function (8) and weights $\mu(m)$ being the moments of the node-degree distribution. This model has an interesting critical behavior for weights which fall off like $\mu(m) \sim m^{-b}$. For example [20], for

$$\mu(m) \propto \Gamma(m+1)/\Gamma(m+b+1) \sim m^{-b}, \quad (15)$$

one observes a condensation of particles when the density of particles $\rho = M/N$ is larger than a critical density $\rho_c = 1/(b-2)$. In the thermodynamic limit, at the critical point $\rho = \rho_c$, fluctuations of the number of particles become scale-free and $\pi(m) = \mu(m)$ displays a power law. Below ρ_c it has an exponential cut-off:

$$\pi(m) = \mu(m) \exp(A - Bm), \quad (16)$$

where the constants A and B are chosen so that the normalization $\sum_m \pi(m) = 1$ and the density of particles $\sum_m m\pi(m) = \rho$ are fixed. Above ρ_c , the distribution $\pi(m)$ is approximately given by $\mu(m)$ but with an additional peak centered around $m_* \approx M - \rho_c N$.

² By a random graph we understand here a graph being maximally random among all graphs with a given sequence of degrees.

One now can ask whether the weights (15) can be obtained in our FZRP by tuning the node-degree distribution of the underlying network. Before we proceed, it is important to notice that the model given by the partition function (8) is invariant with respect to the rescaling:

$$\mu(m) \rightarrow \mathcal{N} \phi^m \mu(m). \quad (17)$$

Indeed, the partition function (8) changes only by a factor: $Z(N, M) \rightarrow \mathcal{N}^N \phi^M Z(N, M)$, which is constant for given N and M , while physical quantities stay intact because the normalization factor cancels out. Thus, we expect that if the moments are given by

$$\mu(m) = \mathcal{N} \frac{\Gamma(m+1)}{\Gamma(m+1+b)} \phi^m, \quad (18)$$

then the degree distribution of node occupation numbers at the critical density should be given by

$$\pi(m) \propto \frac{\Gamma(m+1)}{\Gamma(m+b+1)} \sim m^{-b}. \quad (19)$$

The question we face now is whether there is a node-degree distribution $\Pi(k)$ which has moments given by equation (18). First of all, we observe that the parameter ϕ in equation (18) plays the role of a scale parameter of the distribution $\Pi(k)$ as follows from the definition of the moments (9): under the rescaling $k \rightarrow k/\phi$ the moments change as $\mu(m) \rightarrow \mu(m)\phi^m$. We will use the freedom of choosing the parameter ϕ to fix the average degree \bar{k} and thus also the number of links $L = \bar{k}N/2$. The parameter \mathcal{N} has to be chosen in such a way that $\Pi(k)$ takes the proper normalization of a probability.

The integral in equation (14) is hard to calculate and cannot be easily expressed in terms of elementary functions. However, if we assume $\phi \gg 1$, then the function $M(-iz)$ goes to zero sufficiently fast when $z \rightarrow \pm\infty$ and thus we can extend the limits of integration to $\pm\infty$. In this case the integral can be done analytically. Equation (14) becomes a Fourier transform of the function $M(-iz)$ which is a special case of the Mittag-Leffler functions having a known form of an infinite series expansion (see e.g. [21]). Changing variables $k \rightarrow x\phi$ we obtain

$$\Pi(x\phi) = \frac{\mathcal{N}}{2\pi\phi} \int_{-\infty}^{\infty} dz e^{izx} \sum_{m=0}^{\infty} \frac{(-iz)^m}{\Gamma(m+1+b)}. \quad (20)$$

According to equation (B21) of reference [21], the above integral yields

$$\frac{\mathcal{N}}{\phi} \sum_{m=0}^{\infty} \frac{(-x)^m}{m! \Gamma(b-m)}, \quad (21)$$

and hence

$$\Pi(k) = \frac{\mathcal{N}}{\phi} \sum_{m=0}^{\infty} \frac{(-k/\phi)^m}{m! \Gamma(b-m)} = (\phi - k)^{b-1} \frac{\mathcal{N}}{\Gamma(b)\phi^b}. \quad (22)$$

In Figure 1 we compare $\Pi(k)$ computed numerically using the integral (14) with the original limit $\pm\pi$, and calculated

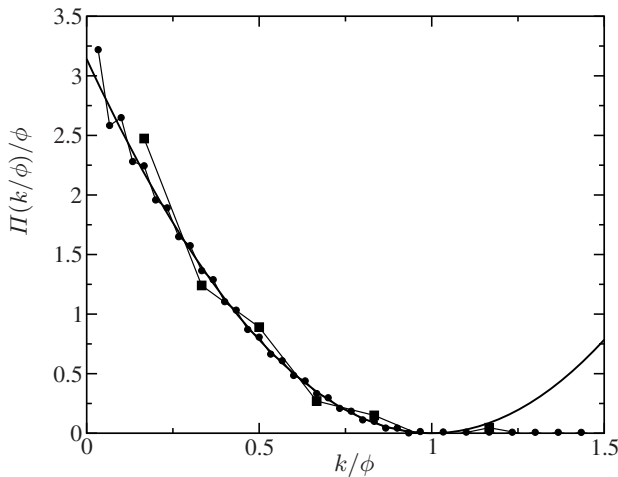


Fig. 1. $\Pi(k)$ calculated from the exact formula (14) (points) and approximated one (Eq. (22), thick line), for $\mathcal{N} = 2\pi$ and $b = 3$. Squares: $\phi = 6$, circles: $\phi = 30$. The approximate solution diverges for $x = k/\phi > 1$ and has to be cut. For $0 < x \leq 1$ the approximation is the better, the larger is ϕ .

by means of equation (22). Because the probability $\Pi(k)$ must be non-negative, the above solution is physical only for $k \leq \phi$ and we have to set $\Pi(k) = 0$ for $k > \phi$. We see that the integer part of ϕ can be interpreted as the maximal degree which can exist in the network. The existence of the upper cut-off in the node-degree distribution is not only a property of the approximate solution. Also when one uses the exact relation (14) to calculate the degree distribution $\Pi(k)$ for the moments of the form (15), one obtains negative values of $\Pi(k)$ for $k > \phi$, so again one has to cut off the solution and set $\Pi(k)$ to zero for $k > \phi$. If one now calculates moments for the distribution (22) with the cut-off directly from the definition (9), one will see that they deviate slightly from those given by equation (18). However, the deviation decreases when ϕ increases and finally becomes negligible for sufficiently large ϕ . In Figure 2 we plot the moments $\mu(m)$ from equation (18) and those calculated from equation (9), for various ϕ . As ϕ increases, the curves tend asymptotically to a power law.

The parameter ϕ is related to the average degree as $\bar{k} = \sum_{k=1}^{\phi} \Pi(k)k$. For large ϕ , the relation between ϕ and \bar{k} is almost linear:

$$\bar{k} = \frac{\sum_{k=1}^{\phi} (k - \phi)^{b-1} k}{\sum_{k=1}^{\phi} (k - \phi)^{b-1}} \approx \frac{\int_0^{\phi} (\phi - k) k^{b-1} dk}{\int_0^{\phi} k^{b-1} dk} = \frac{\phi}{b+1}. \quad (23)$$

This implies that in order to obtain the power-law distribution (19) for $N \rightarrow \infty$, the value of k should increase to infinity. For sparse networks \bar{k} would be finite. We see thus that the price to pay for having a scale-free distribution of particles is to make networks denser when their size increases.

The normalization factor \mathcal{N} must be chosen so that the degree distribution is normalized to unity: $\sum_{k=1}^{\phi} \Pi(k) = 1$. For example, for $b = 2, 3, 4$ we obtain

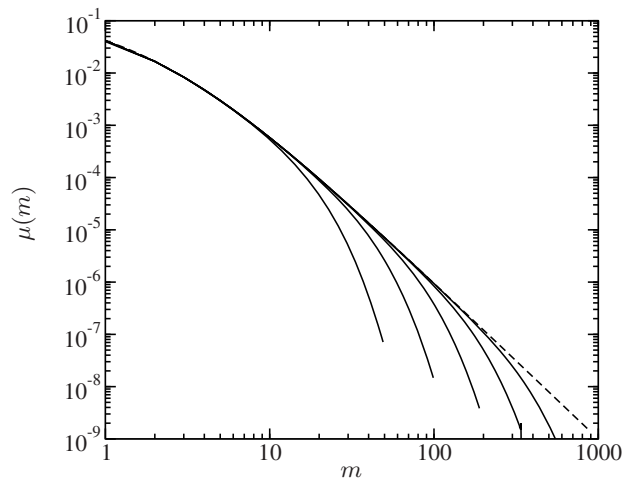


Fig. 2. Desired (dashed line) versus real distribution of particles $\mu(m)$ for networks with finite average degree \bar{k} (solid lines), obtained from equations (22) and (9) for $b = 3$. Lines from left to right: $\phi = 5, 10, 20, 40, 80$ which corresponds to $\bar{k} = 1.67, 2.89, 5.4, 10.4, 20.4$ from equation (28). These plots approximate also $\pi(m)$ at the critical point. The parameter ϕ grows almost linearly with \bar{k} .

the following degree distributions $\Pi(k)$ for $0 < k \leq \phi$:

$$b = 2 : \frac{2(\phi - k)}{\phi(\phi - 1)}, \quad (24)$$

$$b = 3 : \frac{(\phi - k)^2}{\phi(\phi - 1)(2\phi - 1)}, \quad (25)$$

$$b = 4 : \frac{4(\phi - k)^3}{\phi^2(\phi - 1)^2}, \quad (26)$$

and zero for both $k = 0$ and $k > \phi$, with ϕ given by the following formulas for $b = 2, 3$:

$$\phi = 3\bar{k} - 1, \quad (27)$$

$$\phi = \left(-1 + 4\bar{k} + \sqrt{1 - 16\bar{k} + 16\bar{k}^2} \right) / 2, \quad (28)$$

and by the solution of a cubic equation for $b = 4$:

$$\frac{(1 + \phi)(3\phi^2 - 2)}{15\phi(\phi - 1)} = \bar{k}. \quad (29)$$

How does it come about that the power laws are observed in the distribution of particles $\pi(m)$ when one averages it over networks while they are not seen in $\pi(m)$'s for individual nodes, for any single network in the ensemble? The answer is that the effective distribution $\pi(m)$ averaged over networks is a subtle result of a well-tuned superposition of many exponential distributions: for a node with degree k , the distribution of particles is $\pi(m) \propto (k/k_{max})^m$, where k_{max} is the maximal degree in the network [9]. On the node with maximal degree, however, there is a condensation just as for scale-free networks [8], but it disappears in the thermodynamic limit. This happens because the critical density for the condensation becomes larger than $\rho = 1/(b - 2)$ which we assumed to hold in our system, and the system is always in the fluid phase.

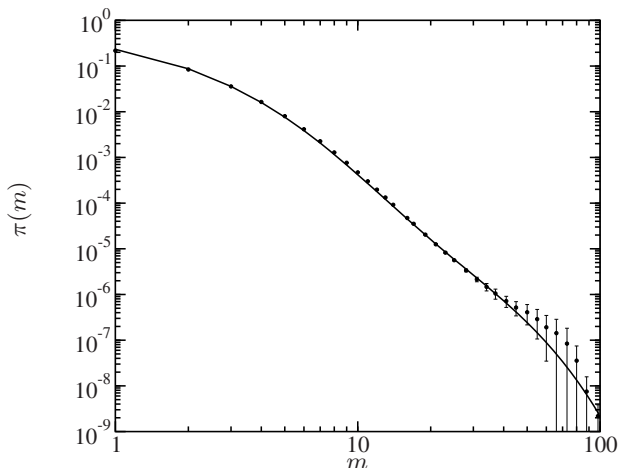


Fig. 3. The particle distribution $\pi(m)$ for ER graphs with $N = 400$ nodes and $\bar{k} = 8$, obtained from MC simulations with $M = 300$ particles and averaged over $n = 200$ graphs (points) and calculated from equation (9) by using a cut Poissonian degree distribution as described in the main text (solid line).

5 Other random graphs

In the previous section we found a node degree distribution for the ensemble of random, uncorrelated networks for which the corresponding FZRP has a power-law particle distribution $\pi(m)$. What happens with FZRP on generic random networks? Can the particle distribution be scale free? Let us begin with what happens in the limit $N \rightarrow \infty$. Consider some typical examples of graphs like random trees [22] or ER graphs for which the limiting shapes of the degree distribution are known: $\Pi(k) = e^{-1}/(k-1)!$ and $\Pi(k) = e^{-\bar{k}}\bar{k}^k/k!$, respectively. So we can calculate the corresponding critical distribution $\pi(m)$. In the first case, the generating function has a closed form $M(z) = \exp(z + e^z)$, as follows from equation (11), and we can deduce the coefficients $\mu(m)$ from the inverse Laplace transform (13). Using the saddle-point method and integrating around $z_0 \approx \log(m/\log m)$ we can find the asymptotic behavior for large m :

$$\log \mu(m) = m(\log m - \log \log m) + O(m). \quad (30)$$

We see that $\mu(m)$ grows over-exponentially for large m . This means that in the thermodynamic limit the condensation always happens, regardless of the density of particles. The distribution $\pi(m)$ in the bulk falls faster than any power law. Similarly, one can estimate that for random ER graphs $M(z) \propto \exp(\bar{k}e^z) - 1$ and the leading term in $\log \mu(m)$ is also $m \log m$. So again it is clear that in the limit $M \rightarrow \infty$ one cannot obtain a power-law distribution of particles.

Surprisingly, in Monte Carlo (MC) simulations of the FZRP on ER graphs one observes distributions $\pi(m)$ which very much resemble those with power-law tails, see Figure 3. The data points were obtained by simulating the FZRP as described in reference [9] for $M = 300$ particles and averaging over $n = 200$ graphs of size $N = 400$ and $\bar{k} = 8$. The same effect appears on random trees. This

quasi-power law can be, however, explained by finite-size effects. In order to mimic these effects in our analytic calculations let us assume that $\Pi(k)$ is Poissonian but has a cutoff at some k_{max} . The position of the cut-off is estimated from the condition that the number of nodes with degree k_{max} obtained in n samples of the ER graph should be of order one: $Nn\Pi(k_{max}) \approx 1$, which gives $k_{max} \approx 22$. Since $\Pi(k)$ falls off rapidly, this is a good estimate. Next, for the distribution with the cut-off k_{max} we calculate the moments from equation (9) and we get rid of the leading exponential behavior of $\mu(m)$, using the freedom (17), by multiplying them by a factor $\exp(-mB)$ with B appropriately chosen. This gives us the distribution of particles $\pi(m)$. As we can see in Figure 3 we obtain almost a straight line in the log-log plot which agrees well with the result of MC simulations. The discussion from the beginning of this section tells us, however, that this quasi-power-law behavior is only a finite-size effect which will disappear in the limit $N \rightarrow \infty$.

6 Conclusion

We analytically investigated the influence of annealed averaging over random networks on the statistical properties of ZRP. In particular, we calculated how the particle distribution depends on the node-degree distribution. We showed that by tuning the node-degree distribution we can make the corresponding zero-range process critical and having a power-law distribution of particles.

We also checked that for random graphs with a truncated Poissonian degree distribution the corresponding particle distribution $\pi(m)$ behaves effectively as a power law in a wide range of m . Generally, such truncated degree distributions occur in considerations of finite-size networks and as we see they usually make the corresponding particle distribution looking like a power law in a certain range. In other words, a superficial power law observed in the empirical data for $\pi(m)$ for typical ensembles of networks can be attributed to the finiteness of the system and the cut-off for the maximal observed degree.

We believe it is the first step towards the analysis of more complex systems, where the topology and the dynamics of the system are coupled to each other and influence mutually. An example of this type of interactions was discussed in the context of 2d statistical systems on random lattices (2d gravity), where a back-reaction of the system on the lattice was observed [2,3,23], which manifested as a change of fractal properties of the underlying geometry when the system became critical. It would be very interesting to see such an adaptation mechanism also for random complex networks.

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