Monte Carlo Studies of Connectivity Disorder

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1 Introduction

The paradigm for studies of the effect of quenched, random disorder on universal properties of critical phenomena are uncorrelated, randomly distributed couplings [1–4]. This includes ferromagnetic random-bond models as well as the physically very different case of spin glasses, where competing interactions complement disorder with frustration [2, 5–9]. For a continuous phase transition in the idealized pure system, the effect of random bonds has been convincingly shown by renormalization group analyses as well as numerical investigations to be able to induce a crossover to a new, disorder fixed point [3, 10–14]. Using phenomenological scaling theory, Harris [6] argued that such a crossover should not occur for systems with a specific-heat exponent $\alpha < 0$. It is now widely believed that also the converse is true, i.e., a crossover *does* occur for systems with $\alpha > 0$ [10, 11, 15]. In the marginal case $\alpha = 0$, realized, e.g., by the Ising model in two dimensions, the regular critical behavior is merely modified by logarithmic corrections [3]. Similarly, for systems exhibiting a first-order phase transition in the regular case, the introduction of quenched disorder coupling to the local energy density can weaken the transition to second (or even higher) order [9]. While this scenario has been rigorously established for the case of two dimensions and an arbitrarily small amount of disorder [7,8,16], the situation for higher-dimensional systems is less clear. For a variety of systems in three dimensions, however, sufficiently strong disorder has been shown numerically [17–19] to be able to soften the transition to a continuous one.

Spatial correlations of the disorder degrees of freedom lead to a modification of the fluctuations present in "typical" patches of the random system with respect to the behavior expected from the central limit theorem for independent random variables, which is implicitly presupposed by Harris' arguments. Such correlations for a random-bond model have been considered occasionally [20–23] and altered relevance criteria have been proposed [20,24]. Luck [24] has considered a class of irregular systems not covered by the random-bond paradigm, namely that of quasi-crystalline or aperiodic structures, and formulated a generalized relevance criterion. Although he did not consider systems with connectivity *disorder* such as the random graph models to be considered here, his reasoning should also apply to these cases, as will be shown below.

In Sect. 2, we define Poissonian Voronoï-Delaunay triangulations [25] and the planar ϕ^3 Feynman diagrams of the dynamical triangulations model [26], and in Sect. 3 we elaborate on a formulation of Luck's relevance criterion appropriate for the considered random graphs and numerically determine the wandering exponents. Section 4 is devoted to a report on results of a high-precision Monte Carlo simulation study of the three-states Potts model on planar, Poissonian Voronoï lattices. Finally, Sect. 5 contains our conclusions.

2 Correlated Random Graph Models

In the following, we present two examples of classes of random graphs exhibiting spatial correlations of the co-ordination number distributions, which are the disorder degrees of freedom for these lattices. In this respect, these graphs are different from "generic" random graph models [27], where bonds are distributed completely at random between a given number of nodes, such producing uncorrelated connectivity disorder.

2.1 Poissonian Voronoï-Delaunay Graphs

The planar Voronoï-Delaunay construction [25] prescribes a segmentation of a patch of the plane into convex polygons compatible with a given set of point positions (generators). The Voronoï cell of a generator is defined as the region of the plane, which is closer to it than to any other generator. The three-valent vertices where these cells meet and the cell edges make up the Voronoï graph associated with the generators. Accordingly, the structure geometrically dual to the Voronoï graph is the Delaunay triangulation of the considered patch of the plane. For regularly placed generators one recovers the Wigner-Seitz elementary cells of regular lattices. If the generators are chosen at random, the resulting Voronoï-Delaunay graph is referred to as *Poissonian random lattice* since the generators can be considered as realization of a Poisson point process [25, 28]. To eliminate surface effects, the Voronoï-Delaunay construction is here applied to generators distributed at random on a sphere; for an illustration see Fig. 1 resp. Fig. **??**.

In what follows we shall focus on the variation of co-ordination numbers q_i of the triangulation resp. loop lengths of the dual graph, i.e., we consider *connectivity disorder* as the only effect of randomness, neglecting the fact of differing edge lengths. From the Euler relations, the average co-ordination number is a topological invariant for a fixed number N of triangles in two dimensions, given for spherical topology by [26]

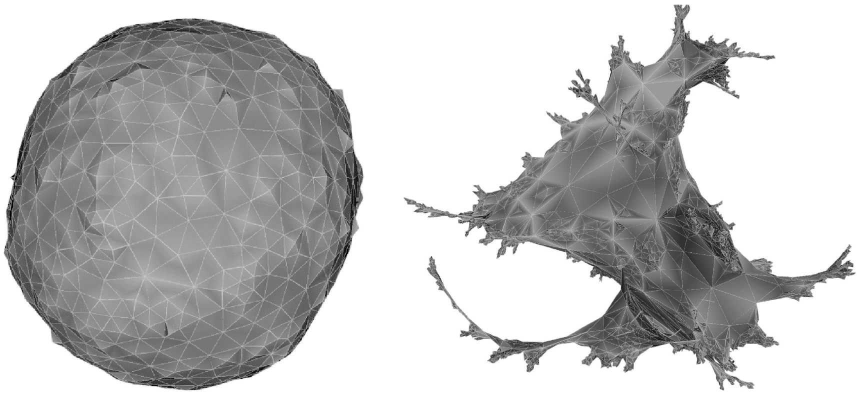

Fig. 1. Snapshots of random Poissonian Delaunay triangulations (left) and dynamical triangulations (right) of spherical topology with $N = 5000$ triangles. The Voronoï resp. ϕ^3 graphs considered numerically are the geometric duals of the shown structures.

$$\bar{q} = \frac{1}{N}\sum_i q_i = 6\frac{N}{N+4} \stackrel{N\to\infty}{\longrightarrow} 6. \tag{1}$$

The variance of co-ordination numbers can be shown numerically to approach [25, 29] $\mu_2 \equiv \langle q_i^2\rangle - \langle q_i\rangle^2 \approx 1.781$, as $N \to \infty$. It turns out that the random variables q_i are not independently distributed, but are reflecting a spatial correlation of the disorder degrees of freedom in addition to the trivial correlation induced by the constraint (1). For nearest-neighbor vertices these correlations are approximately described by the Aboav-Weaire law [25],

$$q\,m(q) = (6-a)q + b, \tag{2}$$

where $q\,m(q)$ is the number of edges of the neighbors of a q-sided cell, and a and b are some parameters [28].

2.2 Dynamical Triangulation Graphs

A different ensemble of random graphs is known from the *dynamically triangulated random surface* (DTRS) model used as a constructive approach to quantum gravity [26], where all possible gluings of N equilateral triangles to the surface of a sphere are counted with equal probability. Independent realizations of this graph ensemble can be generated by a recursive insertion method proposed in Ref. [30], for an example see Fig. 1 resp. Fig. **??**. The dual graphs are planar, "fat" (i.e., orientable) ϕ^3 Feynman diagrams without tadpoles and self-energy insertions, which can be counted analytically by matrix model methods [26,31]. Fluctuations are much more pronounced in these structures than in the Poissonian random graphs. In fact, it can be shown that the asymptotic variance of co-ordination numbers is exactly $\mu_2 = 10.5$ [32],

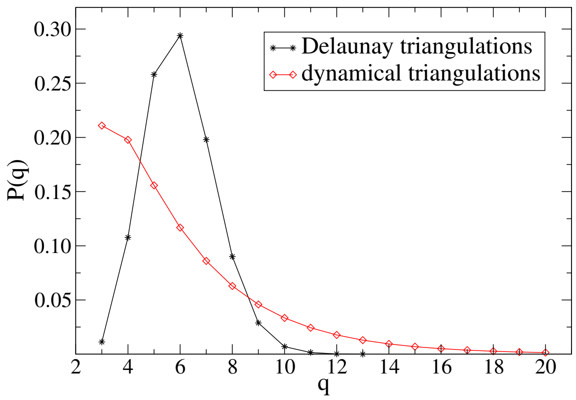

Fig. 2. Comparison of the co-ordination number distributions $P(q)$ of Poissonian Delaunay triangulations and dynamical triangulations in the limit $N \to \infty$.

whereas Eq. (1) still holds. Comparing the distributions of co-ordination numbers [29, 31, 33] shown in Fig. 2, it is seen that for the DTRS model large co-ordination numbers are much more probable than for Poissonian random graphs. In fact, the DTRS ensemble can be described as that of (statistically) self-similar, fractal trees of "baby universes" [34] which branch off from the main surface at vertices with large co-ordination number. The "baby universe" structure is reflected in an exceptionally large internal Hausdorff dimension of $d_h = 4$ [35, 36] as compared to the dimension $d_h = 2$ of Poissonian random graphs. As will be shown below, dynamical triangulations graphs also exhibit spatial correlations between the co-ordination numbers, which are in fact much stronger than those found for Poissonian random lattices.

3 The Harris-Luck Criterion and Wandering Exponents

The relevance of randomness coupling to the local energy density crucially depends on how fast fluctuations of the local transition temperature induced by fluctuations of the random variables in a correlation volume die out as the critical point is approached. For independent random variables, this decay occurs with an exponent of $d/2$ in d dimensions. The comparison of this power with the inverse correlation length exponent $1/\nu$ leads to Harris' celebrated relevance criterion [6, 37].

Following Luck [24], this reasoning can be extended to the correlated random variables present in the random graph models under consideration as follows. Consider a spherical patch P of radius R on a triangulation, containing $B(R)$ vertices. All distances on graphs are to be understood as the number of links in the unique shortest path of links connecting two vertices. Then, the fluctuations of the average co-ordination number in P,

$$J(R) \equiv \frac{1}{B(R)} \sum_{i \in P} q_i, \tag{3}$$

around its expected value $J_0 = \bar{q}$ [cf. Eq. (1)] in general decay in the limit $R \to \infty$ of large patches as

$$\sigma_R(J) \equiv \langle |J(R) - J_0| \rangle / J_0 \sim \langle B(R) \rangle^{-(1-\omega)} \sim R^{-d_h(1-\omega)}, \tag{4}$$

defining the wandering exponent ω of the considered graph type. Here, the averages $\langle \cdot \rangle$ are to be understood as the ensemble averages of the considered class of graphs of a given total size. While for $\omega = 1/2$ the usual $1/\sqrt{\langle B(R) \rangle}$ behavior of uncorrelated random variables is recovered, for random lattices with long-range correlations of the co-ordination numbers one expects $\omega > 1/2$, leading to a slowed-down decay of fluctuations. Near criticality, the fluctuation $\sigma_\xi(J)$ of the average co-ordination number in a correlation volume induces a local shift of the transition temperature proportional to $|t|^{d_h \nu(1-\omega)} \mu_2^{1/2}$. For the regular critical behavior to persist, these fluctuations should die out as the critical point $t = 0$ is approached. This is the case when ω does not exceed the threshold value

$$\omega_c(\nu) = 1 - \frac{1}{d_h \nu} = \frac{1-\alpha}{2-\alpha}, \tag{5}$$

provided that hyper-scaling is applicable. On the other hand, for $\omega > \omega_c(\nu)$ a new type of critical behavior could occur. Re-writing Eq. (5) as

$$\alpha_c = \frac{1-2\omega}{1-\omega}, \tag{6}$$

it is obvious that for $\omega = 1/2$ the Harris criterion is recovered.

Since for graphs with sufficiently long-range correlations of the co-ordination numbers $\omega > 1/2$, this type of disorder is *more* relevant than uncorrelated randomness in the sense that a change of universality class can already be expected for some range of *negative* values of α, cf. Eq. (6). On the other hand, if correlations decay exponentially, the Harris criterion should stay in effect.

In Ref. [38] we determined the wandering exponent ω numerically by sampling the fluctuations defined in Eq. (4) for a series of graph realizations of the considered ensembles. For both ensembles, an average is taken over 100 different graph realizations. To determine the volume $B(R)$ and the average co-ordination number $J(R)$, a vertex of the triangulation is picked at random and the graph is subsequently decomposed into spherical shells of radius R.

The resulting final averaged fluctuations for Poissonian Delaunay triangulations and dynamical triangulations with $N = 500\,000$ triangles are shown in Fig. 3. Note that the range of accessible distances R for a given number of triangles N is much smaller for the case of dynamical triangulations due to the large fractal dimension $d_h = 4$. According to Eq. (4) these plots should show an approximately linear behavior in a logarithmic presentation for large distances R, the slope of the line being given by $1 - \omega$. For an intermediate

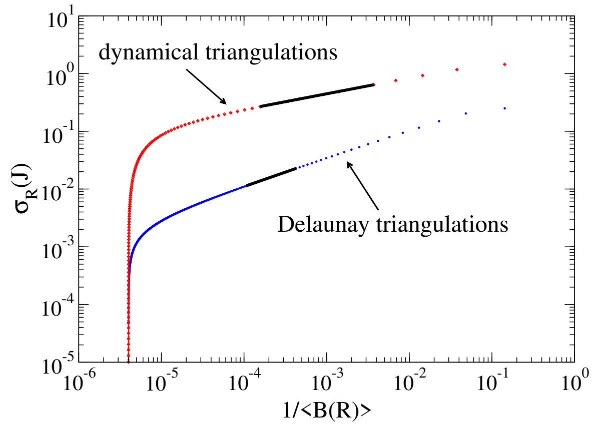

Fig. 3. Numerical estimate of the scaling of the average fluctuation of co-ordination numbers of triangulations of volume $N = 500\,000$ for the two considered ensembles and a fit to the expected functional form (4).

range of volumes $B(R)$ and, consequently, distances R, these expectations are met for both graph types. For $1/B(R) - 2/(N + 4)$ small, however, this behavior is superimposed by an exponential decline of fluctuations, reflecting the topological constraint (1). Hence, taking discretization effects for small R into account, only a rather small window of distances can be used for a reliable determination of the wandering exponent via a fit to the functional form (4). This is done by successively dropping events from either side of the range of R's while monitoring the quality-of-fit parameter[1] Q. Taking these considerations into account, our estimates for the wandering exponent from the largest graphs with $N = 500\,000$ triangles are

$$\omega = 0.50096(55), \quad R = 21, \ldots, 41, \quad \text{Delaunay triangulations,} \tag{7}$$

$$\omega = 0.72492(86), \quad R = 5, \ldots, 14, \quad \text{dynamical triangulations.} \tag{8}$$

The error estimates are calculated by jackknifing over the whole fitting procedure, such as to avoid any bias caused by cross-correlations of the $J(R)$ for different R. Due to the large fractal dimension of the dynamical triangulations graphs, we expect systematic finite-size corrections to be much more pronounced there. We have therefore repeated above analysis for different graph sizes ranging from $N = 1000$ to $500\,000$ and fitted the resulting exponents $\omega(N)$ to the finite-size scaling (FSS) ansatz

$$\omega(N) = \omega_\infty + AN^{-\theta}, \tag{9}$$

where θ is an *a priori* unknown correction exponent. A fit of this form to the data yields [38]

[1]Due to the correlations between values of $J(R)$ for different distances R, the *absolute* values of Q are not immediately meaningful; relative changes, however, are.

$$\omega_\infty = 0.7473(98), \quad \text{dynamical triangulations}, \tag{10}$$

with $A = -0.73(37)$ and $\theta = 0.264(70)$.

The result for Voronoï-Delaunay graphs is well consistent with $\omega = 1/2$ which would result from correlations decaying with a power larger than $d = 2$ (see also Ref. [20]). A direct inspection of the correlation function of co-ordination numbers indicates an even exponential decay [38]. Thus, the relevance criterion (6) reduces to the Harris criterion; Voronoï-Delaunay connectivity disorder should be a relevant perturbation for models with specific-heat exponent $\alpha > 0$. For the dynamical triangulations, on the other hand, the co-ordination number correlations are found to be algebraic (which is consistent with a direct analysis of the correlation function [38]). Our FSS extrapolated estimate for the wandering exponent in (10) suggests that $\omega = 3/4$ in this case. The criterion (6) then implies a relevance threshold of $\alpha_c = -2$, i.e., that these graphs should alter the critical behavior of *all* known ordered models.

4 The Potts Model on Voronoï Graphs

For $q = 3, 4$ Potts models with $\alpha = 1/3,\ 2/3$, the relevance criterion (6) with the found values of ω predicts a change of critical behavior for both random graph types. For the Ising model with $\alpha = 0$, on the other hand, only dynamical triangulations should be a relevant perturbation, whereas Voronoï-Delaunay graphs should at most induce logarithmic corrections.

Indeed, simulations of the $q = 2, 3, 4$ Potts models coupled to dynamical triangulations provide good evidence for a cross-over to new universality classes [39]. Furthermore, the exact solution of the percolation model, which has $\alpha = -2/3$ and corresponds to the limit $q \to 1$ of the Potts model, on dynamical triangulations, also shows a shift to a different universality class [40, 41]. And the first-order case $q = 10$ appears to get softened to a continuous transition [42]. For two-dimensional Delaunay triangulations, on the other hand, simulations of the Ising model yield Onsager exponents; the presence of possible logarithmic corrections could not be detected [43]. For the $q = 3$ Potts model in two dimensions only an exploratory study with rather small graphs is available, which does *not* show a change of critical behavior as compared to regular lattices [44].

We have therefore performed high-precision Monte Carlo simulations of the $q = 3$ Potts model with Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta_{s_i s_j}, \tag{11}$$

where $s_i \in \{1, 2, 3\}$ and the sum runs over all nearest-neighbor pairs of vertices of a Voronoï graph, which we consider as a topological object, with equal distances set to unity between any two nearest-neighbor vertices. All simulations are performed using the Swendsen-Wang (SW) cluster algorithm [45].

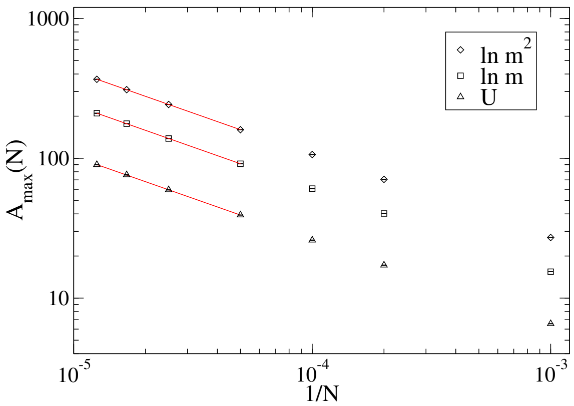

Fig. 4. FSS of the maxima of $\mathrm{d}\ln m/\mathrm{d}K$, $\mathrm{d}\ln m^2/\mathrm{d}K$, and $\mathrm{d}U/\mathrm{d}K$ of the $q = 3$ Potts model on Voronoï graphs. The solid lines show fits to the FSS ansatz (12).

The disorder averages are performed on the level of the free energy and its derivatives using 100 different realizations of Voronoï graphs. As these realizations are completely independent, the simulations can be done in parallel. The sufficiency of this number of copies was checked by performing the same analyses with only half the number of graphs. For the FSS analysis, we considered graphs of sizes $N/1000 = 1$, 5, 10, 20, 40, 60, and 80. For each copy, after thermalization 50 000 measurements were taken, yielding a total statistics of 5×10^6 events per lattice size. By estimating integrated autocorrelation times, we checked that it only takes a few SW updates for all considered graph sizes to create an effectively uncorrelated new configuration. To compute the various considered quantities as continuous functions of the coupling $K = \beta J$, we make use of the reweighting technique [46].

To determine the transition point and the critical exponents of the model, we applied a well tried sequence of FSS analyses, see, e.g., Refs. [39, 47]. First, to determine the correlation length exponent ν, we make use of the fact that the logarithmic derivatives of the order parameter[2] m as well as the derivative of the Binder cumulant $U = 1 - \langle m^4\rangle/3\langle m^2\rangle^2$ at their respective maxima should scale asymptotically as

$$A_{\max}(N) \sim aN^{1/2\nu}, \tag{12}$$

where we restrict ourselves to $A = \mathrm{d}\ln m/\mathrm{d}K$, $\mathrm{d}\ln m^2/\mathrm{d}K$, and $\mathrm{d}U/\mathrm{d}K$. Figure 4 shows the results of the scaling analysis together with fits to the functional form (12). To account for the visible effects of scaling corrections, we include only results for $N \geq 20\,000$. These fits yield $\nu = 0.8328(26)$, 0.8340(26),

[2]To break symmetry explicitly, we use the maximum definition of m, see, e.g. Ref. [49].

and 0.8342(46) for $A = \mathrm{d}\ln m/\mathrm{d}K$, $\mathrm{d}\ln m^2/\mathrm{d}K$, and $\mathrm{d}U/\mathrm{d}K$, respectively, resulting in a weighted average of

$$\nu = 0.8335(26), \tag{13}$$

in excellent agreement with the regular lattice value of $\nu = 5/6 = 0.833\bar{3}$. With this estimate of ν, the transition point can be found from the peak positions of various observables,

$$K(A_{\max}, N) \sim K_c + aN^{-1/2\nu}, \tag{14}$$

with A one of c_V, $\mathrm{d}m/\mathrm{d}K$, $\mathrm{d}\ln m/\mathrm{d}K$, $\mathrm{d}\ln m^2/\mathrm{d}K$, $\mathrm{d}U/\mathrm{d}K$, or χ, where $c_V = K^2 N[\langle e^2\rangle - \langle e\rangle^2]$ and $\chi = N[\langle m^2\rangle - \langle m\rangle^2]$, and e denotes the internal energy per site. An error weighted average of these independent estimates gives $K_c = 1.524\,876(21)$, where the error does not take into account the uncertainty in ν. This value should be compared with $K_c \approx 1.484\,21$ for the (regular) honeycomb lattice [48].

Further critical exponents are determined independently from the FSS relations,

$$c_{V,\max}(N) \sim N^{\alpha/2\nu}, \quad m_{\inf}(N) \sim N^{-\beta/2\nu}, \quad \text{and} \quad \chi_{\max}(N) \sim N^{\gamma/2\nu}, \tag{15}$$

where $m_{\inf}$ denotes the magnetization at its point of inflection. From the fits we obtained the values

$$\alpha/2\nu = 0.2201(27), \quad \beta/2\nu = 0.0617(14), \quad \text{and} \quad \gamma/2\nu = 0.8718(12), \tag{16}$$

which are again close to the regular lattice values of $\alpha/2\nu = 1/5 = 0.2$, $\beta/2\nu = 1/15 = 0.066\bar{6}$, and $\gamma/2\nu = 13/15 = 0.866\bar{6}$.

5 Conclusions

Adapting Luck's formulation [24] for quasi-periodic lattices to the case of random graphs, we have analyzed the applicability of this relevance condition to connectivity disorder. For dynamical triangulations we obtained a large wandering exponent compatible with $\omega = 3/4$, indicating that they should form a relevant perturbation for all models with $\alpha > \alpha_c = -2$, which is in accord with previous explicit results for the q-states Potts model. For Poissonian Voronoï-Delaunay graphs, on the other hand, correlations between the co-ordination numbers seem to decay exponentially, such that the adapted relevance criterion reduces to Harris' threshold of $\alpha_c = 0$.

For the $q = 3$ Potts model with $\alpha = 1/3$, connectivity disorder from Poissonian random lattices should be relevant. The FSS analysis presented above yields, however, a thermal scaling exponent in very good agreement with that for the regular lattice model. Whether the small, but significant difference of the magnetic exponents indicates the onset of a crossover to a new universality class or is merely an effect of neglected corrections to scaling, has to be checked by a more careful scaling analysis including corrections, possibly augmented by simulations for even larger lattices.

Acknowledgements

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