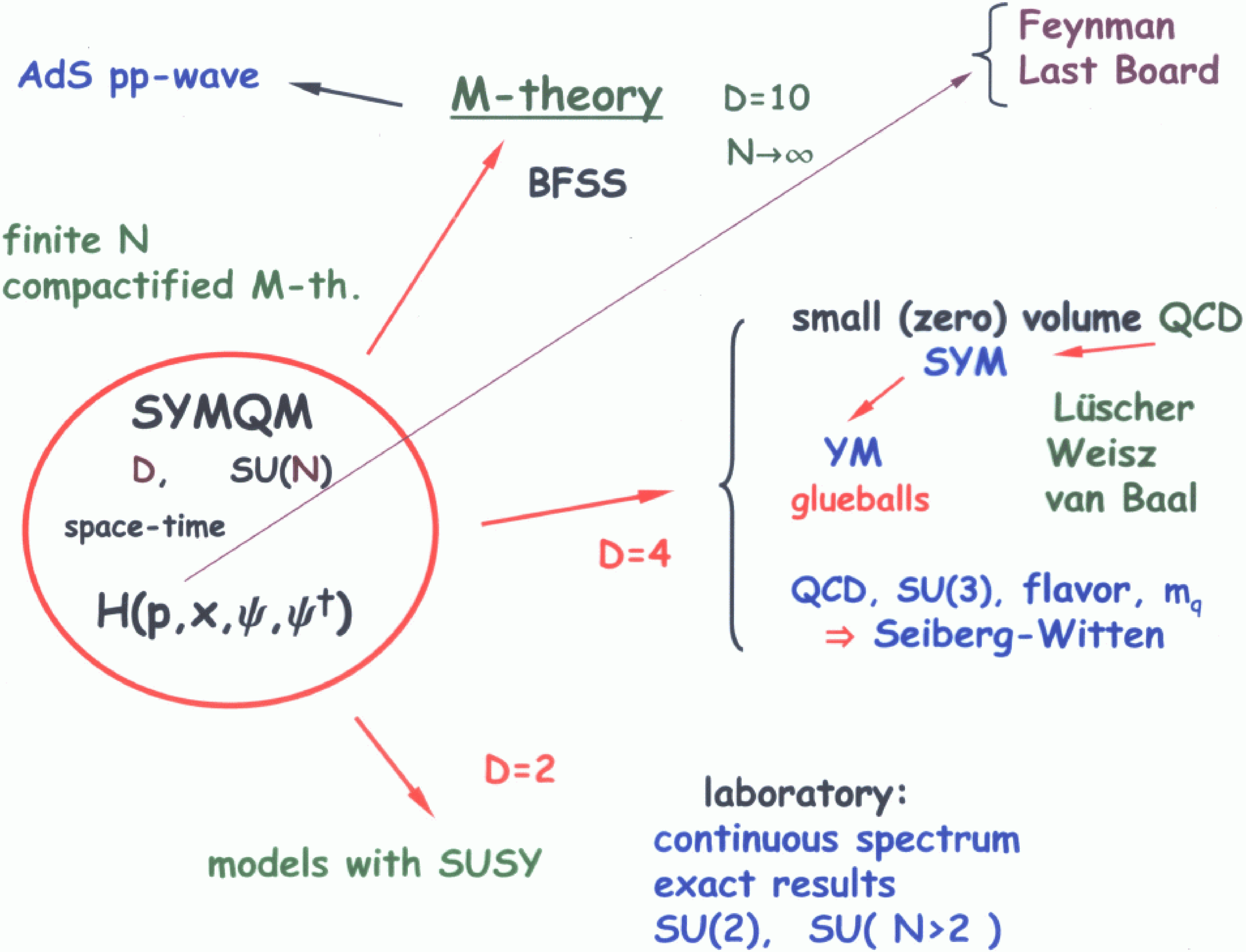

AdS pp-wave
M-theory
D=10
N→∞
BFSS
Feynman
Last Board
finite N
compactified M-th.
SYMQM
D, SU(N)
space-time
H(p,x,ψ,ψ†)
small (zero) volume QCD
SYM
Lüscher
Weisz
van Baal
YM
glueballs
D=4
QCD, SU(3), flavor, m_q
⇒ Seiberg-Witten
D=2
models with SUSY
laboratory:
continuous spectrum
exact results
SU(2), SU(N>2)

Matrix Model of M-theory

- D=10 SUSY YM SU(N)

A_μ (x,t), ψ_α (x,t) — no x dependence → dimensional reduction to D=1 time; space → 1 point; $L_x, L_x, \ldots, L_x \to 0$

A_μ (t) $\mu=1,\ldots,10$
ψ_α (t) $\alpha=1,\ldots,2^{D/2}=32$

- D-2 = 8 bosonic DOF $X^i(t)$
  32/4 = 8 fermionic DOF $\psi_\alpha(t)$ Majorana/Weyl

- Color SU(N) $A_\mu{}^b$, $\psi_\alpha{}^b$, in addjoint repr., $b=1,\ldots,N^2-1$

- $N \to \infty$ ↔ $p_{11} \to \infty$

D=4 SYM $\longrightarrow$ QM

$(A_a^0,\ A_a^i) \longrightarrow A_a^0(t),\ x_a^i(t),\qquad i=1,\ldots,3$

$a=1,\ldots,3,\qquad SU(2)$

fermions: Majorana/Weyl $\quad \psi_a^\alpha \longrightarrow \psi_a^\alpha(t)\ ,\quad \alpha=1,\ldots,4$

$\{\psi_a^\alpha\ ,\ \psi_b^\beta\}=\delta_{ab}\,\delta^{\alpha\beta},\qquad \Rightarrow\qquad f_a^m,\ f_a^{m\dagger},\quad m=1,2\quad$ Majorana

$|\,v\,\rangle = |\,(0,0,0),(0,0,0),(0,0,0),\ (0,0,0),(0,0,0)\,\rangle$

$$H = \frac{1}{2}\vec{p}_i\cdot\vec{p}_i + \frac{g^2}{4}\sum_{i,j}(\vec{x}_i\times\vec{x}_j)^2 + H_F$$

gauge invariant creators

$W_{bmax}(a's)\,|\,v\,\rangle \longrightarrow \{|\text{ basis }\rangle\}$ $\longrightarrow$ sectors

SYMMETRIES

$S_3,\qquad SO(3)\ ,\quad Spin(3)\ ,\quad L^{ik} = x_a^{[i}\ p_a^{k]}$

Quantum Mechanics in a PC

- A basis: $| n \rangle = (a^\dagger)^n | 0 \rangle / \sqrt{n!}$, $n \geq 0$

  elementary state

- a general state

$$| st \rangle = \Sigma_i^{Ns} \alpha_i | n^{(i)} \rangle$$

- in MATHEMATICA

$$st=\{ N_s, \{\alpha_1, \alpha_2, \ldots, \alpha_{Ns} \}, \{n^{(1)} \}, \{n^{(2)} \}, \ldots, \{n^{(Ns)} \}\}$$

- operations on states

$\| st_1 \rangle + \| st_2 \rangle$	$\longleftrightarrow$	add[st1,st2]
$\alpha \| st \rangle$	$\longleftrightarrow$	mult[α,st]
$\langle st_1 \| st_2 \rangle$	$\longleftrightarrow$	sc[st1,st2]

- operators

  $a, a^\dagger, f, f^\dagger$ ...

  H, L, G, Q ...

cutoff $\Sigma_I^{DOF} (a^\dagger a)_I < B_{max}$

$\lim_{B_{max} \to \infty}$

PHYSICS

F	0		1		2		3		
B	N_s	Σ	N_s	Σ	N_s	Σ	N_s	Σ	B - F
0	1	1	-	-	1	1	4	4	0
1	-	1	6	6	9	10	6	10	0
2	6	7	6	12	21	31	42	52	0
3	1	8	36	48	63	94	56	108	0
4	21	29	36	84	111	205	192	300	0
5	6	35	126	210	240	445	240	540	0
6	56	91	126	336	370	815	600	1140	0
7	21	112	336	672	675	1490	720	1860	0
8	126	238	336	1008	960	2450	1500	3360	0
j_{max}	8		17/2		9		19/2		

Table 1: Sizes of the bases generated in each fermionic sector, F. N_s is the number of basis vectors with given number of bosonic quanta, B, while Σ gives the cumulative size up to B. The last column gives the difference between the total number of the bosonic and fermionic states in all seven sectors.

BASES FOR D=4

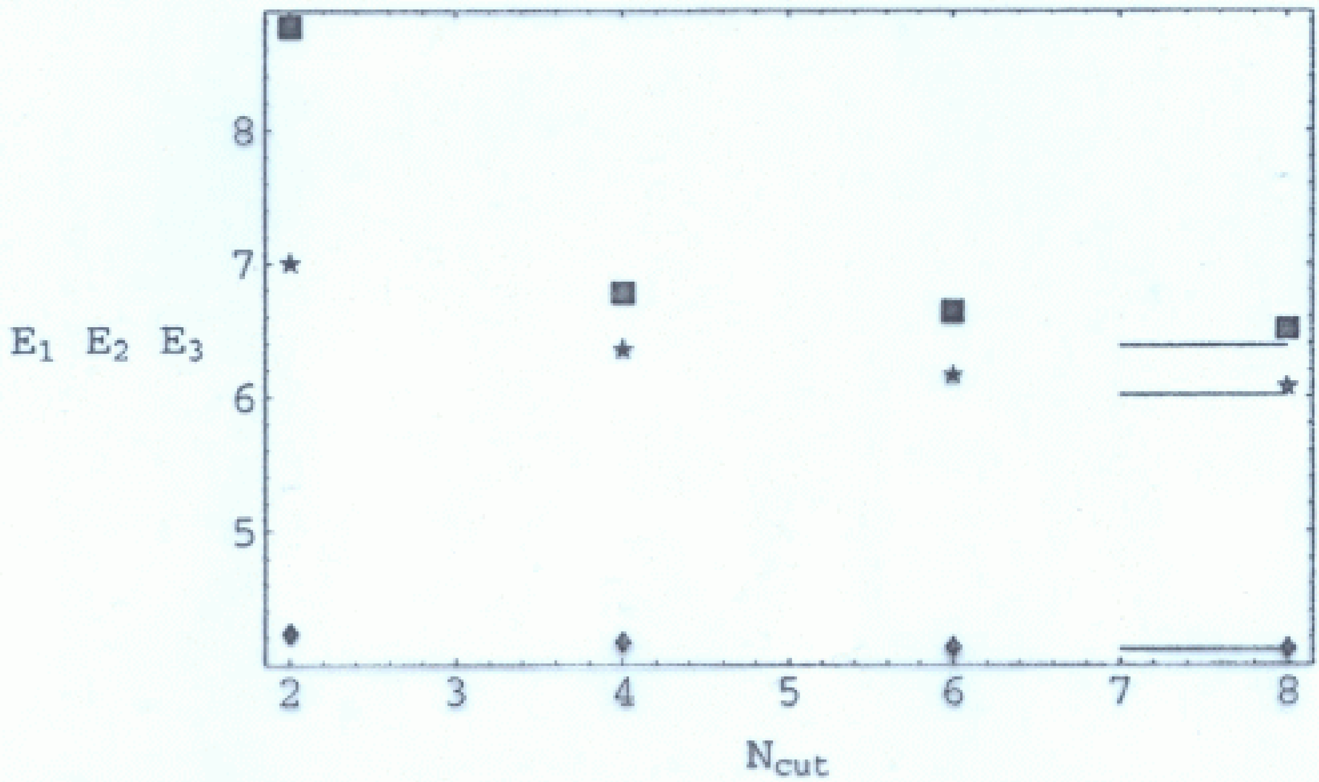


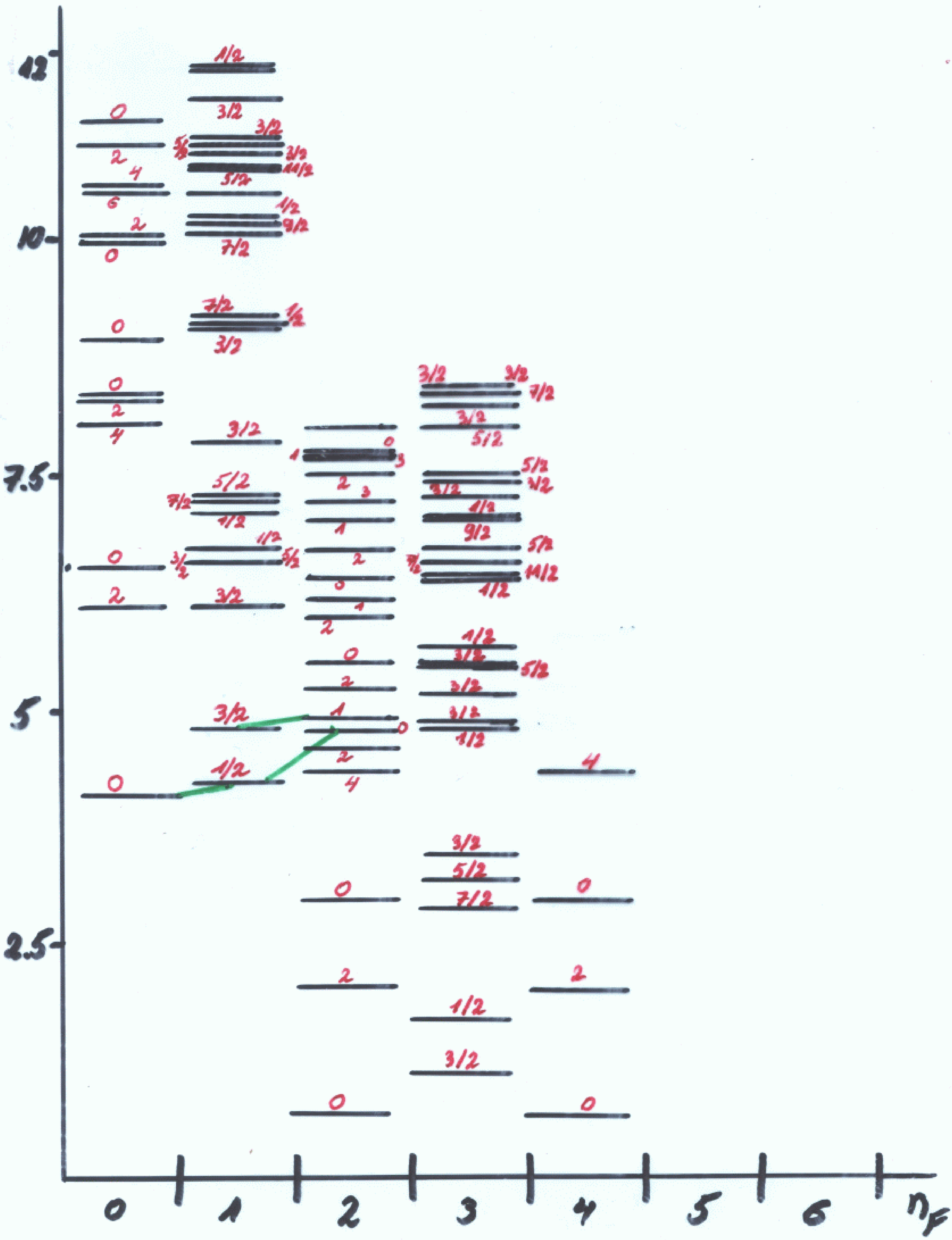
Figure 6: First three energy levels in the $F = 0$ sector of D=4 SYMQM for different cut-offs. Soild lines show the 0-volume results of Ref.[32].

For the localized bound states we conjecture that the rate of convergence with N_{cut} is sensitive to the large x asymptotic of the wave function. We have observed this in a simple anharmonic oscillator and other models. This regularity is also clearly confirmed in the Wess-Zumino and D=2 SYMQM models discussed in previous Sections.

Taking this into account we claim that the low energy spectrum of D=4 SYMQM is discrete in $F = 0, 1, 5, 6$ and continuous in the $F = 2, 3, 4$ sectors. This is an interesting quantification of the result of Ref.[8] which was mentioned earlier. Since the fermionic modes are crucial to provide continuous spectrum, it is natural that it does not show up in the sectors where they do not exist at all, or are largely freezed out by Pauli blocking.

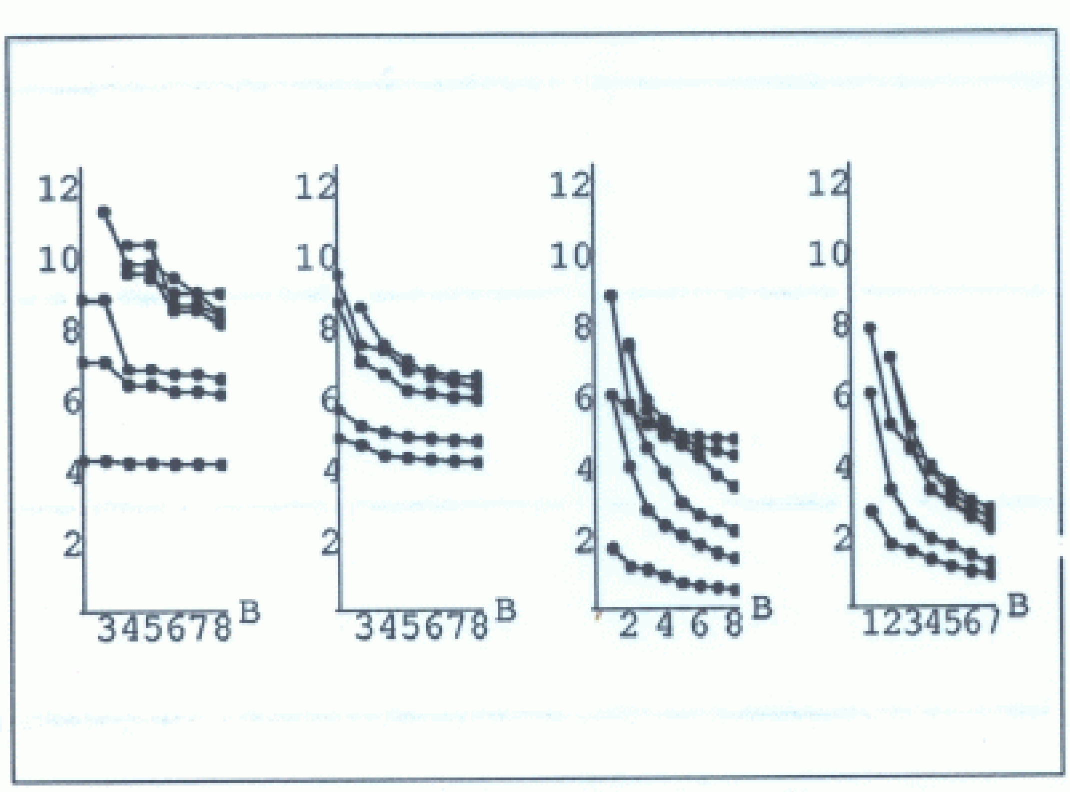
Second result, which is evident from Fig.7, concerns the supersymmetric vacuum in this model. Assuming that indeed the eigen energies have approximately converged in $F = 0, 1$ sectors (none of them to zero) it follows that the SUSY vacuum can not be in empty and filled sectors ($F = 1, 5$ is also ruled out by the angular momentum) [12]. The obvious candidates are the lowest states in $F = 2, 4$ sectors with their energy consistent with zero at

[12] In early attempts, empty and filled sectors were considered as possible candidates for SUSY vacuum.



12
10
7.5
5
2.5
0
1
2
3
4
5
6
n_F

J. Kotański



F=0 F=1 F=2 F=3

Spectrum: continuous vs. discrete

$$V \sim (\vec{x}_m \times \vec{x}_m)^2 \quad \Longrightarrow \quad \text{vacuum valleys}$$

- classically: no localized states
- quantum: transverse oscillations $\Longrightarrow$ a barier

  YM discrete spectrum - glueballs (0-volume)

- SUSY: cancellation of transverse fluctuations

  Lüscher, Nicolai, de Wit

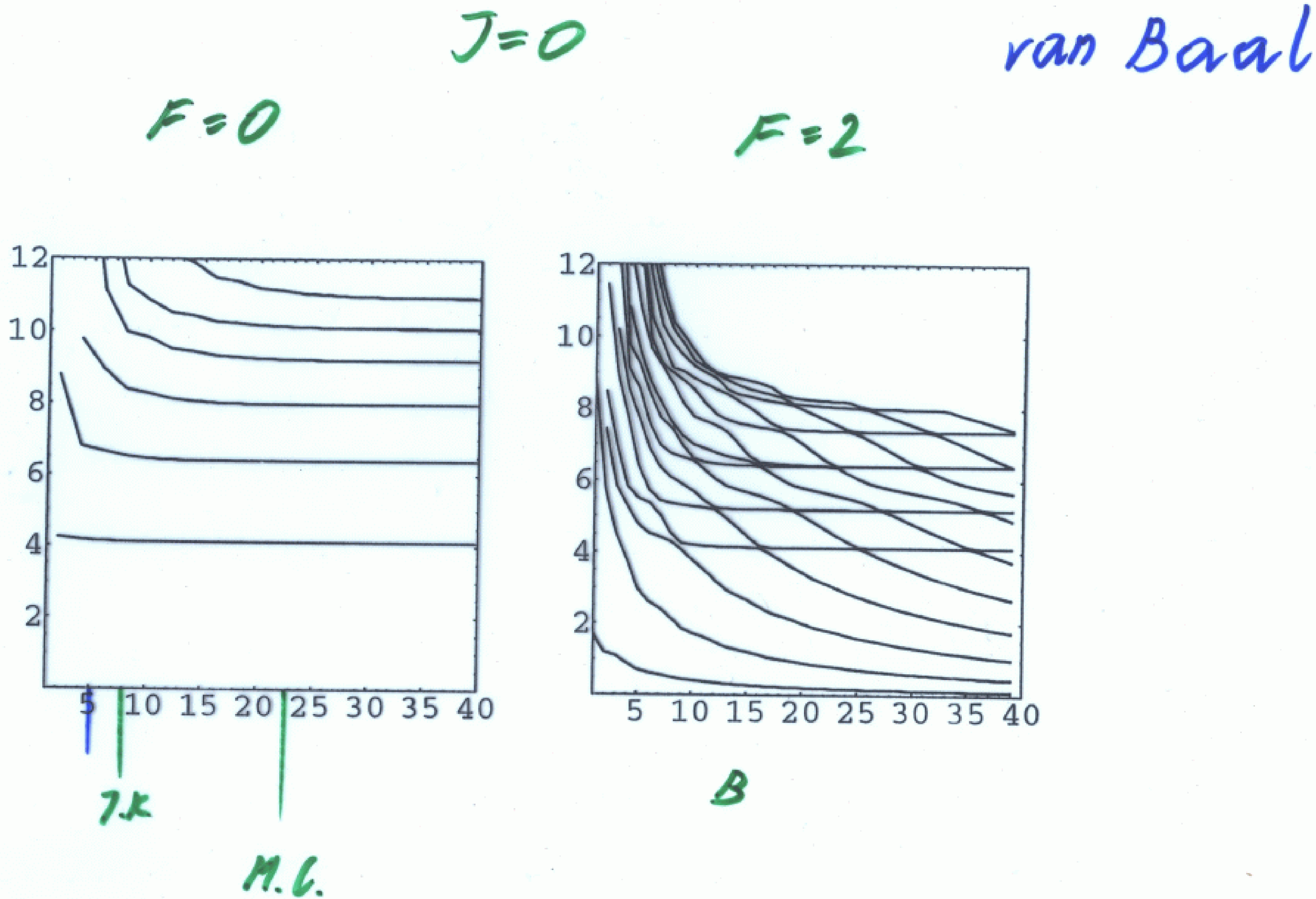
  SYM continuous spectrum

●● D=10 has continuum + threshold bound state !

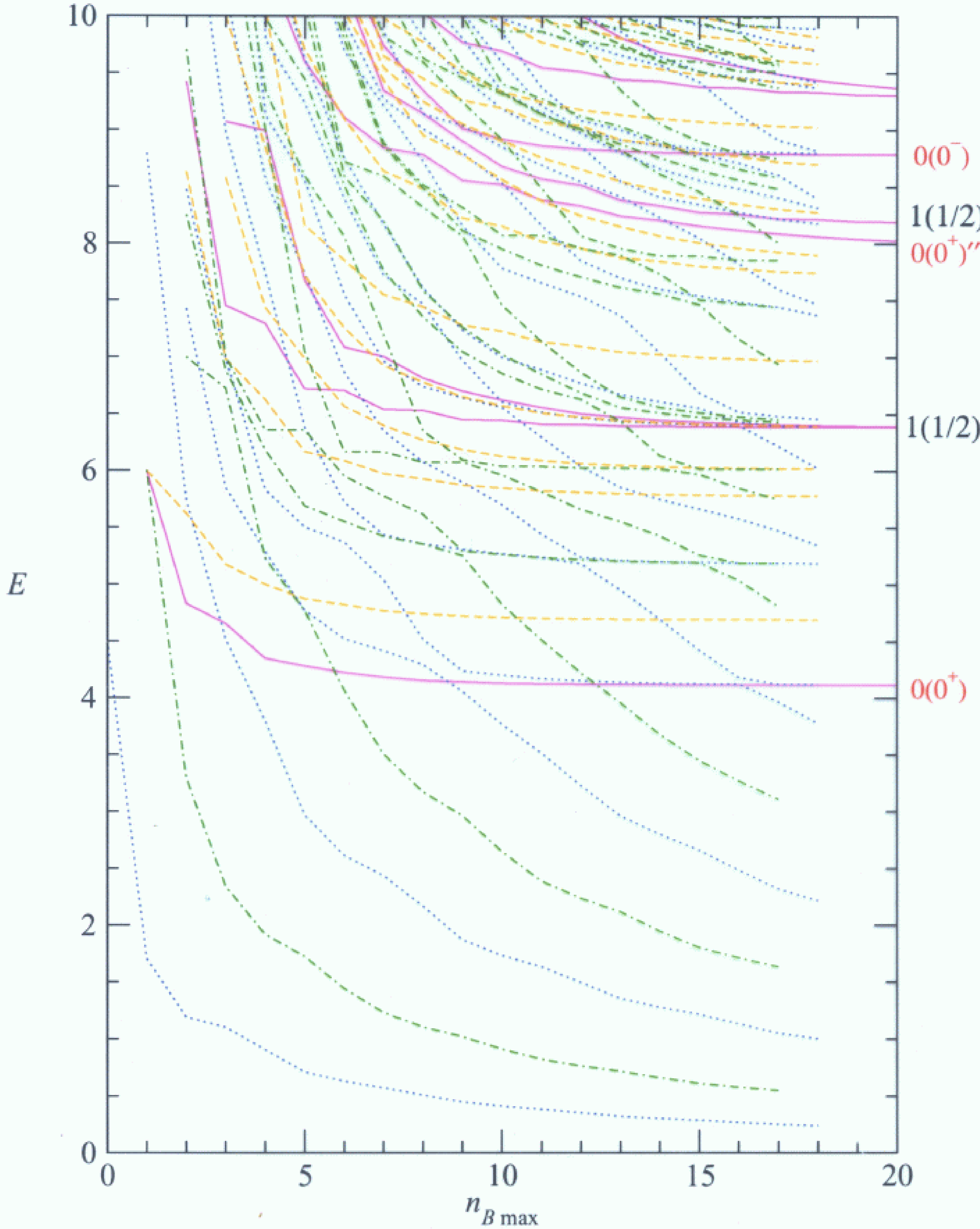
●●● Finite B_{max} ?

expect $\Longrightarrow$ { discrete for F=0,1,5,6; continuous and discrete for F=2,3,4 "fermion rich" }

Claim: continuous and discrete spectra have different B_{max} dependence

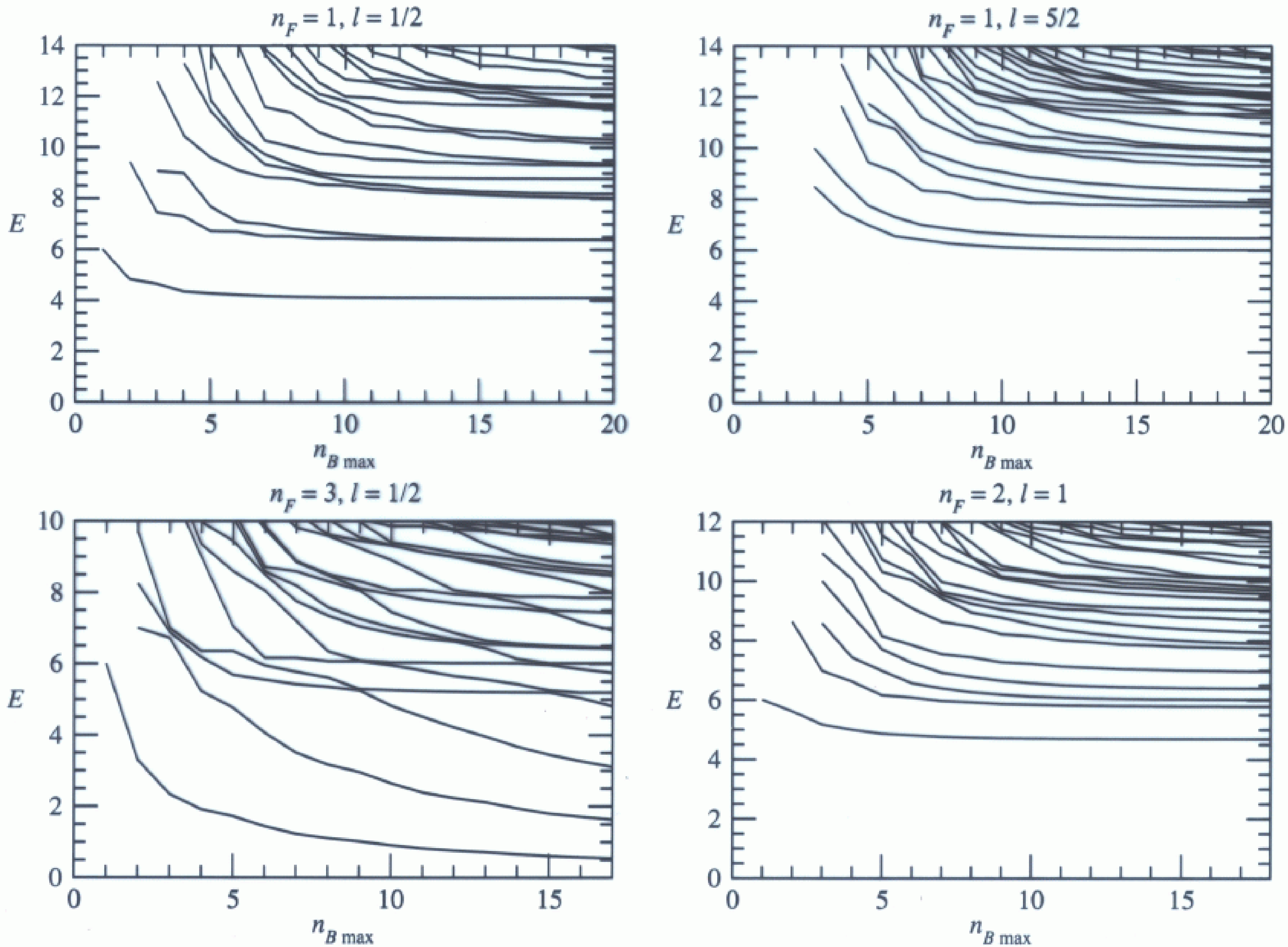

J=0
van Baal
F=0
F=2
12
10
8
6
4
2
5 10 15 20 25 30 35 40
T.K
M.C.
B

M. Campostrini



J / n_B	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{15}{2}$	$\frac{17}{2}$	$\frac{19}{2}$	$\frac{21}{2}$	$\frac{23}{2}$	$\frac{25}{2}$	$\frac{27}{2}$	$\frac{29}{2}$	$\frac{31}{2}$	$\frac{33}{2}$	$\frac{35}{2}$	$\frac{37}{2}$	N_s	Σ
0	0	1																		4	4
1	1	1	0																	6	10
2	3	4	2	1																42	52
3	3	6	3	1	0															56	108
4	7	11	9	6	2	1														192	300
5	8	13	11	8	3	1	0													240	540
6	12	22	21	17	11	6	2	1												600	1140
7	14	24	24	21	13	8	3	1	0											720	1860
8	20	35	38	36	27	19	11	6	2	1										1500	3360
9	21	39	42	40	32	23	13	8	3	1	0									1750	5110
10	29	52	60	61	52	42	29	19	11	6	2	1								3234	8344
11	31	56	65	67	58	48	34	23	13	8	3	1	0							3696	12040
12	39	73	87	92	86	75	58	44	29	19	11	6	2	1						6272	18312
13	42	77	93	100	93	83	66	50	34	23	13	8	3	1	0					7056	25368
14	52	96	119	131	127	118	100	81	60	44	29	19	11	6	2	1				11232	36600
15	54	102	126	139	137	128	109	91	68	50	34	23	13	8	3	1	0			12480	49080
16	66	123	156	176	177	171	153	132	106	83	60	44	29	19	11	6	2	1		18900	67980

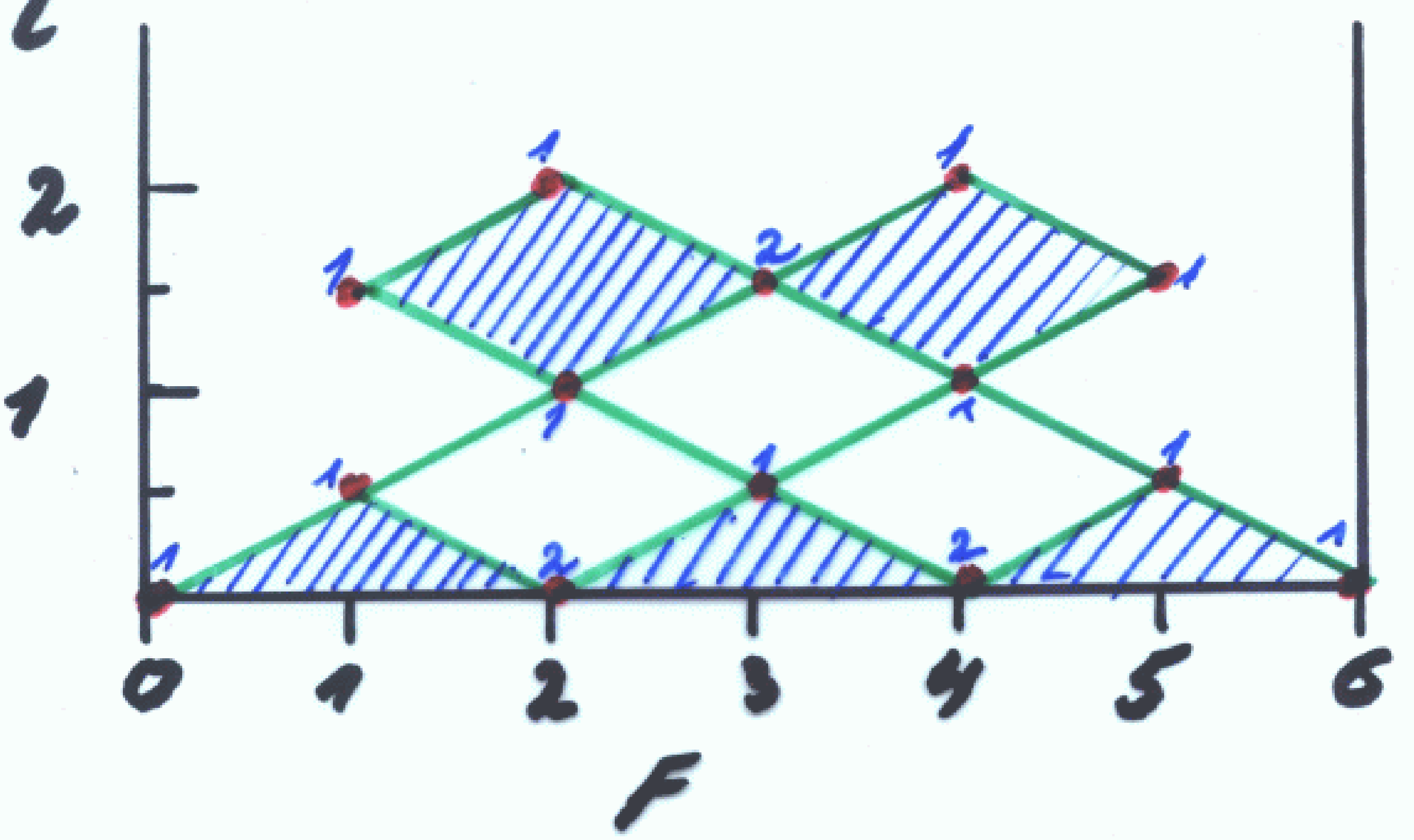
Table 14: Number of SO(3) multiplets with $n_F = 3$ and fixed j and n_B. N_s is the number of basis vectors in all angular momentum channels with given number of bosonic quanta, n_B, while Σ gives the cumulative size up to n_B. tableF3



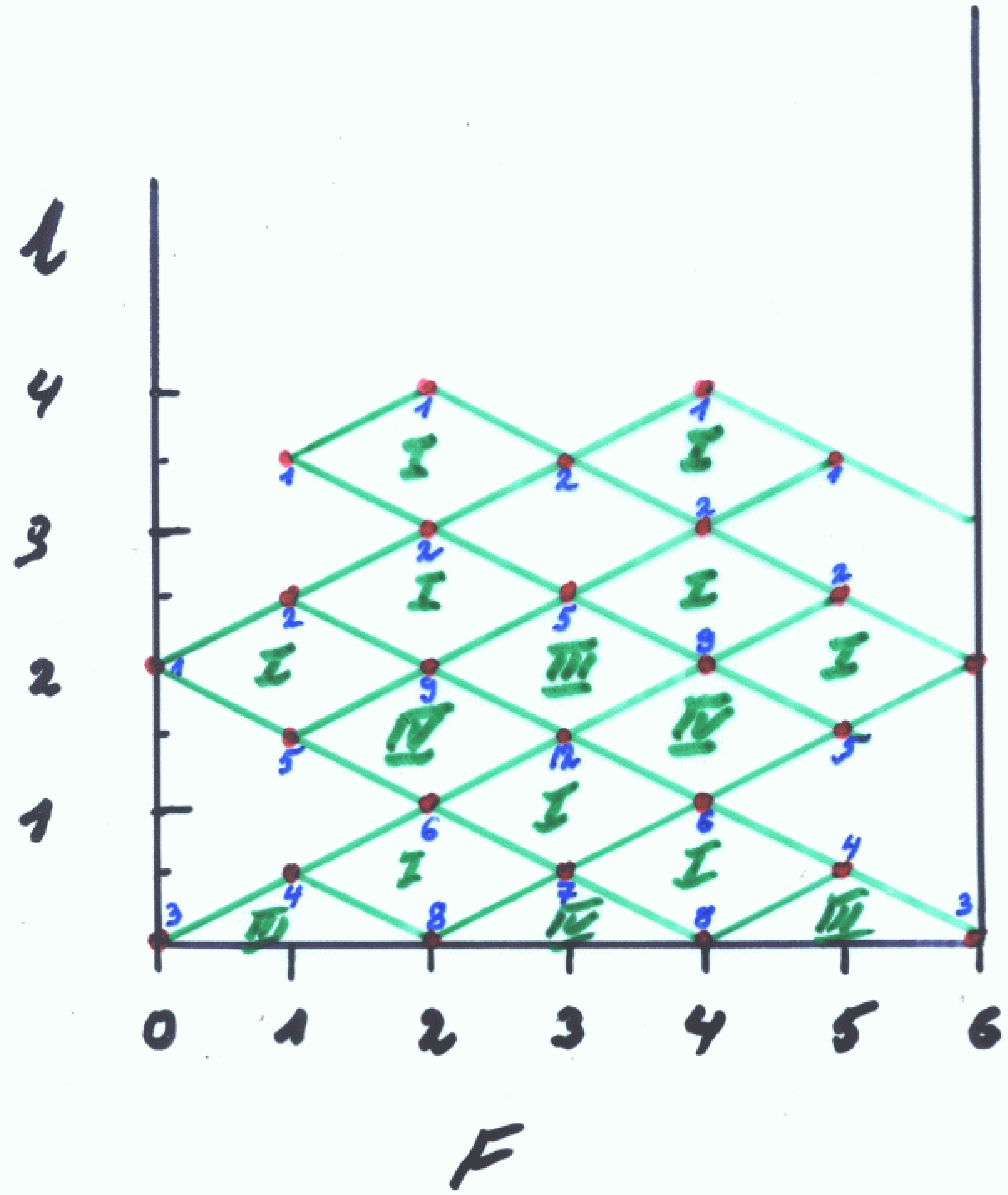
$n_F = 1, l = 1/2$
$n_F = 1, l = 5/2$
$n_F = 3, l = 1/2$
$n_F = 2, l = 1$
E
$n_{B\,\max}$

SUPERMULTIPLETS

$B_{MAX} = 1$



$B_{MAX} = 3$



SUSY generators in Weyl representation

$Q_{1/2}^{\dagger} = \frac{1}{2}(1-i)(\; Q_1 - i\, Q_2 + Q_3 + i\, Q_4)/\sqrt{2},$

$Q_{-1/2}^{\dagger} = \frac{1}{2}(1-i)(i\, Q_1 + Q_2 - i\, Q_3 + Q_4)/\sqrt{2}.$

$Q_{1/2}^{\dagger}$ creates a fermion with $J_z = \frac{1}{2}$

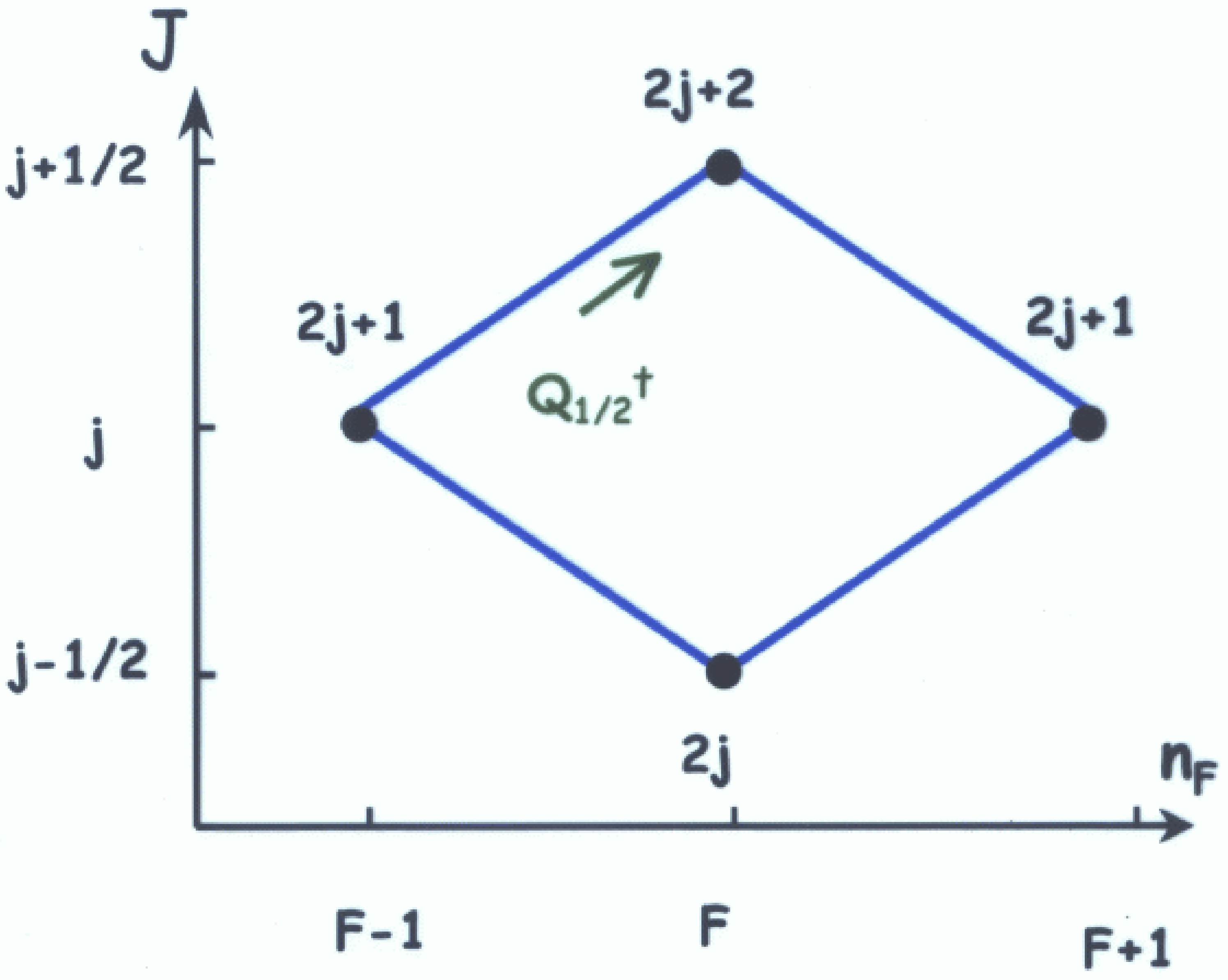
⇒ it carries $J_z = \frac{1}{2}$.

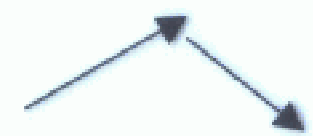
$Q_{1/2} = (Q_{1/2}^{\dagger})^{\dagger}$ annihilates a fermion with $J_z = \frac{1}{2}$

⇒ it carries $J_z = -\frac{1}{2}$ (!).

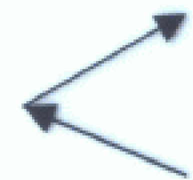
Diamonds Are Forever

$$\{Q_m, Q_n^\dagger\} = 2H\delta_{mn}, \quad \{Q_m^\dagger, Q_n^\dagger\} = 0, \quad \{Q_m, Q_n\} = 0,$$



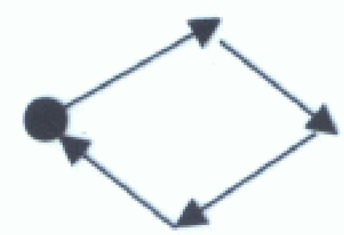


$Q_{-1/2}^\dagger Q_{1/2}^\dagger \propto T_0^0$



$Q_{1/2}^\dagger Q_{-1/2} \propto T_1^1$

$Q_{-1/2} Q_{-1/2}^\dagger |s\rangle = 2E\, |s\rangle$



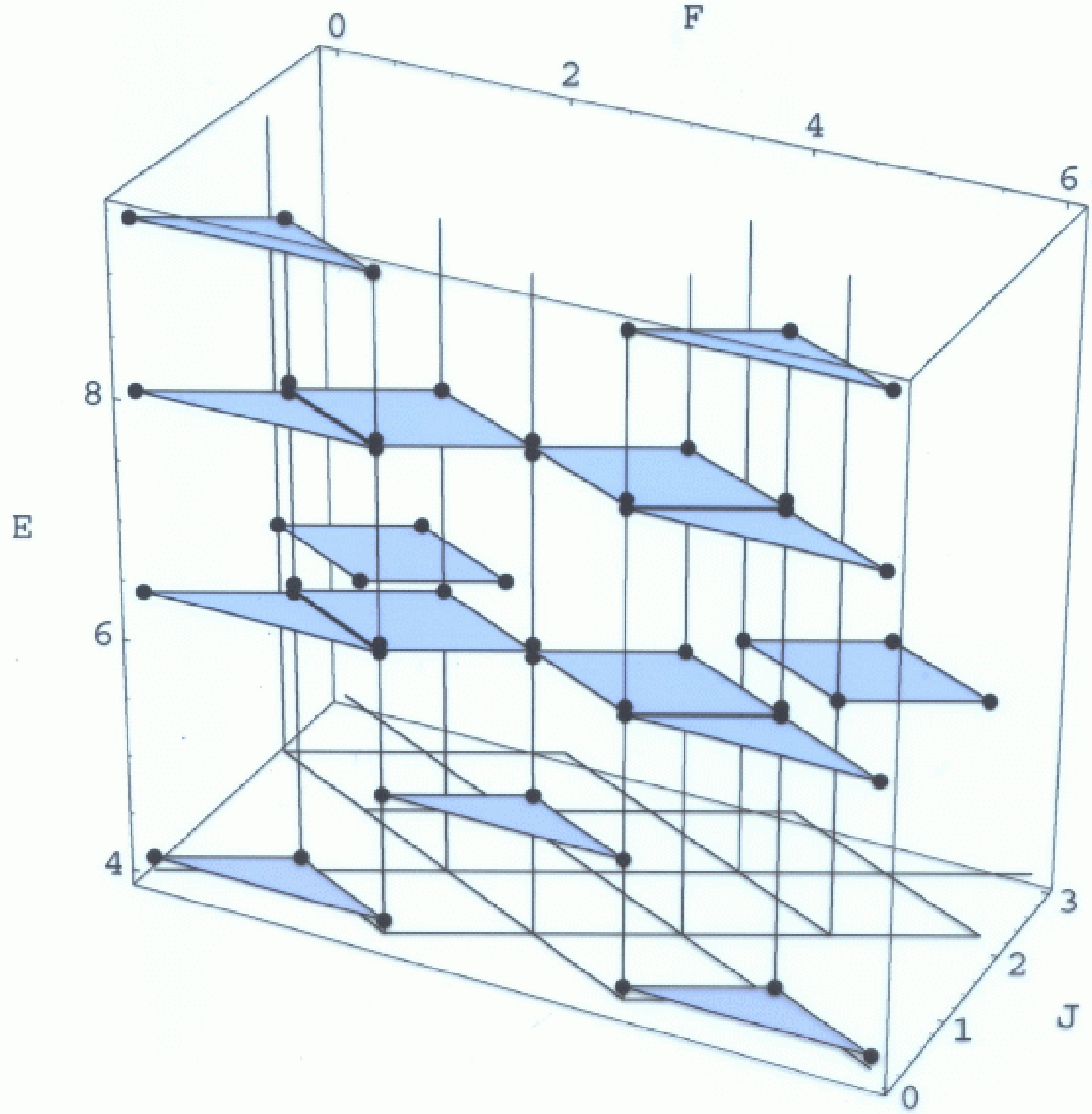
$Q_{-1/2} Q_{1/2} Q_{-1/2}^\dagger Q_{1/2}^\dagger |s\rangle = -4E^2 |s\rangle$

- Any state $|s\rangle$ is annihilated by two of Q's

  $\Longrightarrow$ position (vertex) in a supermultiplet

1000×energies at $n_{B\,\max} = 18$				
$n_{F0}(j_0)$	(n_{F0}, j_0)	$(n_{F0}+1, j_0-\frac{1}{2})$	$(n_{F0}+1, j_0+\frac{1}{2})$	$(n_{F0}+2, j_0)$
$0(0^+)$	4117	—	4117	4119
$0(0^+)'$	6388	—	6388,6401	6394,6459
$0(0^+)''$	7997	—	8063	?
$0(0^-)$	8787	—	8789	8798
$0(2^+)$	6015	6019	6020	6041
$0(2^+)'$	7839	7899	7902	8071
$0(2^-)$	11334	11352	11355	11407
$0(3^+)$	12138	12174	12178	12230
$0(3^+)'$	18140	18395	18663	?
$0(4^+)$	7739	7768	7772	7863
$0(4^-)$	13747	13824	13841	13948
$1(1/2)$	6388,6401	6394,6459	6395	?
$1(1/2)'$	8216	?	?	?
$1(3/2)$	4692	4692	4694	?
$1(3/2)'$	5783	5783	5791	?
$1(3/2)''$	6971	6971	7008	?
$1(5/2)$	6486	6484,6493	6501	?
$1(5/2)'$	7733	7751	7763	?
$1(7/2)$	6591	6593	6611	?
$1(7/2)'$	7515	7518	7553	?
$2(0)$	5188	?	?	5188
$2(1)$	6019	?	?	6019

Spectroscopy of SYMQM.

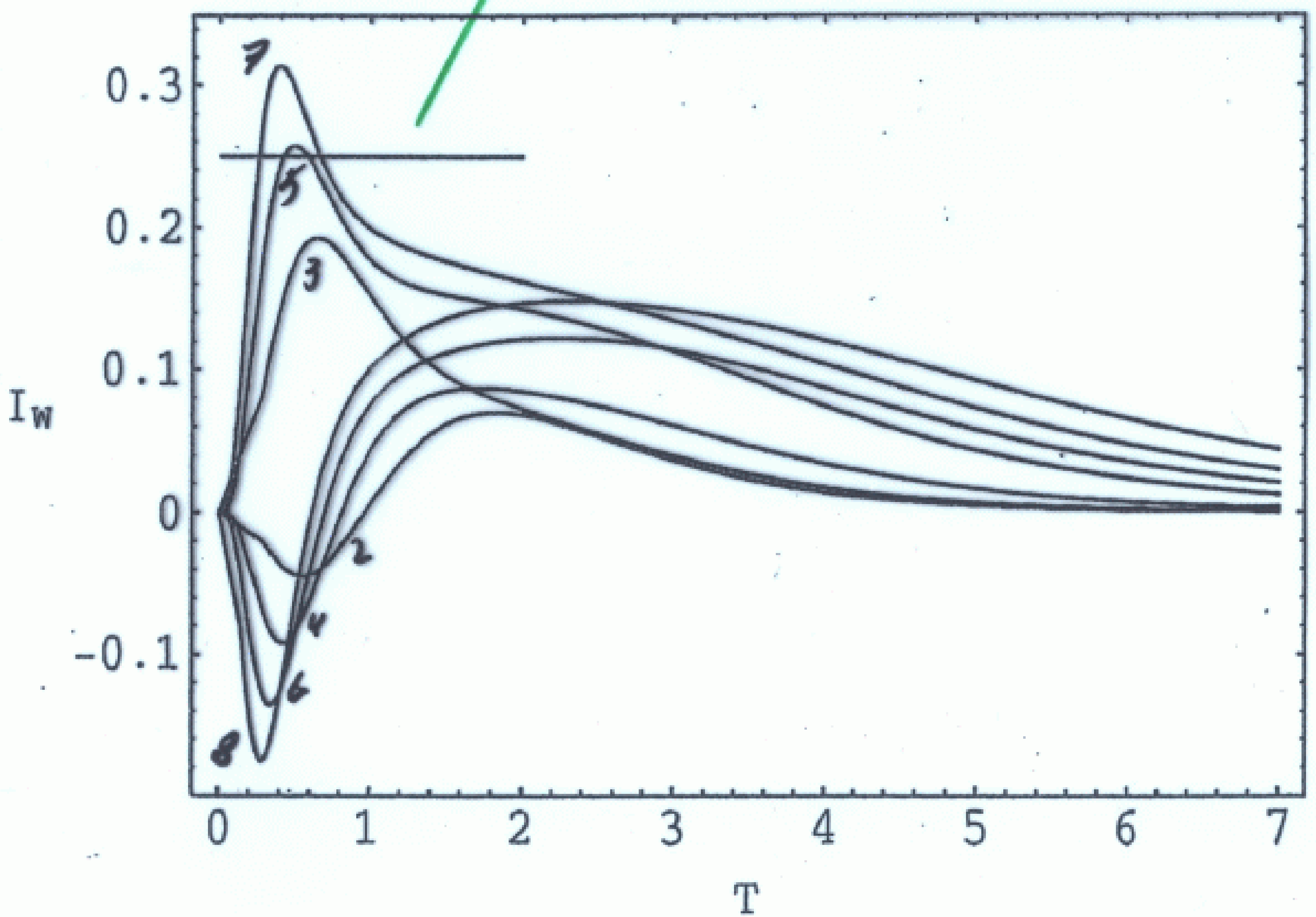

F
0
2
4
6
E
8
6
4
J
0
1
2
3

SYMQM
$N=2$
$D=4$

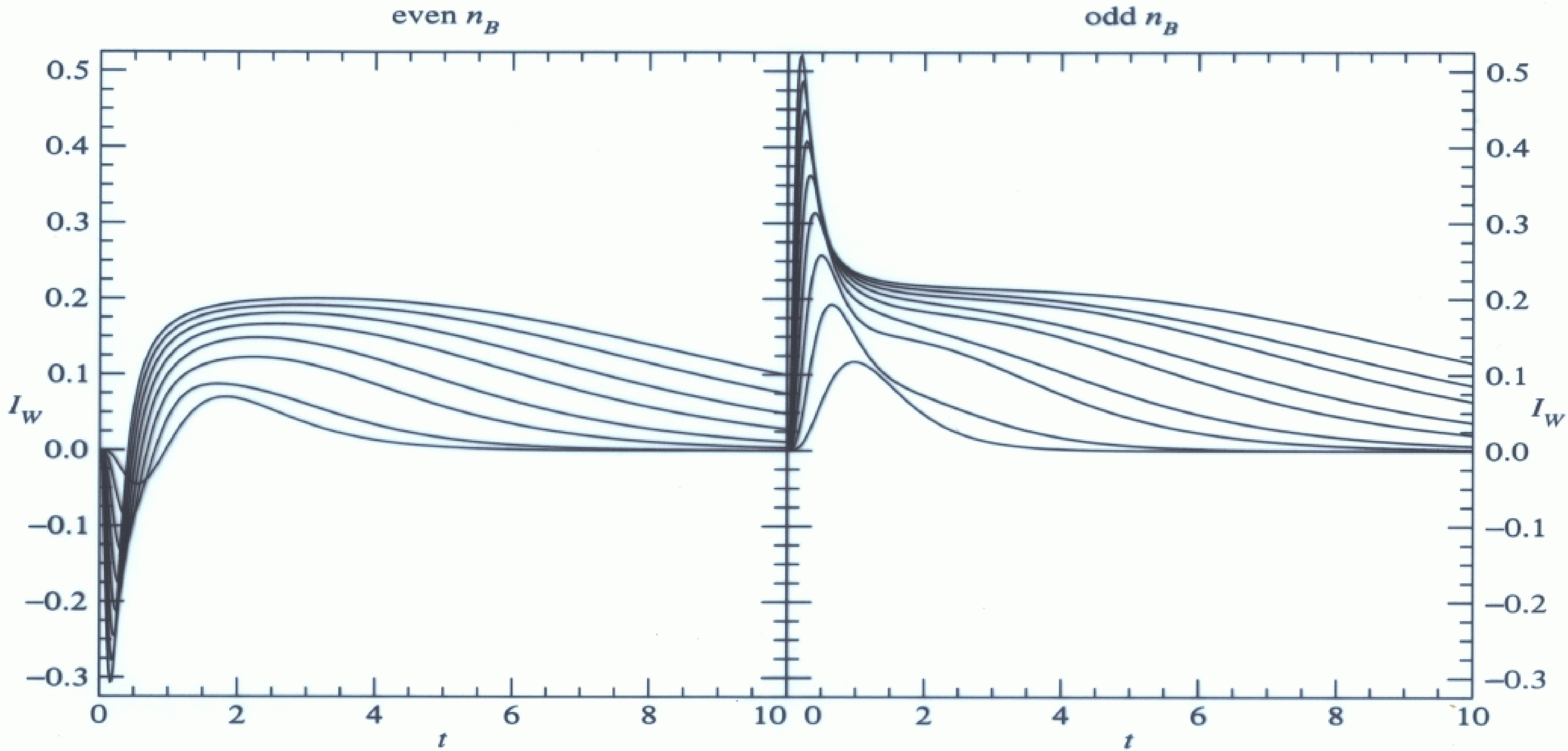
Witten Index

Sethi Stern
Smilga
Green Gutperle
...

$\frac{1}{2^2}$

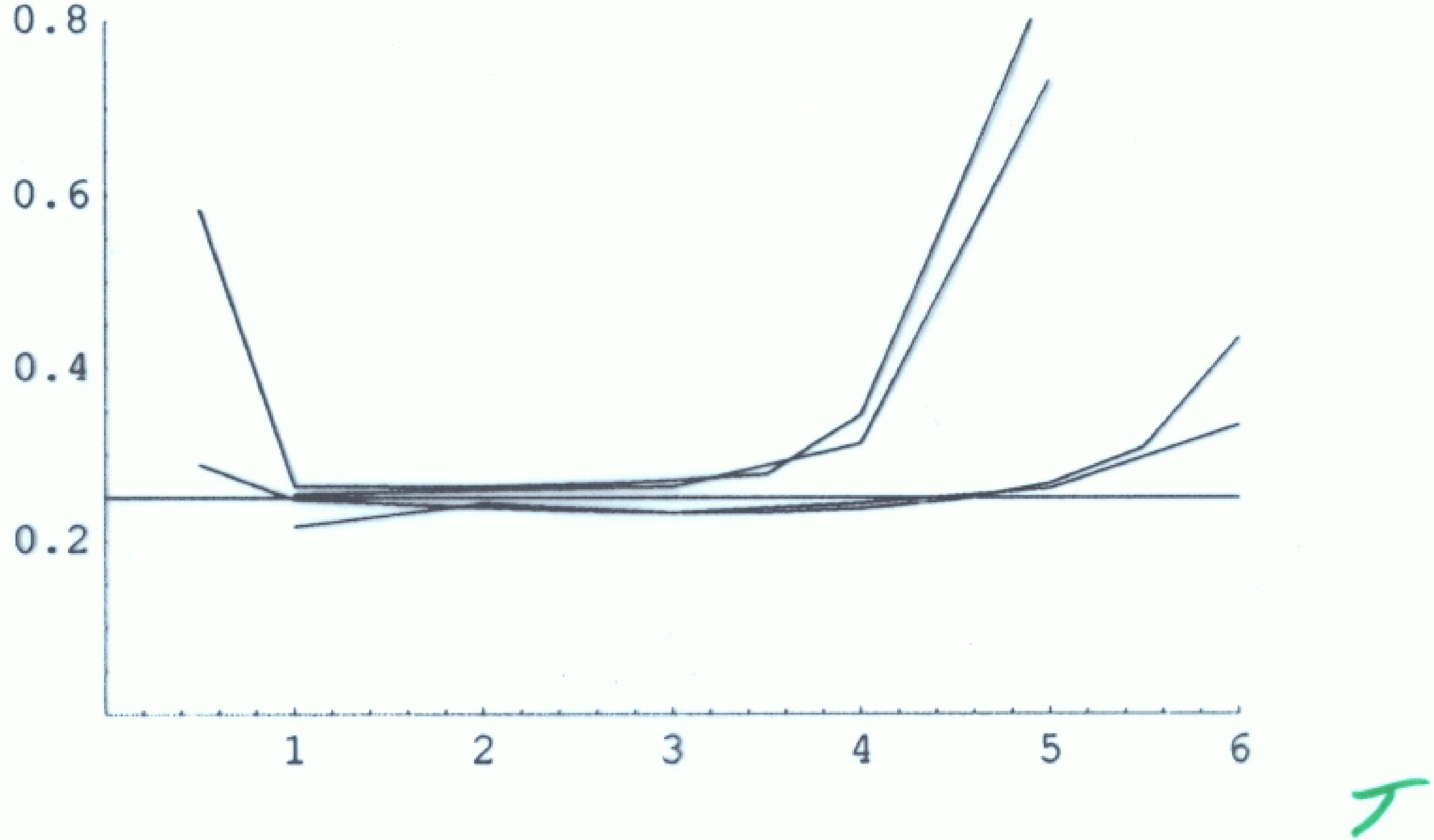


$$I_W = \mathrm{Tr}\,(-)^{\hat{F}}\, e^{-T\hat{H}}$$



even n_B
odd n_B
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
I_W
0
2
4
6
8
10
t

Padé[III](T)



Continuum spectrum - scaling

momentum p $\xrightarrow{\text{cutoff}}$ p_N eigenvalues: z_N^m $\xrightarrow{N,m \text{ large}}$ $\pi\, m / \sqrt{N}$

↗ zeros of the Hermite polynomials

M. Trzetrzelewski

Scaling

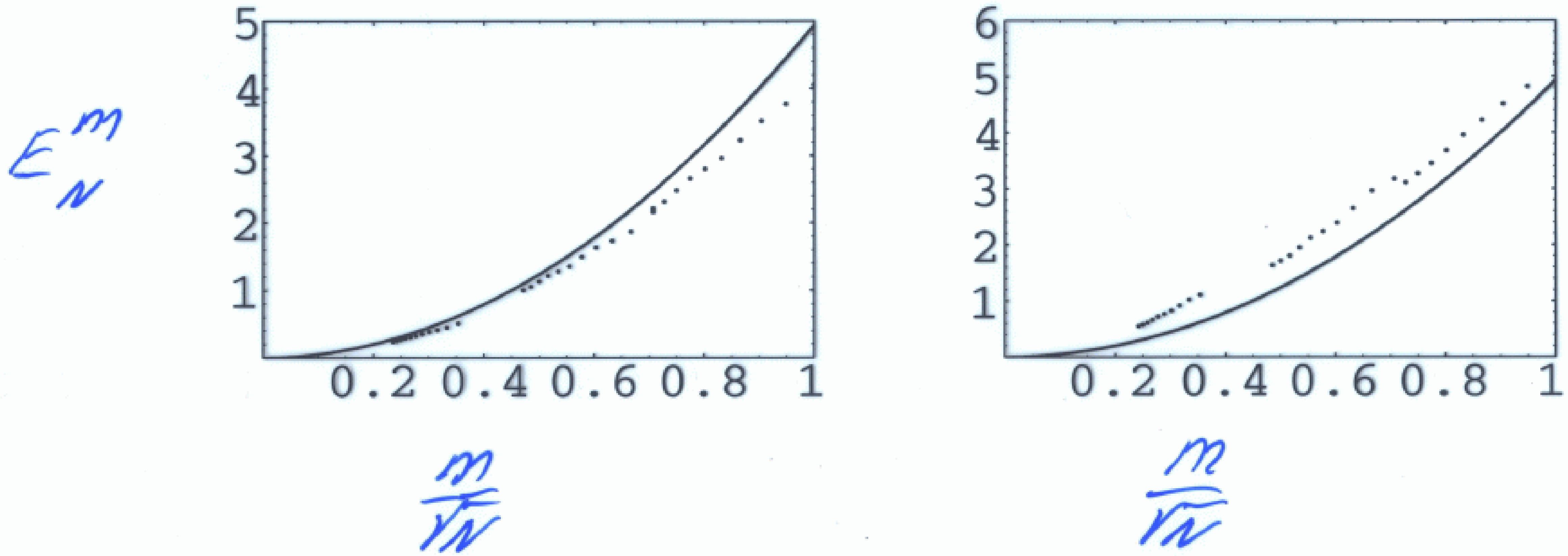
E_N^m - energies of the scattering states with a cutoff

infinte cutoff limit is the <u>scaling</u> limit

$$\lim_{N \to \infty} E_N^{m(N,p)} = E(p), \qquad m = p\sqrt{N}/\pi$$

Example: $H = p^2/2 \xrightarrow{\text{cutoff}} H_N \xrightarrow{\text{diagonalize}} E_N^m \xrightarrow{\text{scaling limit}} p^2/2$

Should work always when one can define p asymptotically


E_N^m
5
4
3
2
1
0.2
0.4
0.6
0.8
1
$\frac{m}{\sqrt{N}}$
6
5
4
3
2
1
0.2
0.4
0.6
0.8
1
$\frac{m}{\sqrt{N}}$

$n_F = 2$ $n_F = 3$

$D=4$

Majorana

$$\begin{pmatrix} f_1 \\ f_2 \\ -f_2^+ \\ f_1^+ \end{pmatrix}$$

$D=10$

Majorana-Weyl

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_8^+ \\ -f_7^+ \\ f_6^+ \\ -f_5^+ \\ -f_4^+ \\ f_3^+ \\ -f_2^+ \\ f_1^+ \end{pmatrix}$$

F	0	2	4	6	8	10	12
B	N_s	N_s	N_s	N_s	N_s	N_s	N_s
0	1	28	406	4 060	17 605	41 392	56 056
1	-	324	9 072	81 648	374 544	908 460	1 205 568
2	45	3 816	89 838				
3	84	23 652					
4	1 035						
5	2 772						
6	16 215						
7							
8	(194 580)						
j_{max}							

F	1	3	5	7	9	11
B	N_s	N_s	N_s	N_s	N_s	N_s
0	-	120	1 512	8 856	29 512	51 520
1	72	2 016	29 232	192 528	626 040	1 126 944
2	288	21 024				
3	3 240					
4	12 960					
5						
6						
7						
8						
j_{max}						

Table 1: Sizes of bases generated in each (B,F) sector for the $D = 10$ system. N_s is the number of basis vectors.

```
degeneracy[eh, Length[eh]]
```

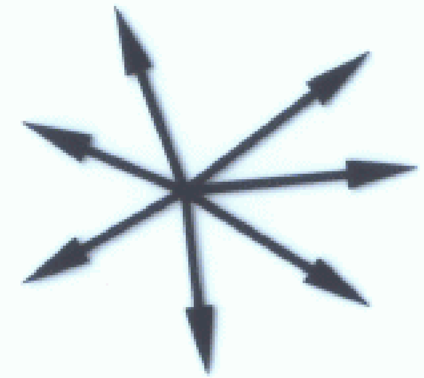
110.993	1
90.1984	1
87.6043	7
86.7109	1
84.5	35
82.0125	21
78.6521	35
76.5	259
75.8341	1
75.1616	7
74.5379	1
67.5	370
65.7341	7
65.3479	35
62.4675	1
61.9875	21
49.7582	1

```
(***     SO7   d              irreps          d        SO(9)

  (0,0,0)     1               scalar          1       (0,0,0,0)
  (1,0,0)     7               vector          9       (1,0,0,0)
  (2,0,0)    27               tensor         44       (2,0,0,0)
  (0,1,0)    21              adjoint         36       (0,1,0,0)
  (0,0,1)     8              fundamental     16       (0,0,0,1)
  (0,0,2)    35              fermion^2      126       (0,0,0,2)           ***)
```

SO(9) structure of the D=10 SYMQM

1) Rotationas in 9 dimensions



- Generators J_{ik} $0<i<k<10$

  9(9-1)/2=36 = no of parameters

- Cartan generators $[J_{ik}, J_{mn}]=0$

  # C.g. = no of disjoint planes = 4 : J_{12} J_{34} J_{56} J_{78}

- Four Casimir operators

  states : $|m_1, m_2, m_3, m_4, j_1, j_2, j_3, j_4 >$

Weight diagrams - four-dimensional
Representations - labelled by four indices
States in a representation - labelled by four indices

etc. , etc.

2) Fock space - states

$$a_b^{\,i}\, a_b^{\,j}\, a_c^{\,k}\, a_c^{\,l}\, a_d^{\,m}\, a_e^{\,n}\, f_e^{\,\alpha}\, f_d^{\,\beta}\, | \, 0 >$$

- SO(9) generators

$$J^{ik} = x_a^{\,i}\, p_a^{\,k} - x_a^{\,k}\, p_a^{\,i} - \tfrac{1}{2}\, \psi_a^{\dagger}\, \Sigma^{ik}\, \psi_a$$

Spin(9)

$$\Sigma^{ik} = -i\, [\, \Gamma^i,\, \Gamma^k\,]/4$$

- Casimir op. (quadratic) $J^2 = \Sigma_{i<k}\, J_{ik}^{\,2}$

••• Surprise $J^2\, |\, 0 > \; = 78\, |\, 0 >$

Empty state $|\, 0_B,\, 0_F >$ is not invariant under rotations !

$\lambda = 78 \Rightarrow |\, 0_B,\, 0_F > \in (1120)$, $d_{1120} = 132132$

WHERE IS SINGLET ???

3) How the generators act on $| 0_B, F >$ states

$$B=0 \Rightarrow J^{ik} = S^{ik} , \qquad S^{ik} \sim \Sigma f_a^I f_a^K$$

!! Four generators are diagonal in our basis

$$S^{23} = \tfrac{1}{2} (F^1 - F^2 + F^3 - F^4 + F^5 - F^6 + F^7 - F^8),$$
$$S^{45} = \tfrac{1}{2} (F^1 + F^2 - F^3 - F^4 + F^5 + F^6 - F^7 - F^8),$$
$$S^{67} = \tfrac{1}{2} (F^1 + F^2 + F^3 + F^4 - F^5 - F^6 - F^7 - F^8),$$
$$S^{89} = \tfrac{1}{2} (F^1 + F^2 + F^3 + F^4 + F^5 + F^6 + F^7 + F^8 - 12).$$

$F^\alpha = f^{\dagger\alpha}{}_b f_b{}^\alpha$ - no. of fermions of type α

- The basis is an eigenbasis of all F^α by construction

In 3 dim

$$S^z = \tfrac{1}{2} (N^+ - N^-)$$

4) Procedure to construct singlets in the B=0 sector

a) Take a full basis for each F.

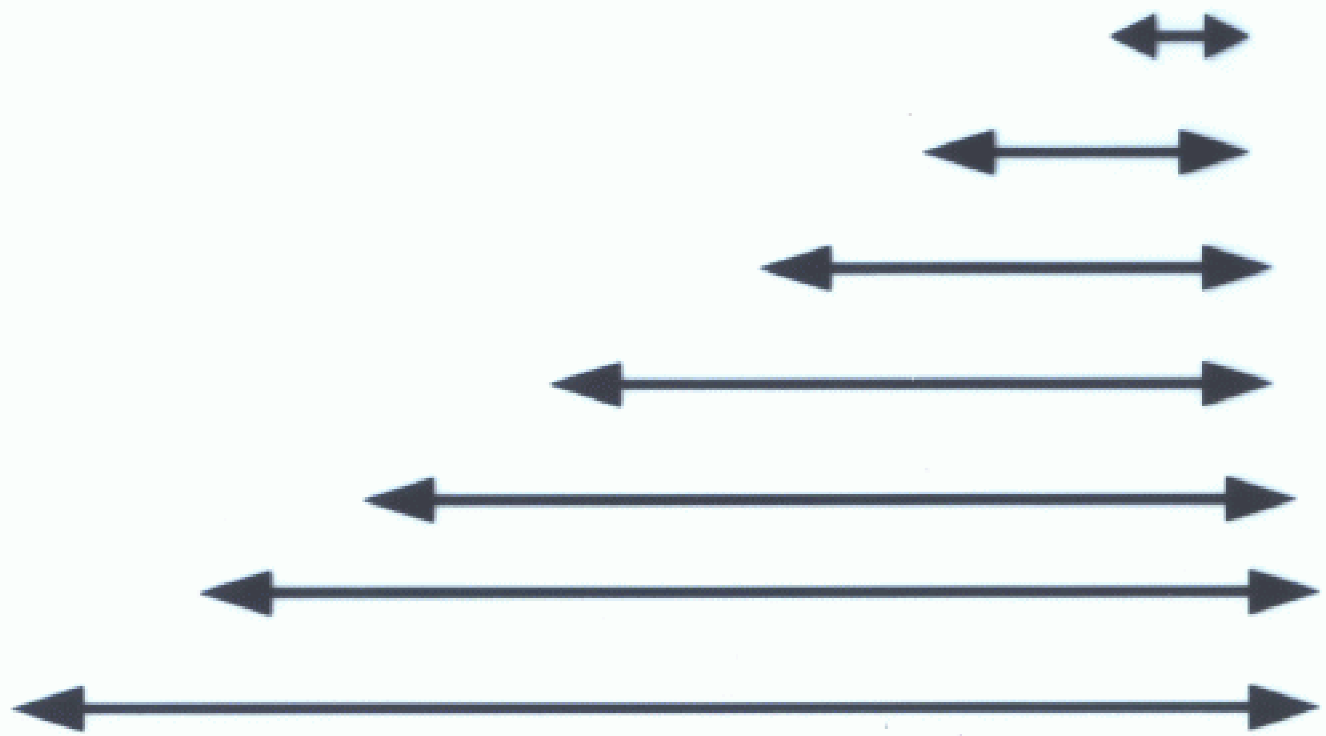
b) Find all states with $M_{23} = M_{45} = M_{67} = M_{89} = 0$,
in general find $\mathcal{N}(M_{23}, M_{45}, M_{67}, M_{89})$.

c) Diagonalize S^2 in the basis $\mathcal{N}(0,0,0,0)$;
eigenstates with $S^2 = 0$ are desired singlets.

d) Can also construct any other representation,
e.g. graviton = **44** - symmetric, traceless tensor
$d = \frac{1}{2} 9(9+1) - 1$ with $J^2 = j(j+7)_{j=2} = 18$

5) A map of the B=0 sector

F 0 2 4 6 8 10 12



irrep	d	J^2
(0000)	1	0
(2000)	44	18
(0010)	84	18
(0110)	1650	36
(1200)	2574	44
(0104)	46332	70
(1120)	132132	78

1 4 28 148 511 1048 1360 $\longleftrightarrow$ $\mathcal{N}(0,0,0,0)$

1 28 406 4060 17605 41392 56056 – tot # | >'s