QCD Dirac operator at nonzero chemical potential: lattice data and matrix model

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work done with Gernot Akemann $\longrightarrow$ PRL 92 (2004) 102002

Outline:

1. Motivation
2. Matrix models and the domains of weak and strong nonhermiticity
3. Comparison with lattice data
4. Recent developments
5. Summary and Outlook

LEILAT04, June 4, 2004

Motivation

- nonhermitian operators appear in many areas of physics:
  - S-matrix theory
  - dissipative quantum maps
  - neural network dynamics
  - disordered systems with imaginary vector potential
  - QCD at nonzero density/chemical potential

  $\longrightarrow$ try to learn something about their (complex) eigenvalue spectra

- QCD at nonzero density is an active research area (RHIC, neutron stars, early universe)
  - rigorous results known at (very) high densities
  - qualitative predictions for QCD phase diagram based on symmetry considerations
  - lattice simulations suffer from sign problem (complex action) some new ideas recently (but $V \to \infty$ limit?)
    - ∗ reweighting along the critical line
    - ∗ combined expansions of weight function and observable
    - ∗ analytic continuation from imaginary μ
    - ∗ factorization method for distribution functions of observables

random matrix theory has been very successful at zero density
(ε-regime of QCD)
$\longrightarrow$ formulate and solve random matrix model for $\mu > 0$

- compare results to lattice data
- improved analytical understanding of QCD at $\mu > 0$
- eigenvalue spectra of nonhermitian operators in general
- algorithmic implications?

NB: first analysis of Dirac spectra at $\mu > 0$ on the lattice:
Markum, Pullirsch, TW, PRL 83 (1999) 484
result: spectral correlations in the bulk show transition
chiral GUE $\to$ weak nonhermiticity $\to$ Ginibre ensemble $\to$ Poisson ensemble

Matrix models at $\mu > 0$

Stephanov (1996): matrix model for Dirac operator

$$Z(\mu) = \int\limits_{\mathbb{C}^{(N+\nu)\times N}} dW \, e^{-\frac{N}{2} \,\mathrm{tr}\, WW^\dagger} \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f & iW + \mu \\ iW^\dagger + \mu & m_f \end{pmatrix}$$

- explains failure of quenched approximation at $\mu > 0$
- difficult to compute eigenvalue correlations on the scale of the mean level spacing (cf. very recent work)

Akemann (2002): complex eigenvalue model (with $1 - \tau^2 \,\widehat{=}\, \mu^2$)

$$Z(\tau) = \int\limits_{\mathbb{C}} \prod_{j=1}^{N} dz_j dz_j^* \; |z_j|^{2\nu+1} \prod_{f=1}^{N_f} (z_j^2 + m_f^2) e^{-\frac{N}{1-\tau^2}\left[|z_j|^2 - \frac{\tau}{2}(z_j^2 + z_j^{*\,2})\right]} \Delta^2(z^2)$$

- complex extension of the chiral Gaussian Unitary Ensemble
- spectral correlations computed for $N_f = 0$ and for $N_f > 0$ "phase-quenched" massless flavors

To what extent are the two models related? $\longrightarrow$ Gernot's talk need to distinguish two different large-N limits:

1. weak nonhermiticity: $V\mu^2 = \mathcal{O}(1)$ or $\lim_{N\to\infty} \lim_{\mu\to 0} N\mu^2 = \alpha^2$

   level spacing $d \propto 1/N$

2. strong nonhermiticity: $N \to \infty$ at fixed μ (or τ)

   level spacing $d \propto 1/\sqrt{N}$

The existence of these two scaling regimes is a prediction for the lattice.

examples of analytical results in Akemann's model (here for $N_f = 0$):

$$\rho_{\rm weak}(\xi) = \frac{\sqrt{\pi\alpha^2}}{\mathrm{erf}(\alpha)}\,|\xi|\,e^{-\frac{(\mathrm{Im}\,\xi)^2}{\alpha^2}}\int_0^1 dt\;e^{-\alpha^2 t}J_\nu(\sqrt{t}\xi)J_\nu(\sqrt{t}\xi^*)$$

with $\xi = \sqrt{2}N\,z$

$$\rho_{\rm strong}(\xi) = \sqrt{2\pi}\,|\xi|\,e^{-|\xi|^2}I_\nu\big(|\xi|^2\big)$$

with $\xi = \sqrt{\frac{N}{1-\tau^2}}\,z$

Comparison with lattice data

- lattice simulations with staggered Dirac operator (Wilson has complex eigenvalues even at $\mu = 0$, Neuberger and DWF too expensive for now)

$$D_{x,y}(U,\mu) = \frac{1}{2} \sum_{\nu=\hat{x},\hat{y},\hat{z}} \left[U_\nu(x)\eta_\nu(x)\delta_{y,x+\nu} - \text{h.c.}\right] + \frac{1}{2}\left[U_{\hat{t}}(x)\eta_{\hat{t}}(x)e^{\mu}\delta_{y,x+\hat{t}} - U_{\hat{t}}^\dagger(y)\eta_{\hat{t}}(y)e^{-\mu}\delta_{y,x-\hat{t}}\right]$$

- need high statistics (20,000 configurations for each parameter set)
- $\beta = 5.0$ strong coupling, but can be justified for this particular purpose:
  - RMT results only describe data below E_c (Thouless energy)
  - E_c is a function of V and β (increases with V, decreases with β)

$\longrightarrow$ for small β, we can get away with small lattices
(for larger β, we simply need larger V)

- $\nu = 0$ because staggered fermions don't have exact zero modes at finite lattice spacing (Smit-Vink 1985)

- gauge fields generated in quenched approximation
$N_f \to 0$ limit is subtle at $\mu > 0$, three possibilities:

  1. take $N_f \to 0$ limit at end of calculation
  2. do a "phase-quenched" calculation and take $N_f \to 0$ limit at the end
  3. set $N_f = 0$ at beginning of calculation

  2. and 3. yield identical results, i.e. $N_f \to 0$ limit of "phase-quenched" theory
  $\longrightarrow$ corresponds to theory with quarks and conjugate quarks
  $\longrightarrow$ should agree with quenched lattice data

  results from 1. (at $N_f \neq 0$) should describe unquenched lattice data (future work)

- no free parameter: scale is set via mean level spacing

(a) weak nonhermiticity:

$$d_{\mathrm{RMT}} = \frac{\pi}{\sqrt{2}N} \quad \longrightarrow \quad \xi = \sqrt{2}N\lambda = \frac{\pi\lambda}{d}$$

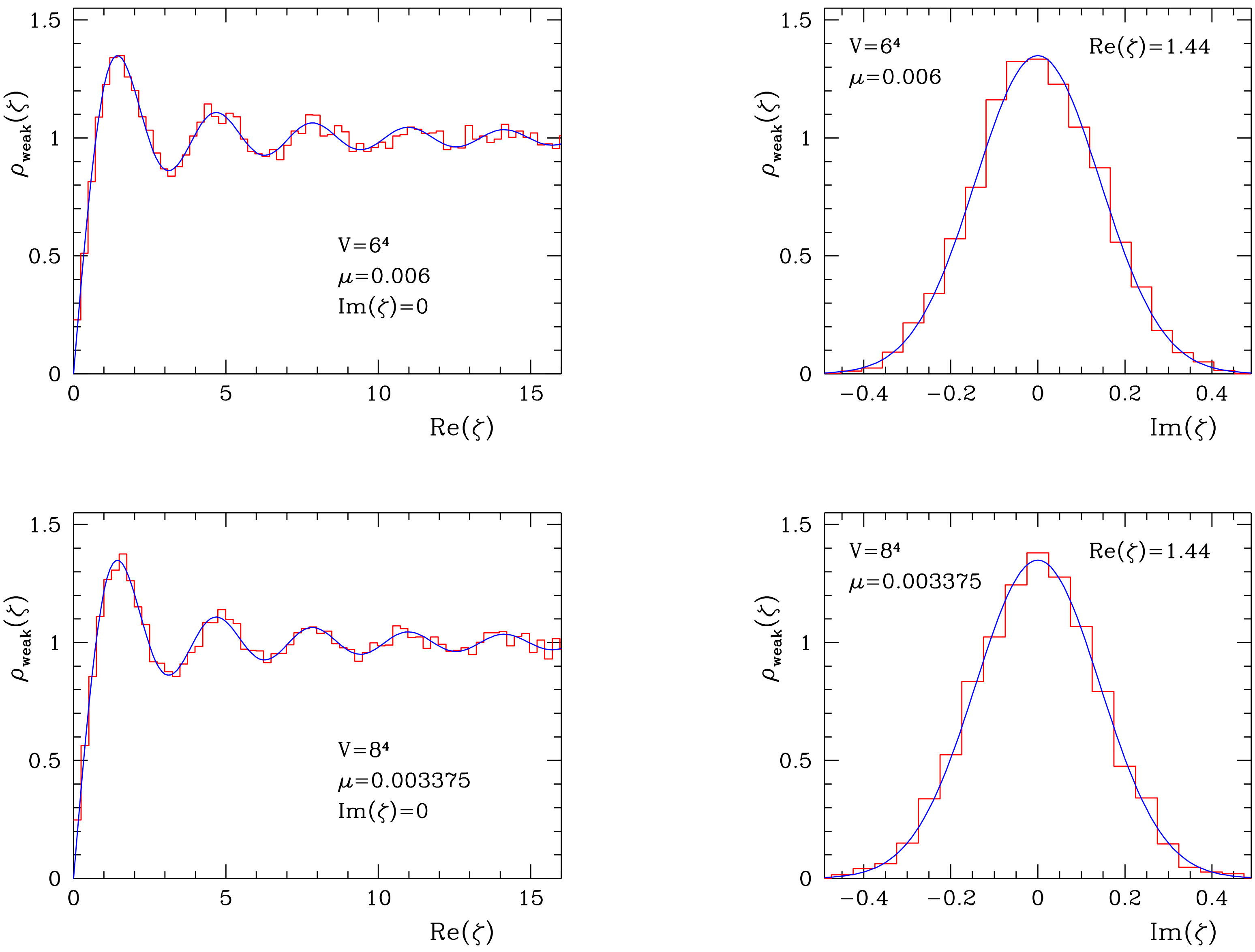
$$\alpha^2 = \mu^2 N = \frac{\pi\mu^2}{\sqrt{2}d}$$

(b) strong nonhermiticity:

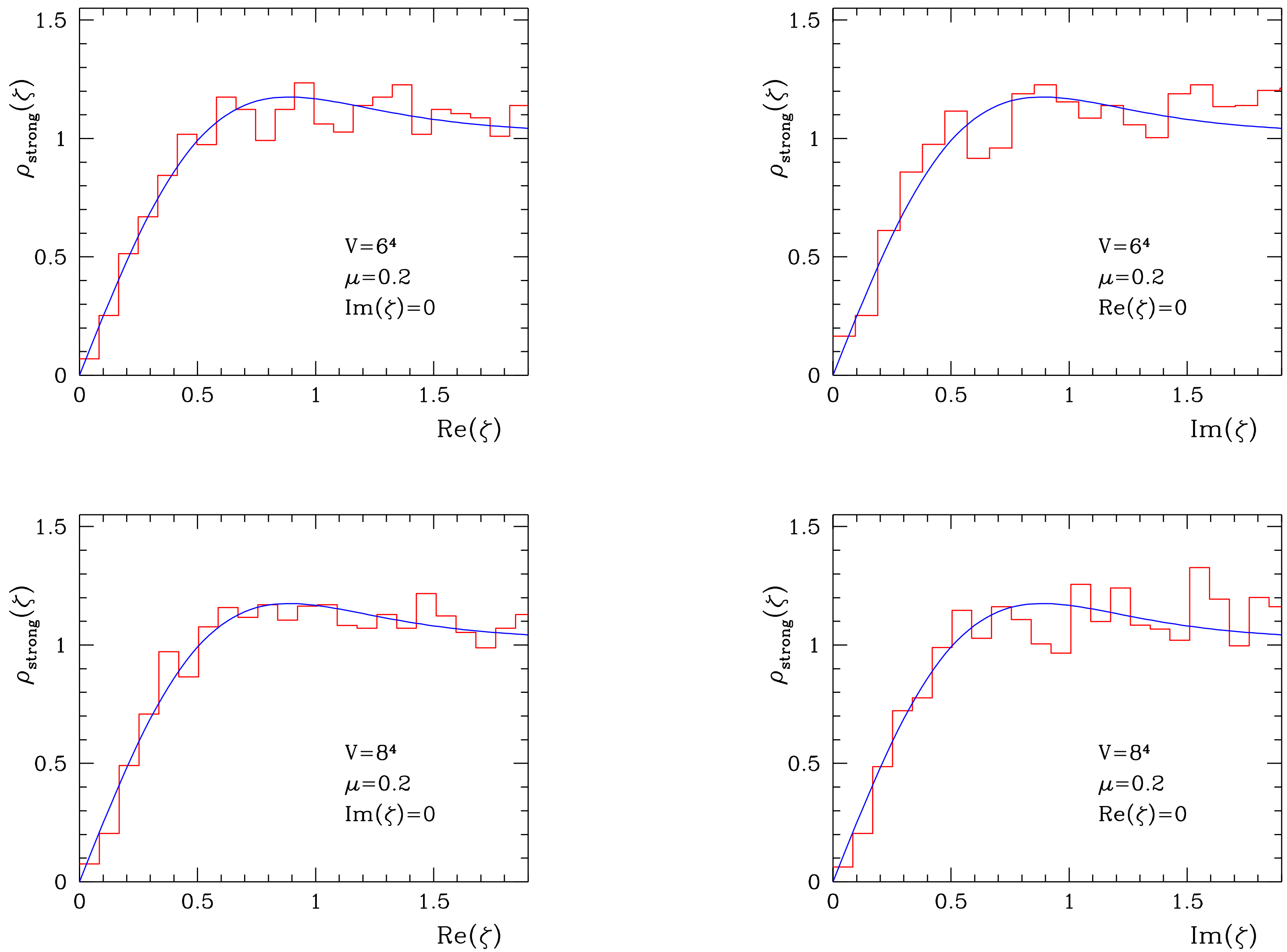
$$\xi = \frac{c\lambda}{d} \text{ with } c = 0.82(5) \text{ independent of } V \text{ and } \mu$$

(NB: We should really be able to compute c analytically.)

Lattice vs RMT: Weak nonhermiticity ($\mu^2 V = \text{const.}$)



Lattice vs RMT: Strong nonhermiticity ($\mu^2 = \text{const.}$)



Recent developments

After our paper was published, analytical results were obtained for Stephanov's model:

Splittorff and Verbaarschot, hep-th/0310271:
replica limit of Toda lattice equation

Osborn, hep-th/0403131:
replace $\mu\gamma_0$ by $\mu \times$ another random matrix:

$$Z(\mu) = \int_{\mathbb{C}^{(N+\nu)\times N}} dW\,dB\, e^{-\frac{N}{2}\,\mathrm{tr}\,(WW^\dagger + BB^\dagger)} \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f & iW + \mu B \\ iW^\dagger + \mu B^\dagger & m_f \end{pmatrix}$$

- identical results in both papers (where they overlap)
- partition functions of Akemann's and Stephanov's model are identical in the weak-nonhermiticity limit
- microscopic spectral correlation functions are (slightly) different!
  $\longrightarrow$ not in the same "universality class"

Example: microscopic spectral density in the weak-nonhermiticity limit ($\nu = 0$ and $\xi = \sqrt{2}N\,z$)

Akemann's model:

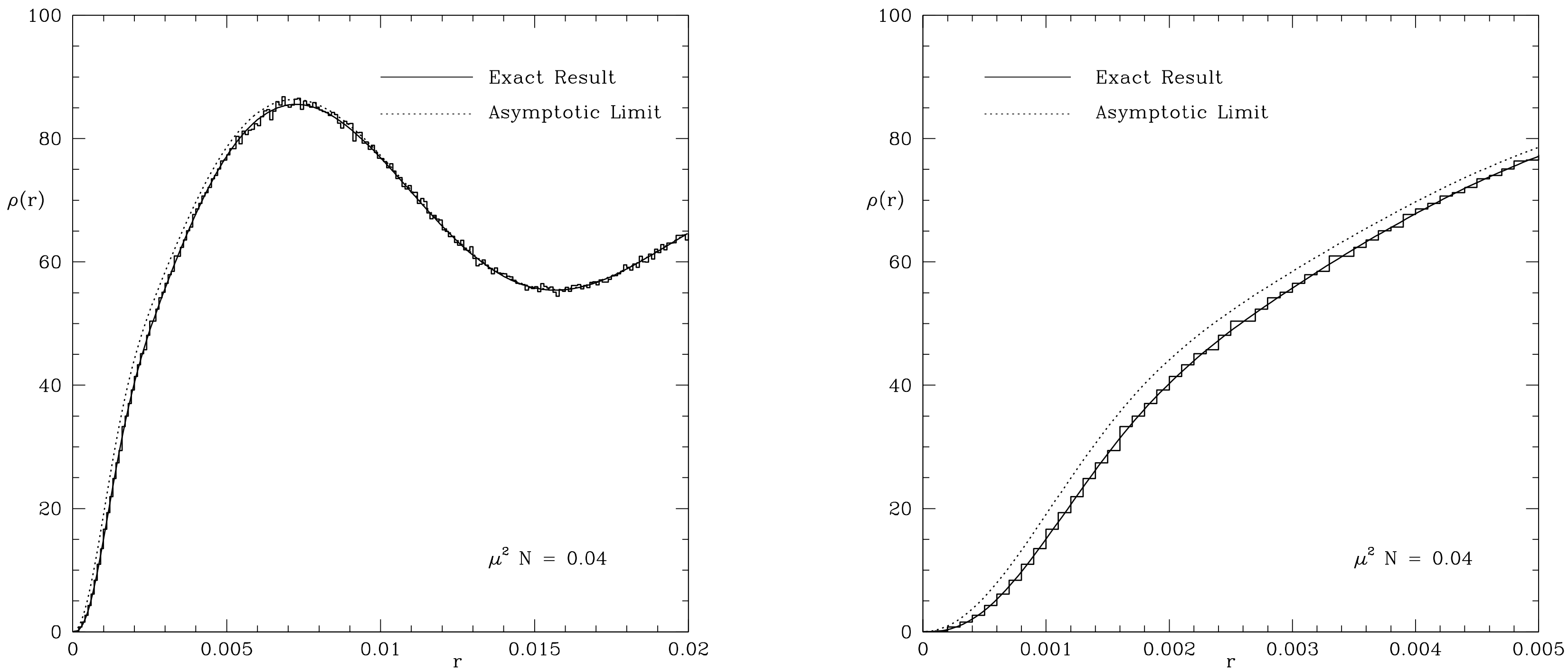
$$\rho_{\rm weak}(\xi) = \frac{\sqrt{\pi\alpha^2}}{\mathsf{erf}(\alpha)}\,|\xi|\; e^{-\frac{(\mathsf{Im}\,\xi)^2}{\alpha^2}} \int_0^1 dt\; e^{-\alpha^2 t} J_0(\sqrt{t}\xi) J_0(\sqrt{t}\xi^*)$$

Stephanov's model:

$$\rho_{\rm weak}(\xi) = \frac{|\xi|^2}{4\mu_s^2}\, e^{\frac{\xi^2+\xi^{*2}}{8\mu_s^2}}\, K_0\Big(\frac{|\xi|^2}{4\mu_s^2}\Big) \int_0^1 dt\; e^{-2\mu_s^2 t} J_0(\sqrt{t}\xi) J_0(\sqrt{t}\xi^*)$$

results agree to lowest order in $\alpha^2 \mathrel{\widehat{=}} 2\mu_s^2$

From Splittorff and Verbaarschot, hep-th/0310271:



histogram: RMT simulation of Stephanov's model
full line: analytical result for Stephanov's model
dotted line: analytical result for Akemann's model

⟶ difference too small to be resolved by our lattice data
Which model corresponds to QCD? Most likely Stephanov's.

Summary and Outlook

- random matrix models are capable of describing complex Dirac spectra at $\mu > 0$
- need to distinguish weak and strong nonhermiticity (different analytical predictions)
- results in weak-nonhermiticity region should be universal
  strong-nonhermiticity results might not be universal
- future work:
  - higher statistics to determine whether lattice data agree with Stephanov's or Akemann's model
  - unquenched simulations (should be doable for $\mu^2 V = \mathcal{O}(1)$)
  - Are the strong-nonhermiticity results universal?
  - What is the Thouless energy? Does it depend on μ?