Fractal Structure of Field Theories

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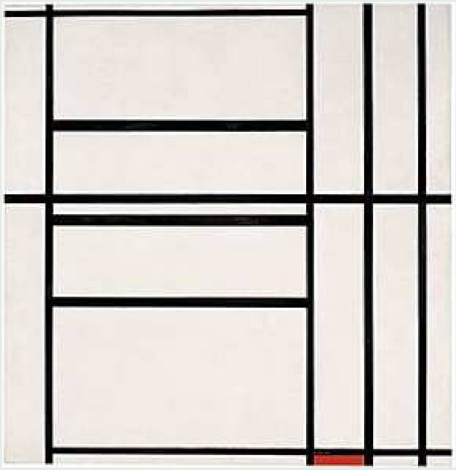
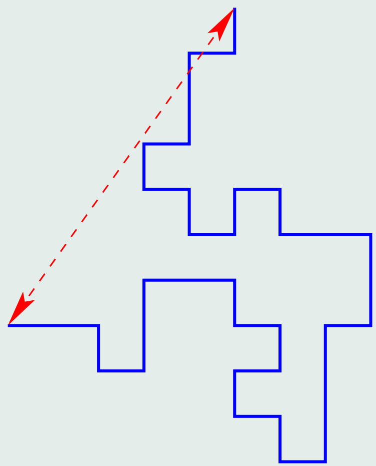


Piet Mondriaan, 1938-39

[de Gennes, 1972]:



- SAW: $\lim_{N\to 0} \mathrm{O}(N)$
- # of walks of length l (configurational entropy):

$$c_l \sim l^{\gamma-1}$$

- End-to-end distance:

$$\langle \mathbf{R}^2 \rangle_l \sim l^{2\nu},$$

implying fractal, or Hausdorff dimension:

$$D_{\mathrm{H}} = 1/\nu = 4/3$$

Correlation function SAW:

$$G(\mathbf{x} \to \mathbf{x}') = \sum_{\text{Paths: } \mathbf{x}\to\mathbf{x}'} \beta^l = \sum_l c_l(\mathbf{x} \to \mathbf{x}')\beta^l$$

Susceptibility:

$$\chi = \sum_{\mathbf{x}'} G(\mathbf{x} \to \mathbf{x}') = \sum_l \underbrace{\sum_{\mathbf{x}'} c_l(\mathbf{x} \to \mathbf{x}')}_{= c_l} \beta^l$$

$$\chi = \sum_l l^{\gamma-1}\left(\frac{\beta}{\beta_c}\right)^l \sim \sum_l l^{\gamma-1}\,\mathrm{e}^{-\theta l} \sim |\beta-\beta_c|^{-\gamma}$$

w/ line tension $\theta \sim |\beta - \beta_c|$

Can this be generalized to arbitrary $-2 \leq N \leq 2$?

Critical exponents O(N) models in 2D:

Model	N	c	γ	η	ν	D_H
Gaussian	-2	-2	1	0	$\frac{1}{2}$	$\frac{5}{4}$
SAW	0	0	$\frac{43}{32}$	$\frac{5}{24}$	$\frac{3}{4}$	$\frac{4}{3}$
Ising	1	$\frac{1}{2}$	$\frac{7}{4}$	$\frac{1}{4}$	1	$\frac{11}{8}$
XY	2	1	∞	$\frac{1}{4}$	∞	$\frac{3}{2}$

For general N:

- Fractal dimension:

$$\nu \neq \frac{1}{D_H}$$

- Configurational entropy:

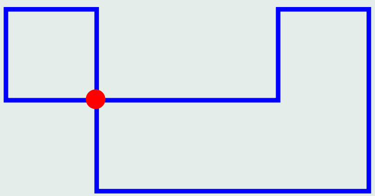
$$c_l \not\propto l^{\gamma-1}$$

Partition function of O(N) models in high-temperature rep. w/ $\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$:

$$Z = \int \prod_{\mathbf{x}} \mathrm{d}\boldsymbol{\phi}(\mathbf{x}) \exp\left(-a^d \sum_{\mathbf{x}} \left\{ \frac{1}{2a^2} \sum_{\mathbf{i}} [\boldsymbol{\phi}(\mathbf{x}+a\mathbf{i}) - \boldsymbol{\phi}(\mathbf{x})]^2 + \frac{m^2}{2}\boldsymbol{\phi}^2(\mathbf{x}) + \frac{\lambda}{4}\boldsymbol{\phi}^4(\mathbf{x}) \right\}\right)$$

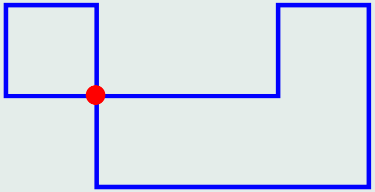
$$\stackrel{(\lambda=0)}{=} \int \prod_{\mathbf{x}} \mathrm{d}\boldsymbol{\phi}(\mathbf{x}) \exp\left[\beta a^{d-2} \sum_{\mathbf{x},\mathbf{i}} \boldsymbol{\phi}(\mathbf{x}+a\mathbf{i}) \cdot \boldsymbol{\phi}(\mathbf{x}) - \tfrac{1}{2} a^{d-2} \sum_{\mathbf{x}} \boldsymbol{\phi}^2(\mathbf{x})\right], \quad \beta = \frac{1}{2d + m^2 a^2}$$

$$= \int \prod_{\mathbf{x}} \mathrm{d}\boldsymbol{\phi}(\mathbf{x}) \left[1 + \beta a^{d-2} \sum_{\mathbf{x},\mathbf{i}} \boldsymbol{\phi}(\mathbf{x}+a\mathbf{i}) \cdot \boldsymbol{\phi}(\mathbf{x}) + \cdots\right] \exp\left[-\tfrac{1}{2} a^{d-2} \sum_{\mathbf{x}} \boldsymbol{\phi}^2(\mathbf{x})\right]$$



Loop gas:

- Each link: β
- $\lambda = 0$: phantom loops
- $\lambda \neq 0$: extra Boltzmann weight for crossings



Loop gas:

- D_{H}: fractal dimension of loops
- Phase transition:
  - line tension θ vanishes
  - proliferation of loops
- $N = 0$: no loops, $Z = 1$

In general:

- Line tension vanishes as:

$$\theta \sim |\beta - \beta_{\mathrm{c}}|^{1/\sigma}, \quad \text{w/ } \sigma \neq 1$$

- Susceptibility:

$$\chi \sim \sum_l l^{\sigma\gamma - 1} \, \mathrm{e}^{-\theta l}$$

As in percolation theory:

$$\nu = \frac{1}{\sigma D_{\mathrm{H}}}$$

Scaling relation:

$$\gamma = \nu(2 - \eta) \quad \Rightarrow \quad \sigma\gamma = \frac{2 - \eta}{D_{\mathrm{H}}}$$

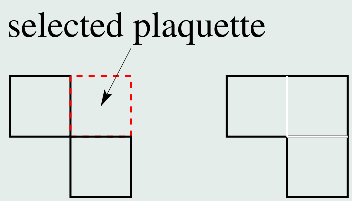
Partition function of Ising model ($N = 1$) in high-temperature rep. ($J = 1$):

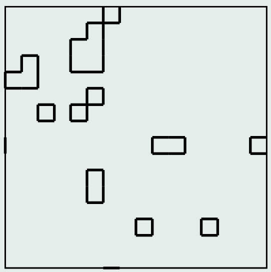
$$Z = (\cosh\beta)^{2N} 2^N \sum_{\text{Closed Graphs}} v^l, \qquad e^{\beta S_i S_j} = \cosh\beta(1 + v S_i S_j)$$

w/ $v = \tanh\beta$, l: # links in graph

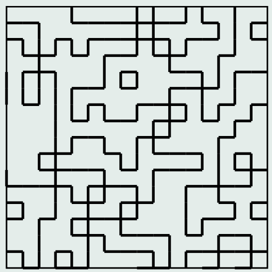
Monte Carlo [Janke & A.S., 2003 (NPB to appear)]:

- Single plaquette □ update
- Acceptance rate: $p_{\text{HT}} = \min\left(1, v^{l'-l}\right)$





$\beta < \beta_c$

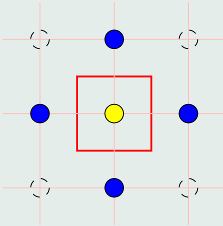


$\beta > \beta_c$

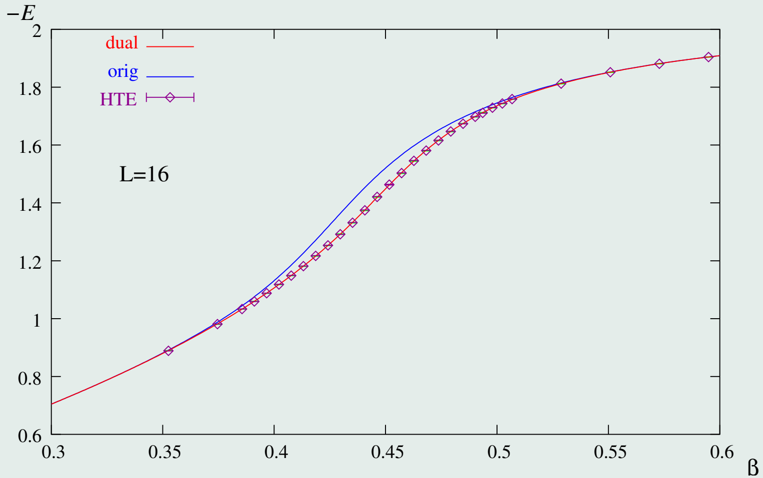
(Periodic Boundary Conditions)

Phase transition: Proliferation of HT graphs

Single plaquette □ update ~ Single spin update on dual lattice:

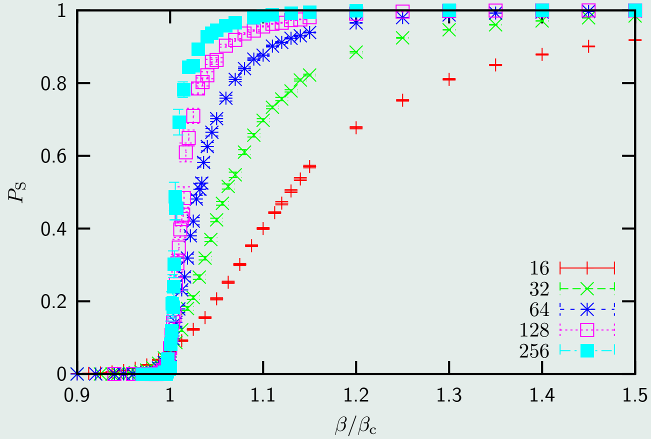


PBC: mismatch w/ $\tilde{\beta} = \frac{1}{2} \ln \coth \beta$ [Kramers & Wannier, 1941]:

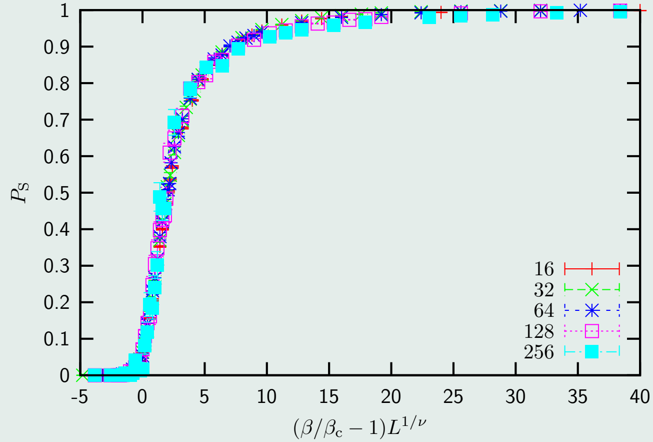


[Peierls, 1936]:

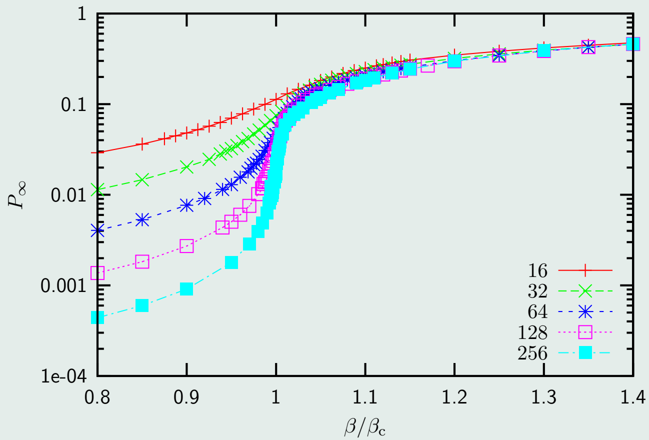
HT graphs ~ Boundaries of geometrical clusters (Domain Walls)



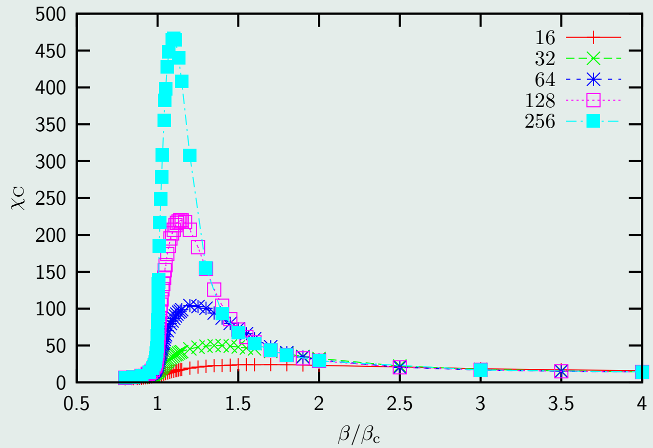
$\therefore$ Graph proliferation @ T_c



$\therefore \nu = 1$



$\therefore \beta_G = 0.627(8)$



$\therefore \gamma_G = 0.748(6)$

MC results consistent w/:

$$\sigma = \frac{8}{11}, D_{\mathrm{H}} = \frac{11}{8} \quad \Rightarrow \quad \nu = \frac{1}{\sigma D_{\mathrm{H}}} = 1$$

2D O(N) models:

Model	N	c	γ	η	ν	D_{H}	σ
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SAW	0	0	$\frac{43}{32}$	$\frac{5}{24}$	$\frac{3}{4}$	$\frac{4}{3}$	1
Ising	1	$\frac{1}{2}$	$\frac{7}{4}$	$\frac{1}{4}$	1	$\frac{11}{8}$	$\frac{8}{11}$
XY	2	1	∞	$\frac{1}{4}$	∞	$\frac{3}{2}$	0

XY-model ($N = 2$):

- $\sigma = 0$: algebraic behavior
- Configurational entropy:

$$c_l \sim l^{\sigma\gamma - 1}$$

w/ $\sigma\gamma - 1 = \frac{1}{6}$