Bosonic and SUSY color-flavor transformation for $\mathrm{SU}(N_c)$ group

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Outline:

1. Review and motivation
2. Bosonic color-flavor transformation for the special unitary group
3. From bosonic to SUSY
4. Application to lattice QCD
5. Summary and Outlook

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"color-flavor" transformation was first derived by Zirnbauer (1996)
(motivation: disordered systems in condensed matter physics)

$$\int_{\mathsf{U}(N_c)} dU \; \exp\left(\bar\psi^i_{x+\hat\mu,a} U^{ij} \psi^j_{x,a} + \bar\psi^j_{x,b} U^{\dagger ji} \psi^i_{x+\hat\mu,b}\right)$$

color indices coupled,
flavor indices diagonal

$$= \int D\mu_{N_c}(Z,\tilde Z) \; \exp\left(\bar\psi^i_{x+\hat\mu,a} Z_{ab} \psi^i_{x+\hat\mu,b} + \bar\psi^i_{x,b} \tilde Z_{ba} \psi^i_{x,a}\right)$$

flavor indices coupled, color indices diagonal

- ψ, $\bar\psi$: $\mathbb{Z}_2$-graded tensors (bosonic and fermionic components)
- $i, j = 1, \ldots, N_c$: "color" indices
- $a = 1, \ldots, n_+$ and $b = 1, \ldots, n_-$: "flavor" indices
- Z, $\tilde Z$ parameterize $\mathsf{U}(n_+ + n_- | n_+ + n_-)/[\mathsf{U}(n_+|n_+) \times \mathsf{U}(n_-|n_-)]$
- $\tilde Z_{BB} = Z^\dagger_{BB}$, $\tilde Z_{FF} = -Z^\dagger_{FF}$
- $D\mu_{N_c}(Z,\tilde Z) = dZ d\tilde Z \, \mathrm{Sdet}(\mathbb{1} - \tilde Z Z)^{N_c}$

Color-flavor transformation for the special unitary group

B. Schlittgen & TW, Nucl. Phys. B 632 (2002) 155

- consider ψ with only fermionic degrees of freedom
- Z parameterizes the coset space $\mathrm{U}(2N_f)/[\mathrm{U}(N_f)\times\mathrm{U}(N_f)]$

$$\bar\psi \mathrel{\widehat{=}} \bar\psi(x+\hat\mu) \qquad \bar\varphi \mathrel{\widehat{=}} \bar\psi(x)$$
$$\psi \mathrel{\widehat{=}} \psi(x+\hat\mu) \qquad \varphi \mathrel{\widehat{=}} \psi(x)$$

$$\int_{\mathrm{SU}(N_c)} dU \exp\left(\bar\psi^i_a U^{ij}\varphi^j_a + \bar\varphi^i_a U^{\dagger ij}\psi^j_a\right)$$

$$= \tilde C \int_{\mathrm{Gl}(N_f,\mathbb{C})} \frac{dZ dZ^\dagger}{\det(\mathbb{1}+ZZ^\dagger)^{2N_f+N_c}} \exp\left(\bar\psi^i_a Z_{ab}\psi^i_b - \bar\varphi^i_a Z^\dagger_{ab}\varphi^i_b\right) \sum_{Q=0}^{N_f} \chi_Q$$

where

$$\chi_0 = 1\,, \qquad \chi_{Q>0} = \mathcal{C}_Q\left[\det(\mathcal{M})^Q + \det(\mathcal{N})^Q\right] \qquad (Q \text{ baryons})$$

$$\mathcal{M}^{ij} = \bar\psi^i_a(\mathbb{1}+ZZ^\dagger)_{ab}\varphi^j_b\;, \qquad \mathcal{N}^{ij} = \bar\varphi^i_a(\mathbb{1}+Z^\dagger Z)_{ab}\psi^j_b$$

$$\tilde C = \frac{1}{\pi^{N_f^2}} \prod_{n=0}^{N_f-1} \frac{(N_c+N_f+n)!}{(N_c+n)!}\;, \qquad \mathcal{C}_Q = \frac{1}{(Q!)^{N_c}(N_c!)^Q} \prod_{n=0}^{Q-1} \frac{(N_c+n)!(N_f+n)!}{n!(N_c+N_f+n)!}$$

Application to lattice QCD

- color-flavor transformation corresponds to a single link of the lattice
  $\longrightarrow$ apply it to all links

- Dirac indices on ψ and $\bar{\psi}$ have to be treated as flavors as well
  $\longrightarrow \dim(Z) = 4(N_f + N_h) \equiv 4N_q$

- details in Schlittgen and TW, hep-lat/0208044

Induced QCD or How to generate the plaquette action

- so far, the gauge fields are noninteracting
  $\longrightarrow$ need to find a way to include plaquette (Yang-Mills) action

Idea 1 goes back to Kazakov & Migdal (1992)

- couple a number of additional heavy fermions to the gauge field
  integrate out these auxiliary fermions
  expand in powers of 1/mass
- for definiteness, use N_h heavy Wilson fermions:

$$D_{yx} = \delta_{yx} - \kappa \sum_{\mu=\pm 1}^{\pm 4} \delta_{y,x+\hat{\mu}}(r+\gamma_\mu)U_\mu(x)$$

$$\kappa = \frac{1}{2Ma+8r} \to 0 \quad \text{as } M \to \infty$$

now write $D = \mathbb{1} - \kappa A$ and expand in powers of κ

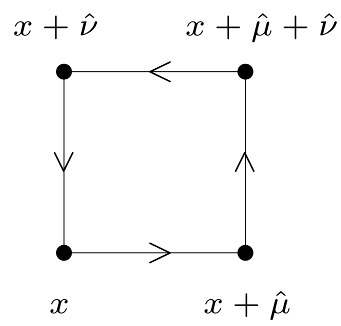
- after integrating out the quark fields we obtain

$$\det^{N_h} D = \exp(N_h \mathrm{Tr} \log D) = \exp(N_h \mathrm{Tr} \log(\mathbb{1} - \kappa A))$$

$$= \exp\left[-N_h \left(\kappa \mathrm{Tr} A + \frac{\kappa^2}{2} \mathrm{Tr} A^2 + \frac{\kappa^3}{3} \mathrm{Tr} A^3 + \frac{\kappa^4}{4} \mathrm{Tr} A^4 + \ldots\right)\right]$$

$\kappa \mathrm{Tr} A \downarrow$ 0 $\quad$ $\mathrm{Tr} A^2 \downarrow$ const. (0 for $r=1$) $\quad$ $\mathrm{Tr} A^3 \downarrow$ 0

- $\mathrm{Tr} A^4 = \mathrm{Tr} A_{xy} A_{yz} A_{zw} A_{wx}$ contains many constant terms as well as terms of the form



$$\sim \mathrm{Tr}\,(r - \gamma_\nu)(r - \gamma_\mu)(r + \gamma_\nu)(r + \gamma_\mu)\, U_\nu^\dagger(x) U_\mu^\dagger(x + \hat{\nu}) U_\nu(x + \hat{\mu}) U_\mu(x)$$

$$= -\,4(1 + 2r^2 - r^4)\,\mathrm{Tr}\, U_p \qquad p = (x; \mu\nu)$$

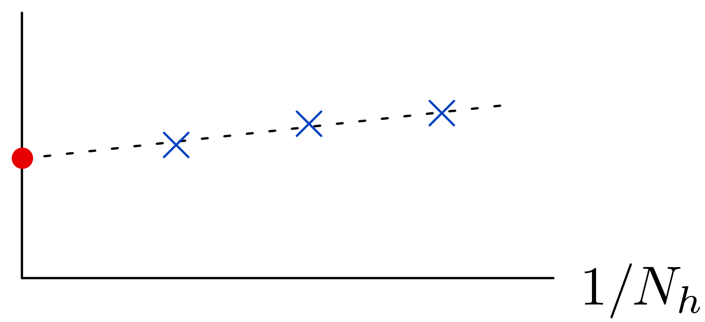
(correct sign!)

- collecting all terms (for $r = 1$) yields $16 N_h \kappa^4 \sum_p \mathrm{Re}\,\mathrm{Tr} U_p$, which is just the familiar plaquette action
  $\longrightarrow$ we can identify $\dfrac{1}{g^2} = 8 N_h \kappa^4$

- to kill the higher-order terms in the exponent ($\sim N_h \kappa^6$, $\sim N_h \kappa^8$, etc.), let $\kappa \to 0$ and $N_h \to \infty$ such that $N_h \kappa^4 =$ const.

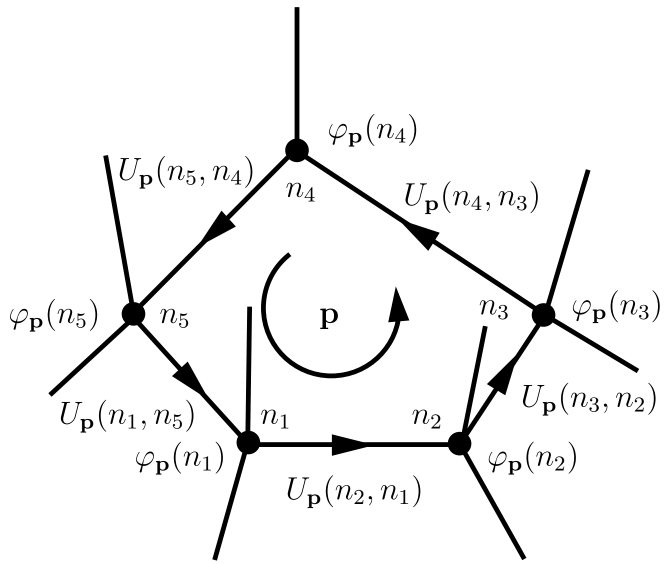
- how large N_h has to be in practice must be determined numerically (one-loop calculation by A.+P. Hasenfratz: $N_h > 11 N_c/2$)

- "safe" procedure:
  - fix a
  - fix $8 N_h \kappa^4 = \dfrac{1}{g^2}$, let $N_h \to \infty$
  - let $a \to 0$



Idea 2 uses a few (N_c) auxiliary bosons instead of (infinitely) many fermions
Budczies and Zirnbauer, math-ph/0305058

- auxiliary boson fields don't propagate all over the lattice but hop along the boundary of a plaquette



- action:

$$S_b(\varphi, \bar{\varphi}, U) = \sum_{\pm p} \sum_{j=1}^{L_p} \Big[m_b \bar{\varphi}_p(n_j) \varphi_p(n_j) - \bar{\varphi}_p(n_{j+1}) U_p(n_{j+1}, n_j) \varphi_p(n_j) \Big]$$

- after integration over auxiliary bosons

$$Z_{\mathsf{aux}} = \int dU \prod_p |\det(m_b - U(\partial p))|^{-2N_b}$$

- Recall how continuum limit is taken:

  weight function $w_t(U)$ such that $\lim_{t\to t_c} w_t(U) = \delta(I - U)$

  (usual plaquette action: $t = \beta$ and $\beta_c = \infty$,
  continuum limit is Yang-Mills theory)

- Using the Peter-Weyl theorem, they show that

$$w_\alpha(U) = \frac{|\det(1 - \alpha U)|^{-2N_b}}{\int_G dU |\det(1 - \alpha U)|^{-2N_b}}$$

  goes to $\delta(I - U)$ as $\alpha \to 1$ for any $N_b \geq N_c$ (thus $m_c = 1$).

  I.e. the boson-induced gauge theory admits a continuum limit

- What is this continuum limit?

They analyze the $d = 1 + 1$ case and find

- for $N_b > N_c$: Yang-Mills theory (by matching to a combinatorial result of Witten 1991)
- for $N_b = N_c$: a different, exotic theory (Cauchy distribution)

For $d \geq 4$, the continuum limit is Yang-Mills theory for all $N_b \geq N_c$.

Note: This approach requires a SUSY version of the color-flavor transformation.

bosonic CFT for the special unitary group

$$\int_{\mathrm{SU(N_c)}} dU \exp\left(\bar{\psi}^i_a U^{ij} \psi^j_a + \bar{\varphi}^i_a U^{\dagger ij} \varphi^j_a\right)$$

$$= \sum_{Q\geq 0} \frac{\int_{|Z|<1} D(Z,Z^\dagger) \exp(\bar{\psi}^i_a Z^\dagger_{ab} \varphi^i_b + \bar{\varphi}^i_a Z_{ab} \psi^i_b)\ \mathcal{C}_Q}{N_Q \int_{|Z|<1} D(Z,Z^\dagger)\ \mathrm{Det}^Q_{N_c}(1 - Z^\dagger Z)}$$

$$= C_0 \sum_{Q=0}^{\infty} \chi_Q \int_{U(N_f)} dU \int_{U(N_f)} dV \,\mathrm{Det}^{-Q+\delta}\left[\bar{\varphi}^i_b (UAV)_{ac} \psi^i_c\right]$$

$$\exp\left[\bar{\varphi}^i_a (UAV)_{ab} \psi^i_b + \bar{\psi}^i_a (V^\dagger D U^\dagger)_{ab} \varphi^i_b\right]$$

$$D(Z,Z^\dagger) = \frac{dZ dZ^\dagger}{\mathrm{Det}^{2N_f - N_c}(1 - ZZ^\dagger)}, \qquad C_0 = \left[\frac{\prod^\delta s! \prod^{N_c} m!}{\prod^{N_f} n!}\right]$$

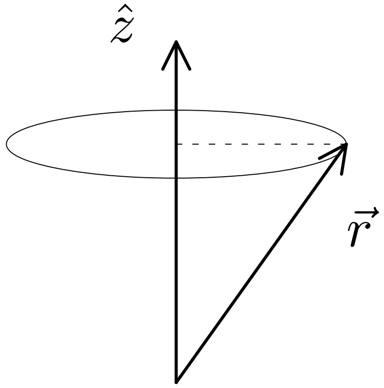
$$\mathcal{C}_{Q>0} = \mathrm{Det}^Q(\bar{\psi}^i_a (1 - Z^\dagger Z)_{ab} \psi^j_b) + \mathrm{Det}^Q(\bar{\varphi}^i_a (1 - Z^\dagger Z)_{ab} \varphi^j_b)$$

$$\delta = N - N_c \quad AD = I_N, \quad \chi_0 = 1, \ \chi_Q = \mathrm{Det}^Q(\psi^i_a \bar{\psi}^j_a) + \mathrm{Det}^Q(\varphi^i_a \bar{\varphi}^j_a).$$

Z parameterize the coset space $U(N_f, N_f)/U(N_f) \times U(N_f)$

Outline of the proof of bosonic CFT:

- analogy: averaging over rotations of a vector in $\mathbb{R}^3$ around the z-axis projects out the z-component of that vector, i.e. the part that is invariant under such a rotation



- basic idea:
  - define a Fock space with up to $2N_fN_c$ species of bosons
  - construct two different implementations of a projection operator onto the color-neutral sector of the Fock space (corresponding to LHS and RHS of the transformation)
  - identify the two implementations

- projection to a manifold could be done either all at once or divide the manifold into a few pieces, project onto each piece then sum up. For bosonic and SUSY, could be either one or infinity!

integrate over color group — projection 1

integrate over flavor group — projection 2

- define bosonic creation and annihilation operators $\bar{c}_a^i$, c_a^i $(a = 1, \ldots, 2N_f;\ i = 1, \ldots, N_c)$
- Fock space is obtained by acting on vacuum $|0\rangle$ with all the $\bar{c}_a^i$ (dimension $= 2N_f N_c$)
- define

$$\begin{cases} |\psi_{Q\geq 0}\rangle = (\epsilon_{i_1,\ldots,i_{N_c}} \bar{c}_1^{i_1} \bar{c}_2^{i_2} \ldots \bar{c}_{N_c}^{i_{N_c}})^Q |0\rangle, \\ |\psi_{Q<0}\rangle = (\epsilon_{i_1,\ldots,i_{N_c}} \bar{d}_1^{i_1} \bar{d}_2^{i_2} \ldots \bar{d}_{N_c}^{i_{N_c}})^Q |0\rangle. \end{cases}$$

Main steps of the proof:

$$\int_{\mathrm{SU(N_c)}} dU \exp\left(\bar\psi^i_a U^{ij}\psi^j_a + \bar\varphi^i_a U^{\dagger ij}\varphi^j_a\right)$$

action of Fock operators on $|0\rangle$ and anticommutation relations

$$= \int_{\mathrm{SU(N_c)}} dU \langle 0|\exp(\psi^i_a c^i_a + \varphi^i_a d^i_a)\exp\left(\bar c^i_a U^{ij}\psi^j_a + \bar\varphi^i_a U^{\dagger ij}\bar d^j_a\right)|0\rangle$$

projection onto color-neutral sector by SU(N_c) rotation

$$= \langle 0|\exp(\psi^i_a c^i_a + \varphi^i_a d^i_a)\hat P \exp(\bar c^i_a \psi^i_a + \bar d^i_a \bar\varphi^i_a)|0\rangle$$

projection onto color-neutral sector using coherent states

$$= \sum_Q \langle 0|\exp(\psi^i_a c^i_a + \varphi^i_a d^i_a)\mathbb{1}_Q \exp(\bar c^i_a \psi^i_a + \bar d^i_a \bar\varphi^i_a)|0\rangle$$

parameterization of G/H, properties of Fock operators, and more algebra

$$= \sum_{Q\geq 0} \frac{\int_{|Z|<1} D(Z,Z^\dagger)\exp(\bar\psi^i_a Z^\dagger_{ab}\varphi^i_b + \bar\varphi^i_a Z_{ab}\psi^i_b)\ \mathcal{C}_Q}{N_Q \int_{|Z|<1} D(Z,Z^\dagger)\ \mathrm{Det}^Q_{N_c}(1 - Z^\dagger Z)}$$

complete with character expansion

Outline of the method:

- basic idea:
  - LHS: we want to split the integration over $SU(N_c)$ to the same Q.
  - RHS: don't integrate over Λ, put an arbitrary matrix instead. do two integration over to $U(N_f)$.
  - identify the two sides.
  - We notice a difference between the representation of GL(N), U(N) and SU(N).

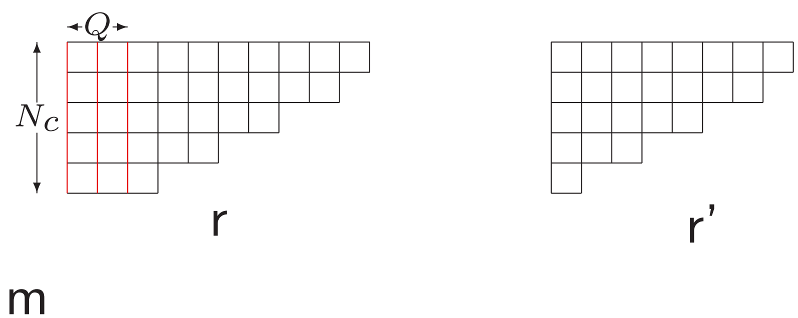


Figure 1: Irreducible representations r' and $r = r' + N_c Q$.

SUSY CFT for the special unitary group

Our result for SUSY CFT:

$$\int_{SU(N_c)} dU \exp\left(\bar{\psi}^i_{+a} U^{ij} \psi^j_{+a} + \bar{\psi}^j_{-b} \bar{U}^{ij} \psi^i_{-b}\right)$$

$$= \int D\mu_{N_c}(Z, \tilde{Z}) \exp\left(\bar{\psi}^i_{+a} Z_{ab} \psi^i_{-b} + \bar{\psi}^j_{-b} \tilde{Z}_{ba} \psi^j_{+a}\right) \left\{1 + \sum_{Q=1} \mathcal{C}_Q \left[\mathrm{Det}^Q \mathcal{M} + \mathrm{Det}^Q \mathcal{N}\right]\right.$$

$$\begin{aligned} D\mu_{N_c}(Z, \tilde{Z}) &= \mathrm{SDet}^{N_f + N_c - N_b}(1 - \widetilde{Z} Z) \\ \mathcal{M} &= \bar{\psi}^i_{+a}(1 - Z\widetilde{Z})_{ab} \psi^j_{+b} \\ \mathcal{N} &= \bar{\psi}^i_{-a}(1 - \widetilde{Z} Z)_{ab} \psi^j_{-b} \end{aligned}$$

$Z, \tilde{Z}$ parameterize $U(n_{b+}, n_{b-}|n_{f+} + n_{f-})/[U(n_{b+}|n_{f+}) \times U(n_{b-}|n_{f-})]$

Outline of the proof for SUSY CFT

- Positive and negative particles. From operators to generators.

$$\bar{c}^i_A = \begin{pmatrix} \bar{b}^i_+ \\ \bar{f}^i_+ \\ b^i_- \\ f^i_- \end{pmatrix}, \quad c^i_A = \begin{pmatrix} b^i_+ \\ f^i_+ \\ -\bar{b}^i_- \\ \bar{f}^i_- \end{pmatrix} \longrightarrow E^{ij}_{AB} = \begin{pmatrix} \bar{b}^i_{+a} b^j_{+b} & \bar{b}^i_{+a} f^j_{+b} & -\bar{b}^i_{+a} \bar{b}^j_{-b} & \bar{b}^i_{+a} \bar{f}^j_{-b} \\ \bar{f}^i_{+a} b^i_{+b} & \bar{f}^i_{+a} f^i_{+b} & -\bar{f}^i_{+a} \bar{b}^i_{-b} & \bar{f}^i_{+a} \bar{f}^i_{-b} \\ b^i_{-a} b^j_{+b} & b^i_{-a} f^j_{+b} & -b^i_{-a} \bar{b}^j_{-b} & b^i_{-a} \bar{f}^j_{-b} \\ f^i_{-a} b^j_{+b} & f^i_{-a} f^j_{+b} & -f^i_{-a} \bar{b}^j_{-b} & f^i_{-a} \bar{f}^j_{-b} \end{pmatrix}$$

Lie superalgebra $gl(N_b|N_f)$, where $N_b = N_{b_+} + N_{b_-}$, $N_f = N_{f_+} + N_{f_-}$.

- Choose Hermitian basis, we get unitary representation of $U(N_{b_+}, N_{b_-}|N_{f_+} + N_{f_-})$.

$$E^{ij}_{AB} = \begin{pmatrix} BB & BF \\ FB & FF \end{pmatrix}, \quad BB = \begin{pmatrix} \bar{b}^i_{+a} b^j_{+b} & -\bar{b}^i_{+a} \bar{b}^j_{-b} \\ b^i_{-a} b^j_{+b} & -b^i_{-a} \bar{b}^j_{-b} \end{pmatrix}, \quad FF = \begin{pmatrix} \bar{f}^i_{+a} f^i_{+b} & \bar{f}^i_{+a} \bar{f}^j_{-b} \\ f^i_{-a} f^j_{+b} & f^i_{-a} \bar{f}^j_{-b} \end{pmatrix}$$

- There are two sets of subgroups here!
  1. BB block $\longrightarrow U(N_{b_+}, N_{b_-})$, FF block $\longrightarrow U(N_{f_+} + N_{f_-})$.
  2. B_+F_+ block $\longrightarrow U(N_{b_+}|N_{f_+})$, B_-F_- block $\longrightarrow U(N_{b_-}|N_{f_-})$.

Summary and Outlook

- The bosonic CFT for $U(N_c)$ is to be used in the duality transformation of the boson induced U(N) Yang-Mills theory. While in Zirnbauer's original paper, only the $N_f = N_c$ and $2N_f < N_c$ cases are solved. Our results apply to all values of N_f for U(N) and SU(N) induced theory!

- Our result for SUSY CFT is aimed at boson induced SU(N) Yang-Mills gauge theory. And the U(N) theory corresponds to Q=0.

- Super symmetry can cue the divergent problem of the first method. $N_{fermion} + N_c - N_{boson} \geq 0$, which is well satisfied in the boson induced theory!

- Currently we have met a problem in constructing the color-neutral projector with generalized super-coherent states. That is about Schur's lemma for infinity dimensional unitary representation of non-compact supergroup. This may be caused by boundary terms since we are integrating over non-compact supermanifold.

- Due to the difference between representation theory of classical and super group, the character expansion method may not work for SUSY.