Ageing phenomena in non-equilibrium systems

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- Some experiments
- The typical scenario
  dynamical correlation and response functions
- Local scale invariance and ageing
- Autoresponse and space-time response at and below criticality

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Matthias Paessens (Jülich)

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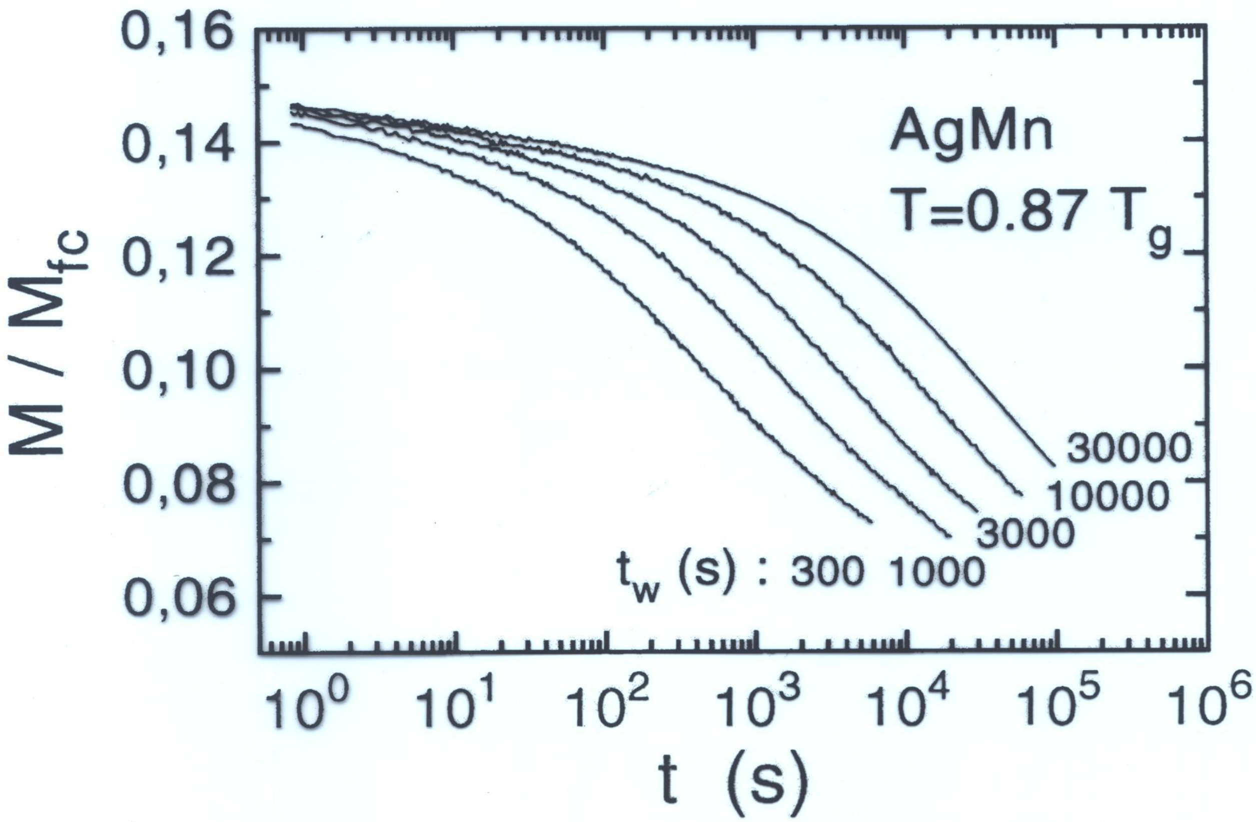
Some experiments

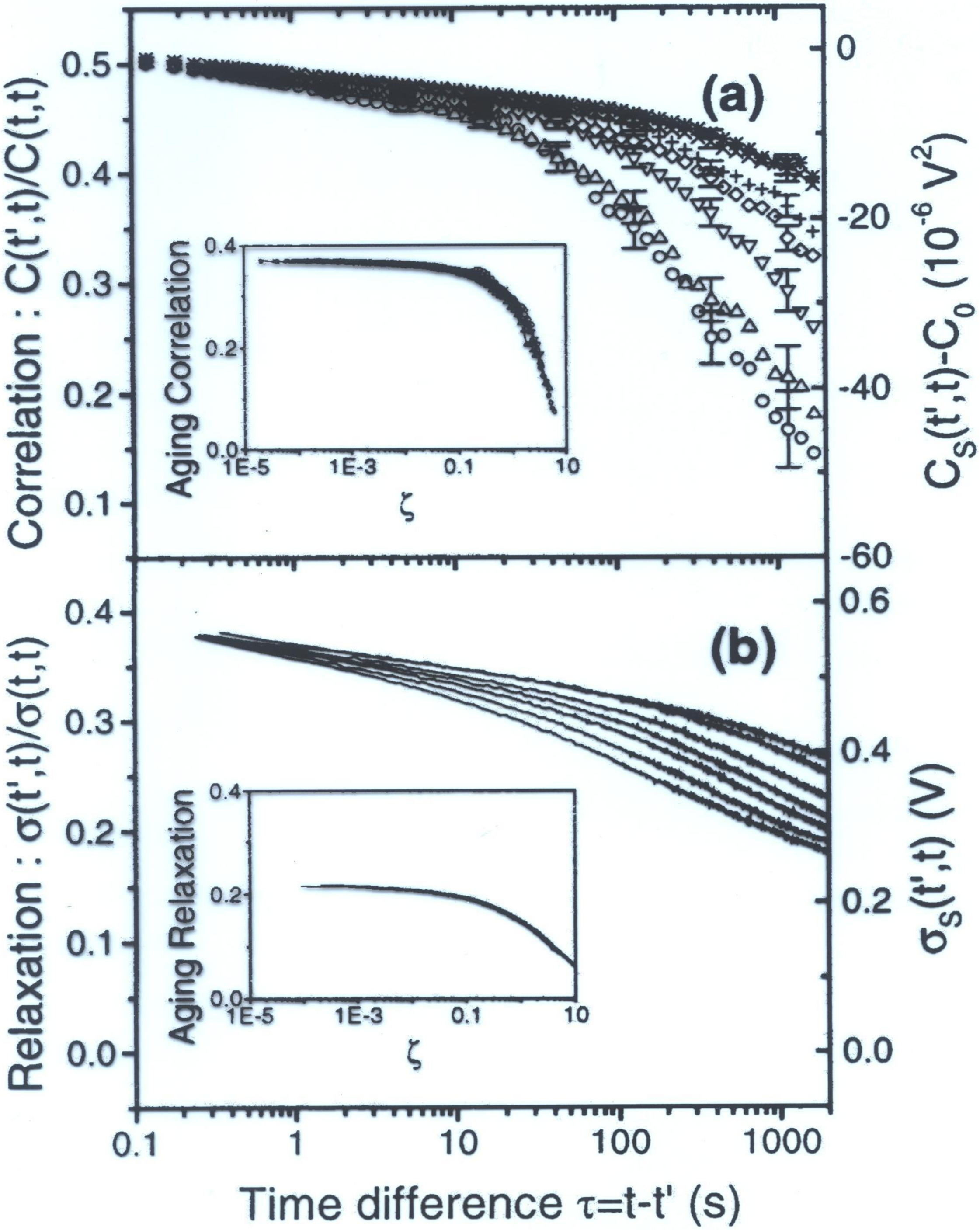
Ageing behaviour in spin glasses

thermoremanent magnetization
(Vincent et al. '95):
quench to low temperature with an external field
field is switched off after the waiting time t_W



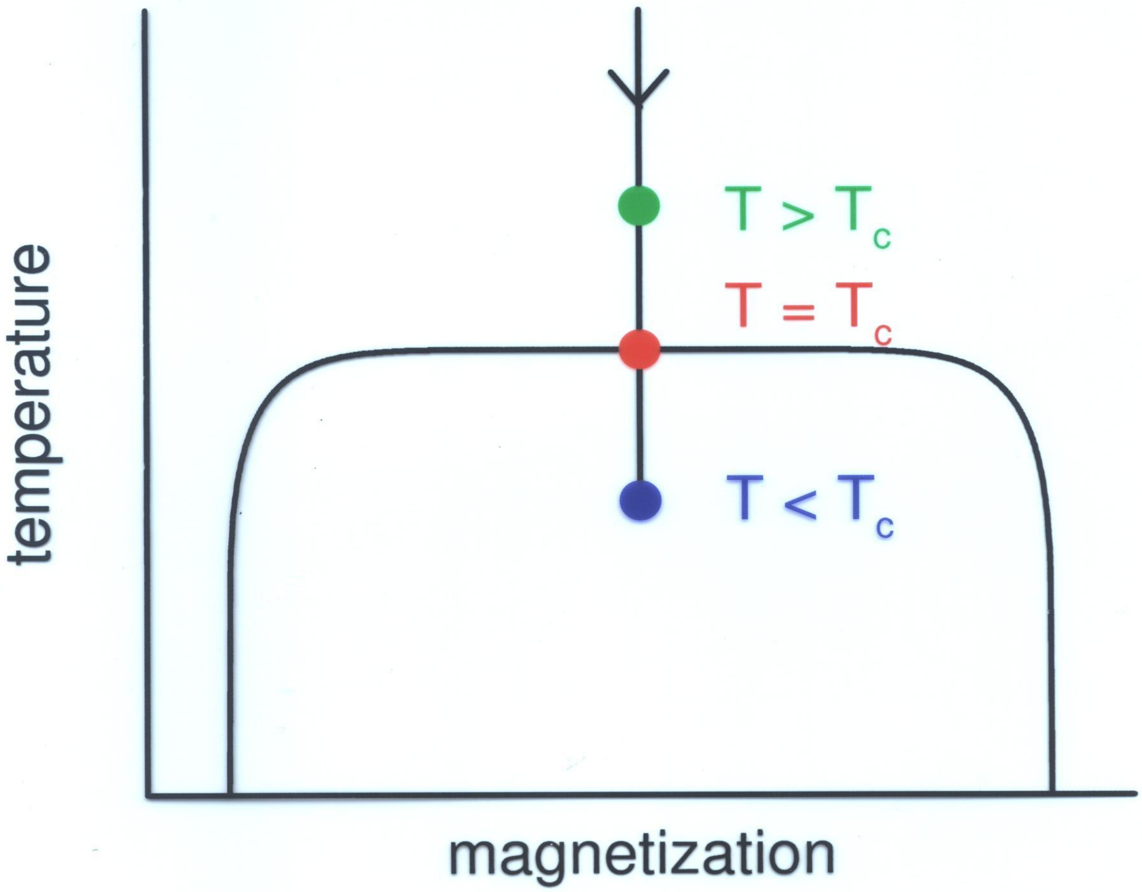
The response of the system is slower for larger waiting times: it ages!

$CdCr_{1.7}In_{0.3}S_4$ (Hérisson/Ocio ’02)



Ageing in a ferromagnetic system: the typical scenario

Quench from a disordered initial state

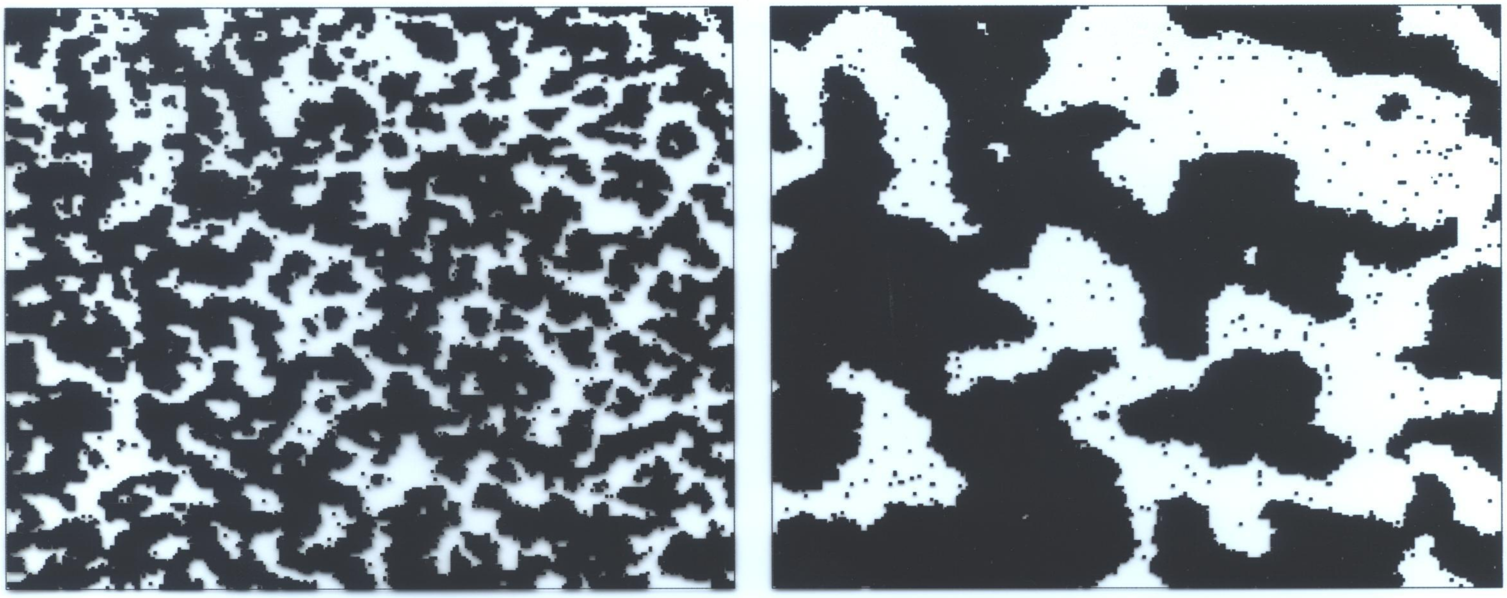


$T > T_c$: system relaxes rapidly to equilibrium

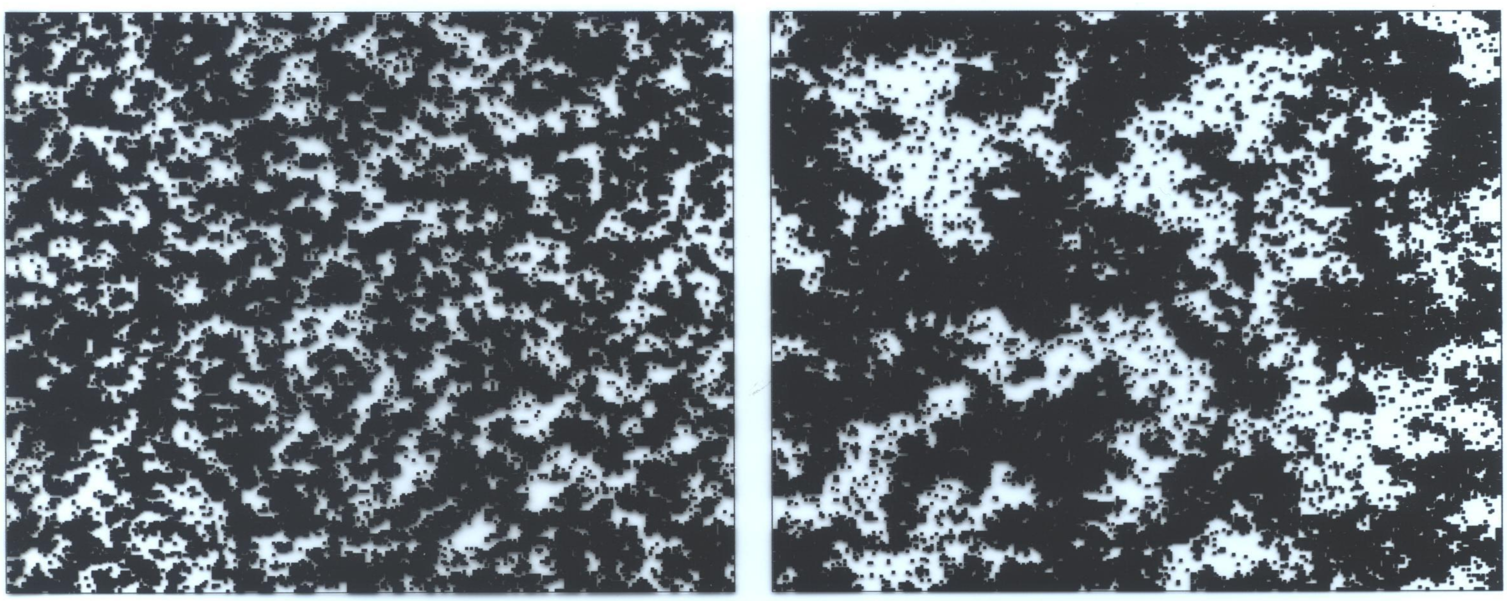
$T = T_c$ and $T < T_c$:
equilibrium is never reached in an infinite system

$T < T_c$: phase ordering

Class S: equilibrium correlations decay exponentially



Class L: long-range (algebraic) equilibrium correlations



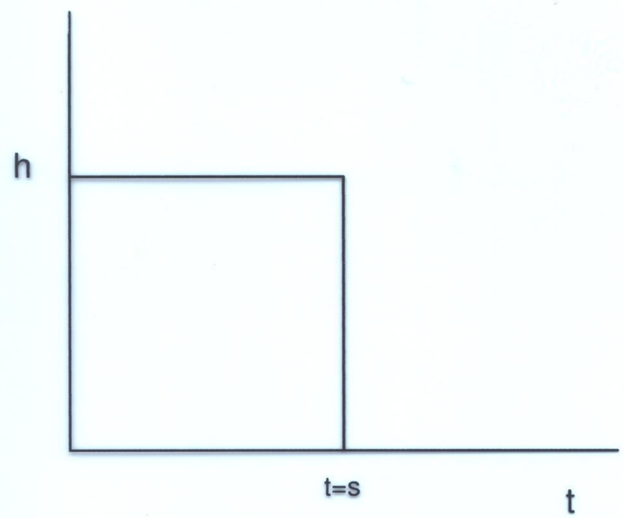
Correlated domains grow as a function of time:

$$L(t) \sim t^{1/z}$$

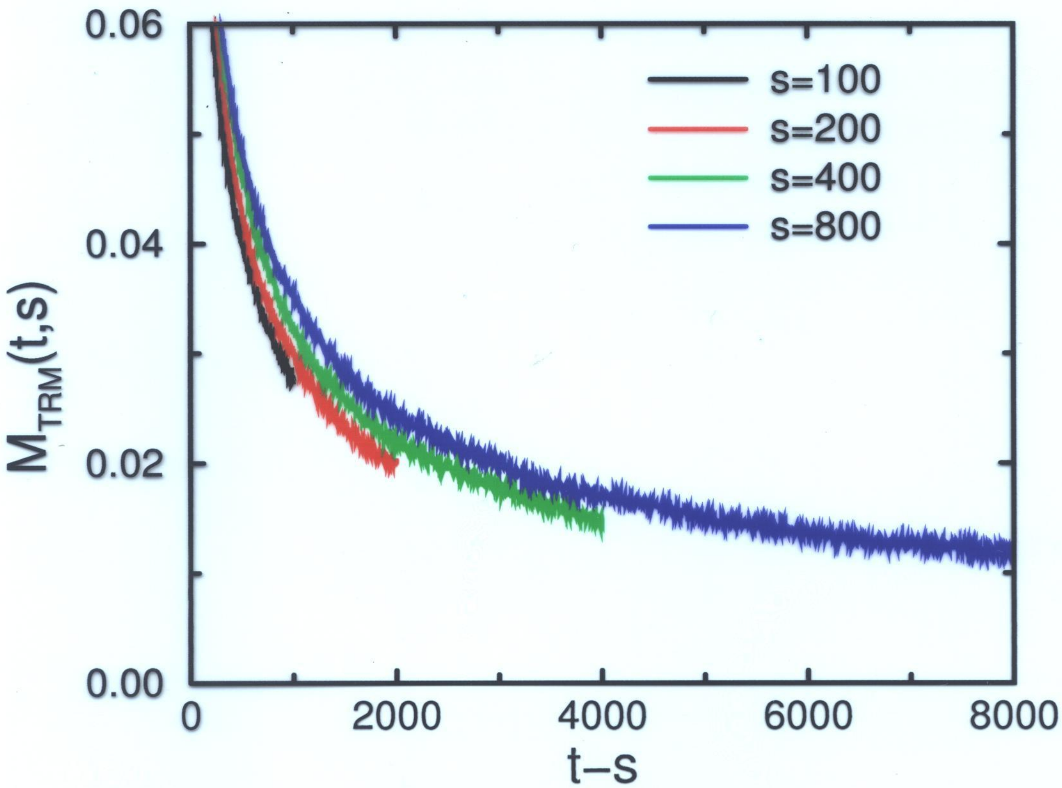
$z =$ dynamical exponent

In the case of non-conserved order parameter and $T < T_c$: $z = 2$

Example of ageing behaviour:
thermoremanent magnetization



2d Ising model ($T = 0.66\, T_c$)



Quantities of interest:

- dynamical correlation function

$$C(t, s; \mathbf{r} - \mathbf{r}') = \langle \sigma_{\mathbf{r}}(t)\, \sigma_{\mathbf{r}'}(s) \rangle$$

- response function

$$R(t, s; \mathbf{r} - \mathbf{r}') = \left. \frac{\delta \langle \sigma_{\mathbf{r}}(t) \rangle}{\delta h_{\mathbf{r}'}(s)} \right|_{h=0}$$

$\sigma_{\mathbf{r}}(t)$: value of the spin at site $\mathbf{r}$ at time t

$h_{\mathbf{r}}(t)$: external field acting at site $\mathbf{r}$ at time t

$s =$ waiting time, $t =$ observation time

autocorrelation: $C(t, s) = C(t, s; \mathbf{0})$

autoresponse: $R(t, s) = R(t, s; \mathbf{0})$

Dynamical scaling behaviour

$$C(t, s) \sim s^{-b}\, f_C(t/s)$$

$$R(t, s) \sim s^{-1-a}\, f_R(t/s)$$

$(s \gg \tau_{micro},\ t - s \gg \tau_{micro})$

$f_C(x)$, $f_R(x)$: scaling functions

For $x \gg 1$: $f_C(x) \sim x^{-\lambda_C/z}$ and $f_R(x) \sim x^{-\lambda_R/z}$

(Picone/Henkel '02)

Local scale invariance and ageing

Isotropic scaling behaviour

two-point functions transform covariantly under a global scaling transformation $\mathbf{r} \longrightarrow b\,\mathbf{r}$

$$G(b\,\mathbf{r}_1, b\,\mathbf{r}_2) = b^{-(x_1+x_2)}\,G(\mathbf{r}_1, \mathbf{r}_2)$$

x_1, x_2: scaling dimensions

Examples: usual (isotropic) equilibrium critical points

Anisotropic scaling behaviour:

dynamical correlation and response functions under a global scaling transformation $\mathbf{r} \longrightarrow b\,\mathbf{r}$

$$G(b^z\,t_1, b\,\mathbf{r}_1; b^z\,t_2, b\,\mathbf{r}_2) = b^{-(x_1+x_2)}\,G(t_1, \mathbf{r}_1; t_2, \mathbf{r}_2)$$

Examples:

- ageing in non-equilibrium systems

  M.Henkel, M.P., C. Godrèche and J.-M. Luck,
  Phys. Rev. Lett. **87**, 265701 (2001)

- strong anisotropic equilibrium critical points

  M.P. and M. Henkel, Phys. Rev. Lett. **87**, 125702 (2001)

- critical dynamics
- non-equilibrium phase transitions

Question:

Is it possible to generalize this anisotropic scaling behaviour with $b = const$ to local scale transformations $b = b(t, \mathbf{r})$?

Well-known cases where this has been done

- $z = 1$: conformal invariance (Polyakov '70)
- $z = 2$: Schrödinger invariance
  (Niederer '72, Hagen '72)

Henkel '97, '02:
A generalization to arbitrary values of the dynamical exponent $z \neq 1$ is possible!

Central assumption of the theory of local scale invariance:

Möbius (special conformal) transformation in the time direction

$$t \longrightarrow t' = \frac{\alpha t + \beta}{\gamma t + \delta} \quad \text{with} \quad \alpha\delta - \beta\gamma = 1$$

$\Longrightarrow$ conformal properties of the transformation are in the time direction

$\Longrightarrow$ construction of infinitesimal scaling transformations $b = b(t, \mathbf{r})$

Response function $R(t,s;\mathbf{r}-\mathbf{r}')$

request that the response function transforms covariantly under the generators

$$\mathcal{S} = \left\{ X_0, X_1, Y_{-1/z}, \ldots \right\}$$

of a Lie algebra

X_0: global scale transformation

X_1: Möbius transformation

$Y_{-1/z}$: space translation

For the autoresponse $R(t,s)$ we have automatically that $Y_{-1/z}\,R=0$, the two remaining conditions $X_0\,R=0$ and $X_1\,R=0$ leading to a system of differential equations

$$[t\,\partial_t + s\,\partial_s + \zeta_1 + \zeta_2]\;R(t,s)=0$$
$$\left[t^2\,\partial_t + s^2\,\partial_s + 2\,\zeta_1\,t + 2\,\zeta_2\,s\right]\;R(t,s)=0$$

with $\zeta_i = x_i/z$

Solution: $\quad R(t,s) = r_0\;(t/s)^{\zeta_2-\zeta_1}\;(t-s)^{-\zeta_1-\zeta_2}$

Comparison with the expected asymptotic behaviour $R(t,s)\sim s^{-1-a}\,f_R(t/s)$ with $f_R(x)\sim x^{-\lambda_R/z}$ when $x\gg 1$ leads to the final expression

$$R(t,s) = r_0\,s^{-1-a}\,x^{1+a-\lambda_R/z}\,(x-1)^{-1-a}\quad \textbf{with}\quad x=t/s$$

Phys. Rev. Lett. **87**, 265701 (2001)

Central prediction of the theory

autoresponse only depends on the values of the exponents a and λ_R/z in case $L(t)\sim t^{1/z}$

Prediction for the space-time response:

Phys. Rev. E **68**, 065101(R) (2003)

$$R(t,s;\mathbf{r}) = R(t,s)\,\Phi\left(r/(t-s)^{1/z}\right)$$

where $\Phi(u)$ is a known function

special case $z=2$:

$$R(t,s;\mathbf{r}) = R(t,s)\,\exp\left(-\frac{\mathcal{M}}{2}\frac{r^2}{t-s}\right)$$

$\mathcal{M}$: non-universal constant

Prediction for mixed responses:

selection rules

$$R_{\sigma\tau}(t,s;\mathbf{r}-\mathbf{r}') = \frac{\delta\langle\varphi(t,\mathbf{r})\rangle}{\delta\tau(s,\mathbf{r}')} = 0$$

$$R_{\varepsilon h}(t,s;\mathbf{r}-\mathbf{r}') = \frac{\delta\langle\varepsilon(t,\mathbf{r})\rangle}{\delta h(s,\mathbf{r}')} = 0$$

Prediction for the autocorrelation with $z=2$:

submitted to Europhys. Lett., cond-mat/0404464

$$C(t,s) = a_0\,x^{\lambda_C/2}\,(x-1)^{-\lambda_C}\,\Psi\left(\frac{x+1}{x-1}\right) \quad \textbf{with} \quad x = t/s$$

where $\Psi(y)$ is known

Autoresponse and space-time response

kinetic Ising models with Glauber dynamics:

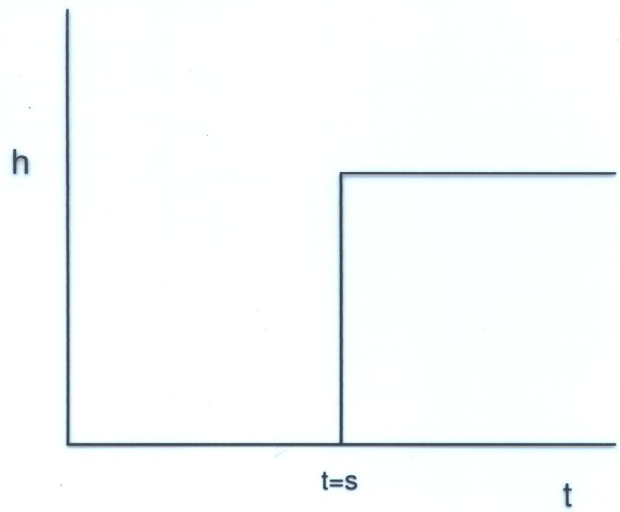
$$\sigma_i(t+1) = \pm 1 \ \textbf{with probability} \ \frac{1}{2}\left[1 \pm \tanh\left(H_i(t)/T\right)\right]$$

local field: $H_i(t) = h_i + \sum_{j(i)} \sigma_j(t)$

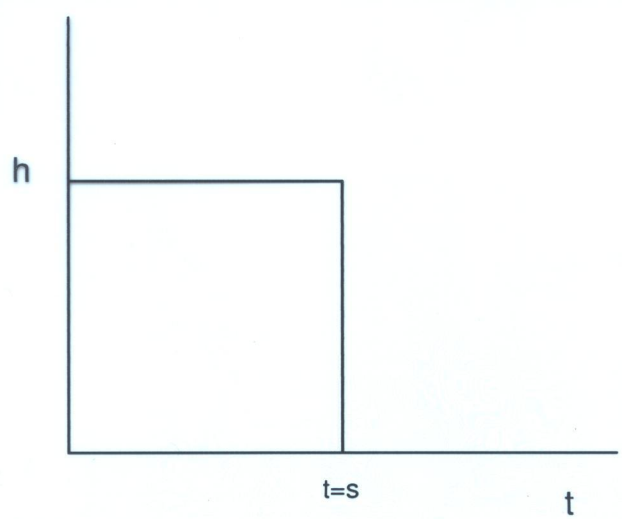
random external field: $h_i = \pm h$

Two possible scenarios:

- ZFC (zero-field cooling)



- TRM (thermoremanent magnetization)



Autoresponse function

The thermoremanent magnetization is an integrated response

$$M_{TRM}(t,s)/h = \int_0^s du\, R(t,u) = r_0\, s^{-a}\, f_M(t/s)$$

with the theoretical prediction

$$R(t,s) = r_0\, s^{-1-a}\, (t/s)^{1+a-\lambda_R/z}\, (t/s-1)^{-1-a}$$

Proposed values for the exponent a

(M. Henkel, M. Paessens and M. P., Europhys. Lett. **62**, 664 (2003))

$$a = \begin{cases} (d-2+\eta)/z & \text{for class } L \\ 1/z & \text{for class } S \end{cases}$$

where η is an equilibrium critical exponent ($C_{eq}(r) \sim r^{-(d-2+\eta)}$)

The proposed value for class L agrees with ALL known analytical results:

- spherical model with short-range interactions
- spherical model with long-range interactions
- 2d XY-model
- non-equilibrium critical dynamics

$\underline{T = T_c}$

(M. Henkel, M. P., C. Godrèche and J.-M. Luck, PRL **87**, 265701 (2001))

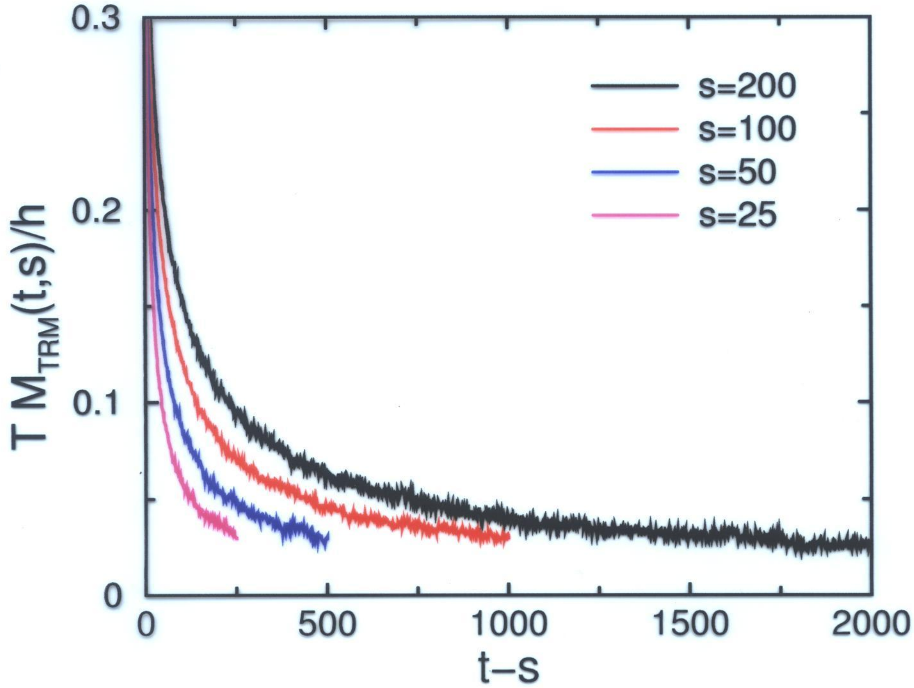
At T_c the autoresponse exponent λ_R is related to the initial-slip exponent of the magnetization Θ (Janssen '92)

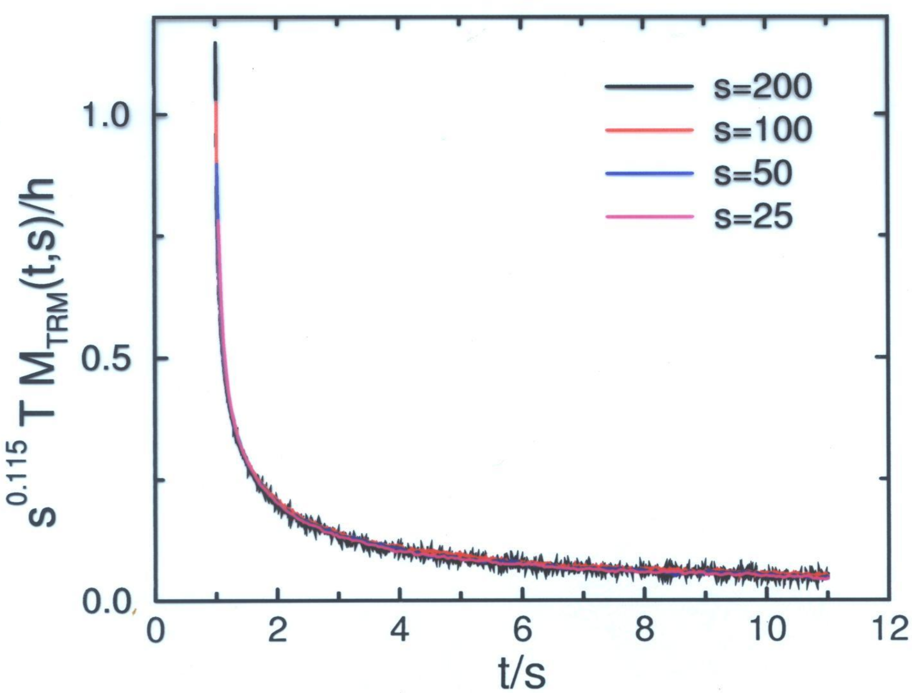
$$\lambda_R = d - z\,\Theta$$

$\Longrightarrow d = 2$: $a = 0.115$ **and** $\lambda_R = 1.59$

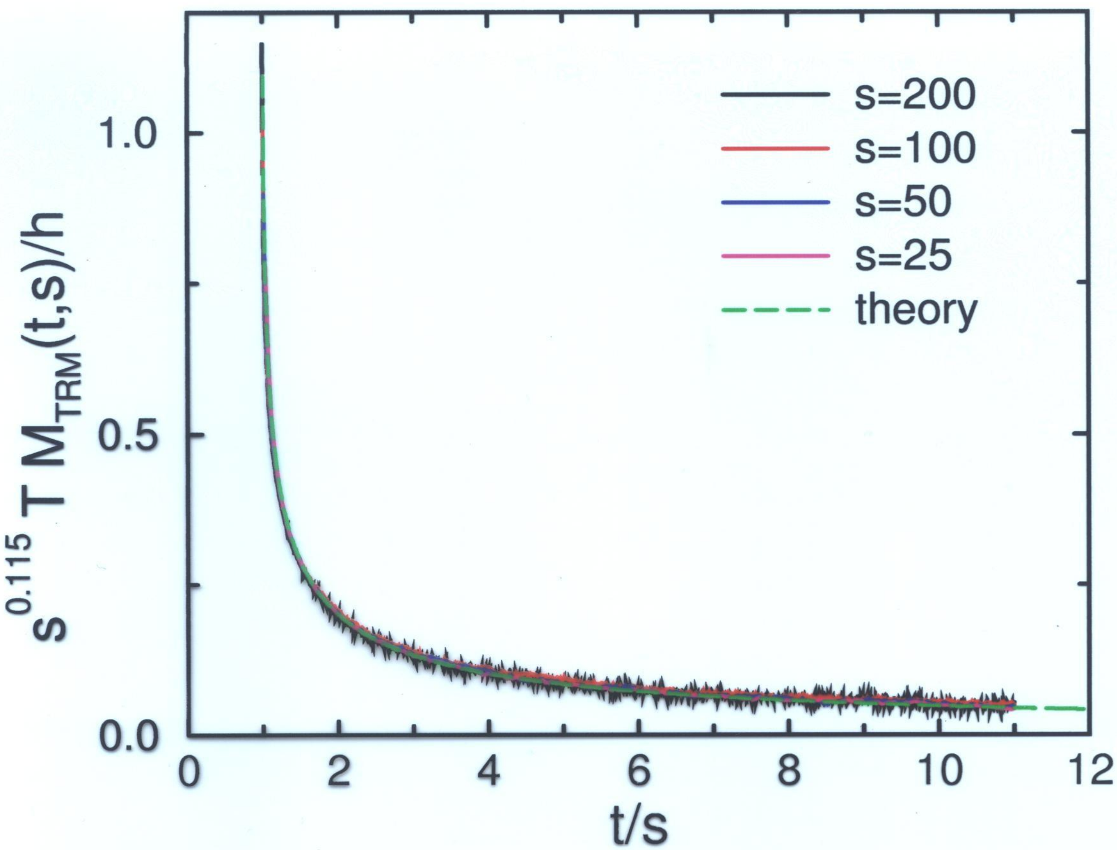
$d = 3$: $a = 0.596$ **and** $\lambda_R = 2.78$

$d = 2$



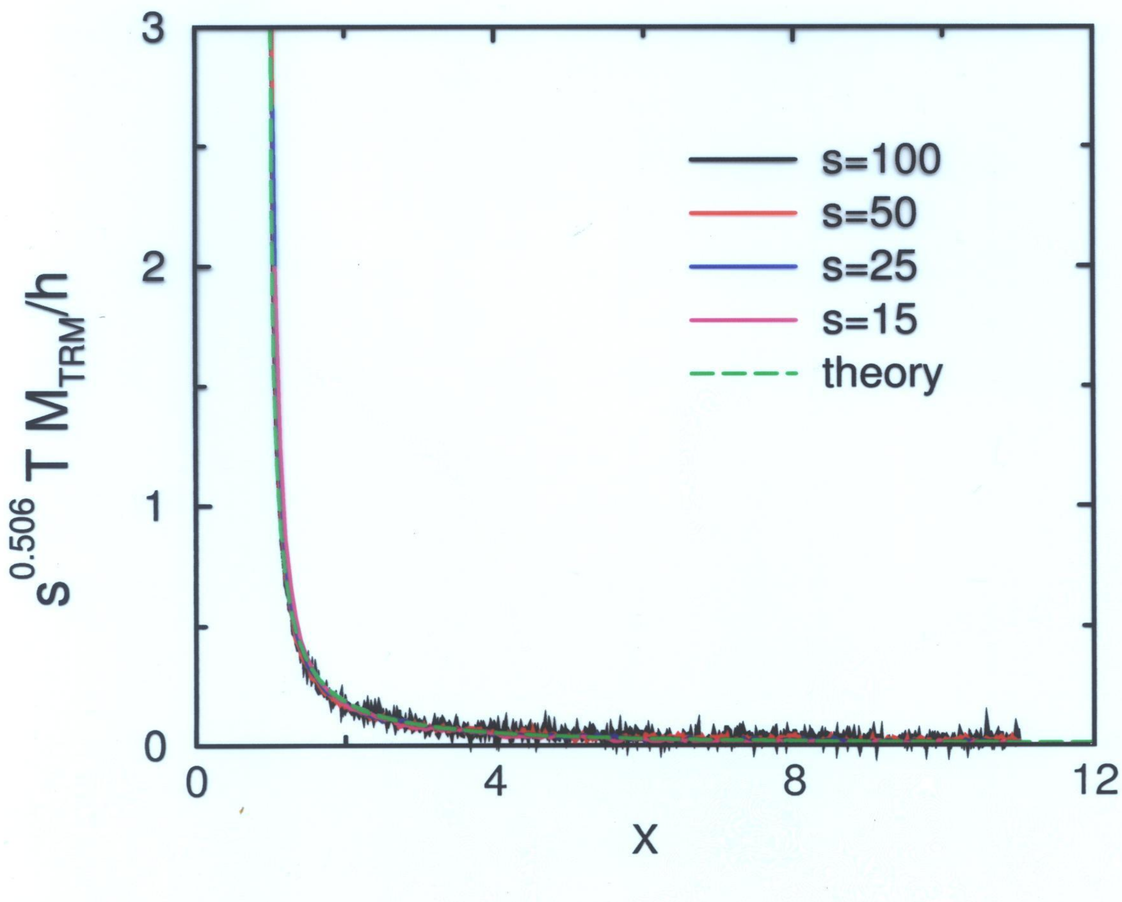


$d = 2$



s=200
s=100
s=50
s=25
theory
1.0
0.5
0.0
0
2
4
6
8
10
12
t/s
s^0.115 T M_TRM(t,s)/h

$d = 3$

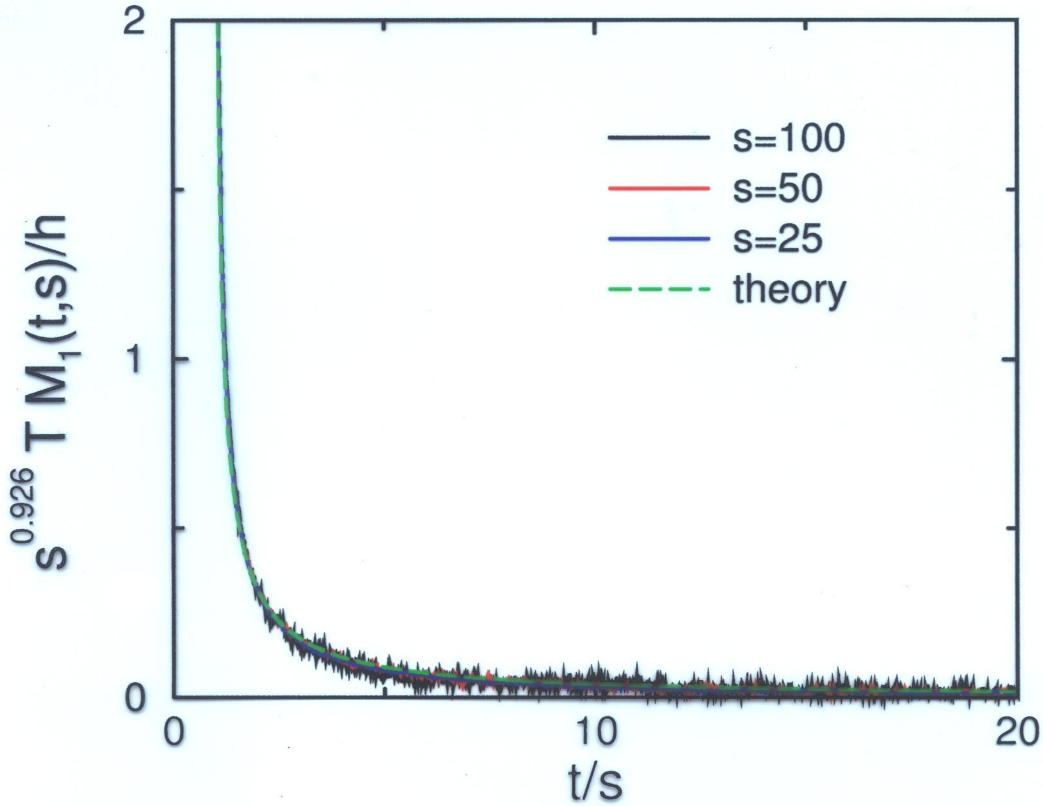


s=100
s=50
s=25
s=15
theory
3
2
1
0
0
4
8
12
x
s^0.506 T M_TRM/h

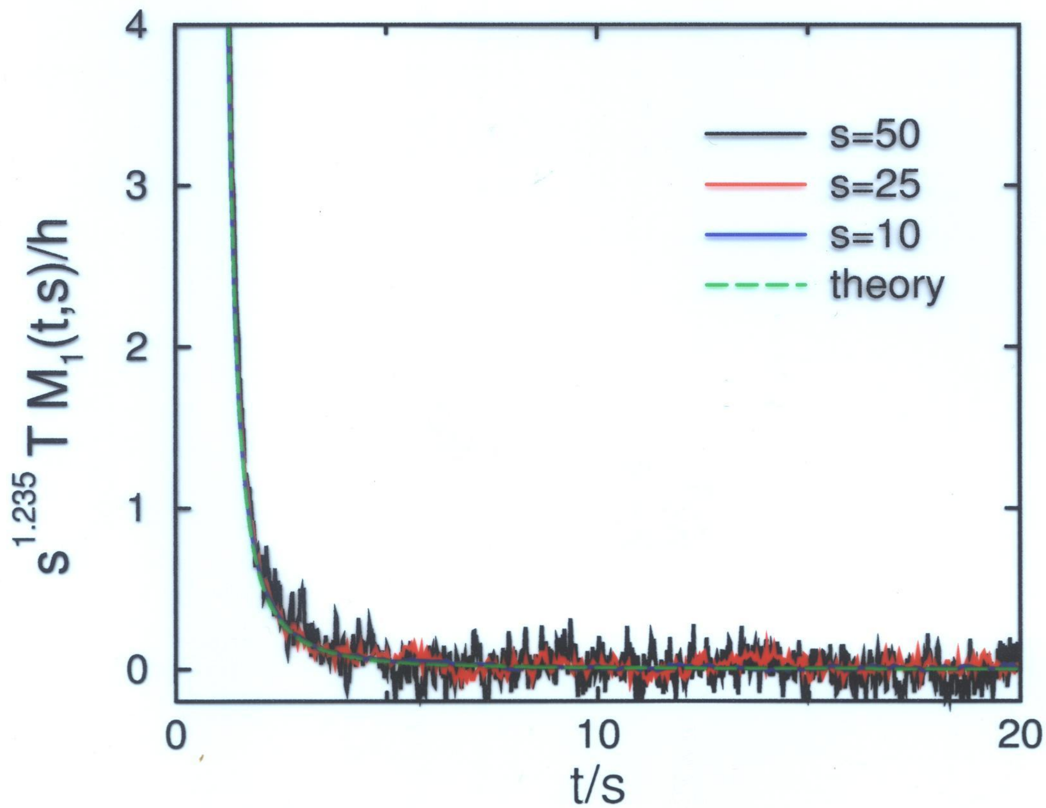
Semi-infinite critical systems

Phys. Rev. Lett. **92**, 145701 (2004)
submitted to Phys. Rev. B, cond-mat/0404203

$d = 2$

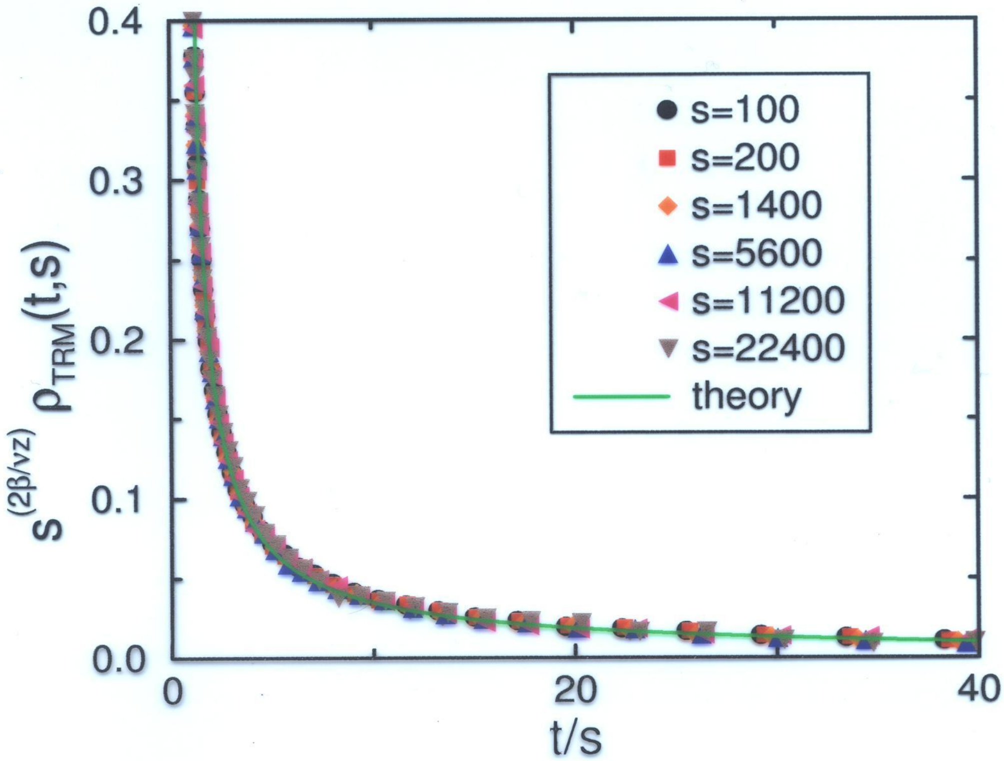


$d = 3$

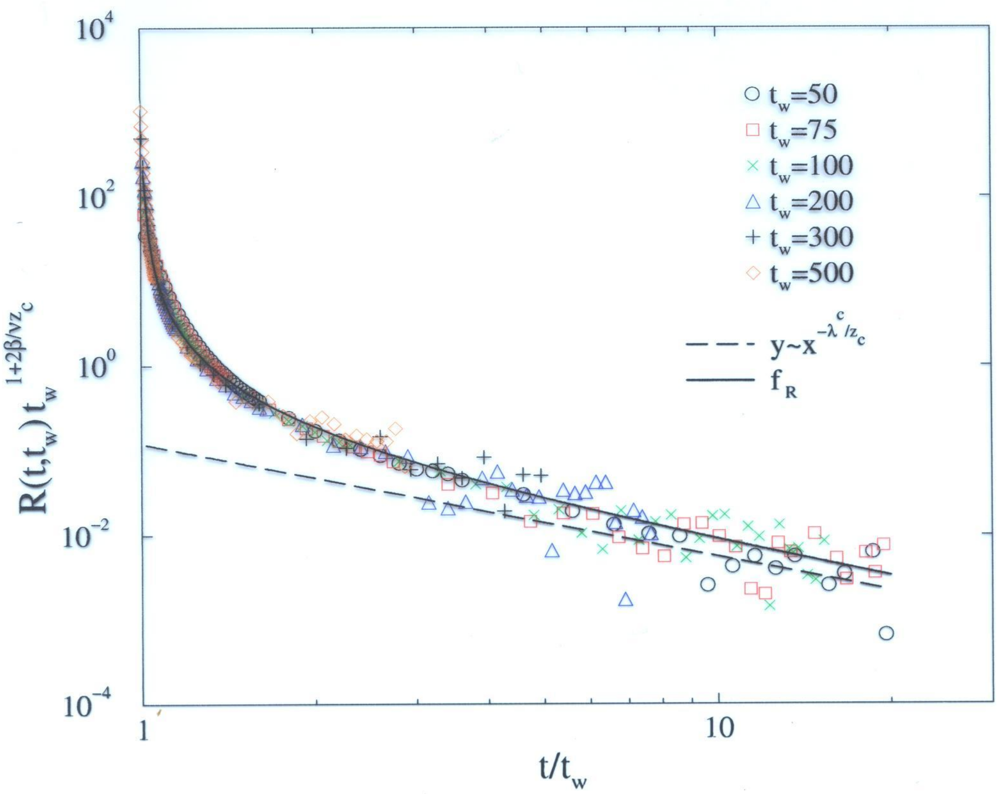


Other results

Ising model with Kawasaki dynamics, i.e. with conserved order parameter (Krzakala '04)



three-dimensional XY-model (Abriet/Karevski '04)



other response functions:

$$\varrho_{\sigma\tau}(t,s) \;=\; \int_0^s du\, R_{\sigma\tau}(t,u) = \frac{\delta\langle\varphi(t)\rangle}{\delta\tau(s)}$$

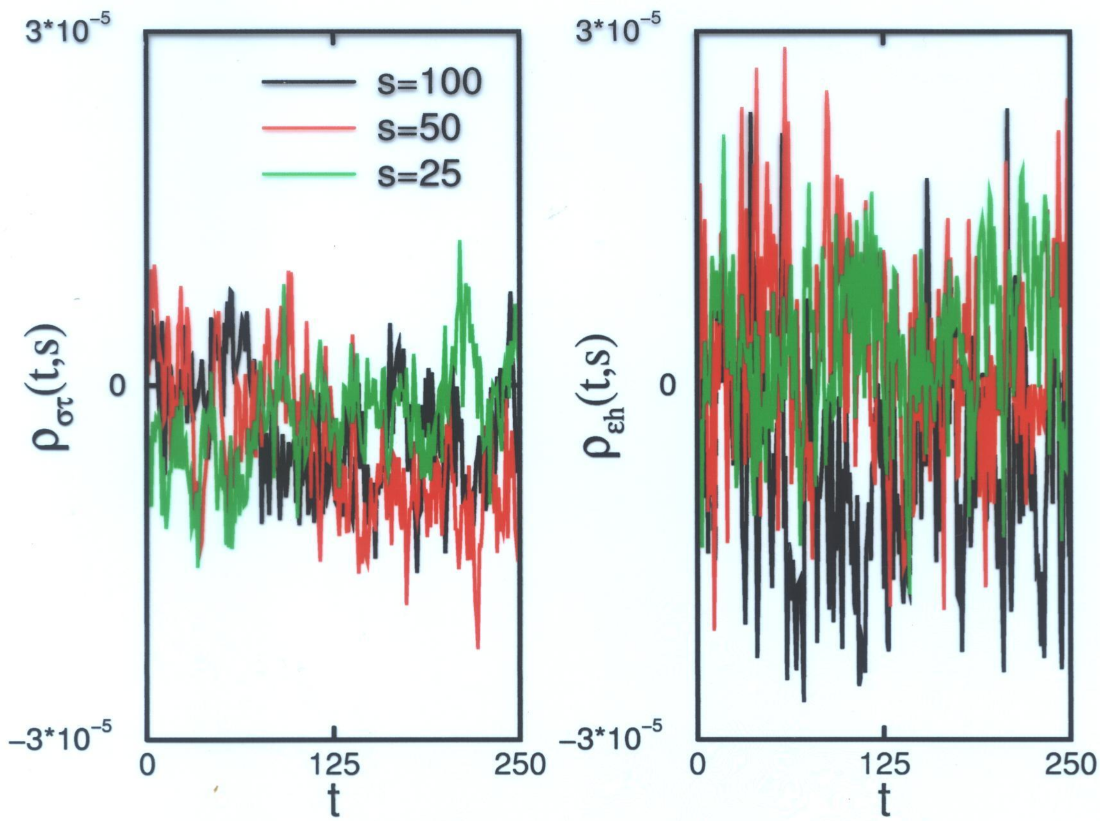
$$\varrho_{\varepsilon h}(t,s) \;=\; \int_0^s du\, R_{\varepsilon h}(t,s) = \frac{\delta\langle\varepsilon(t)\rangle}{\delta h(s)}$$

$$\varrho_{\varepsilon\tau}(t,s) \;=\; \int_0^s du\, R_{\varepsilon\tau}(t,s) = \frac{\delta\langle\varepsilon(t)\rangle}{\delta\tau(s)}$$

where ε is the energy density and τ is a temperature field

Prediction from local scale invariance: $R_{\sigma\tau} = 0 = R_{\varepsilon h}$

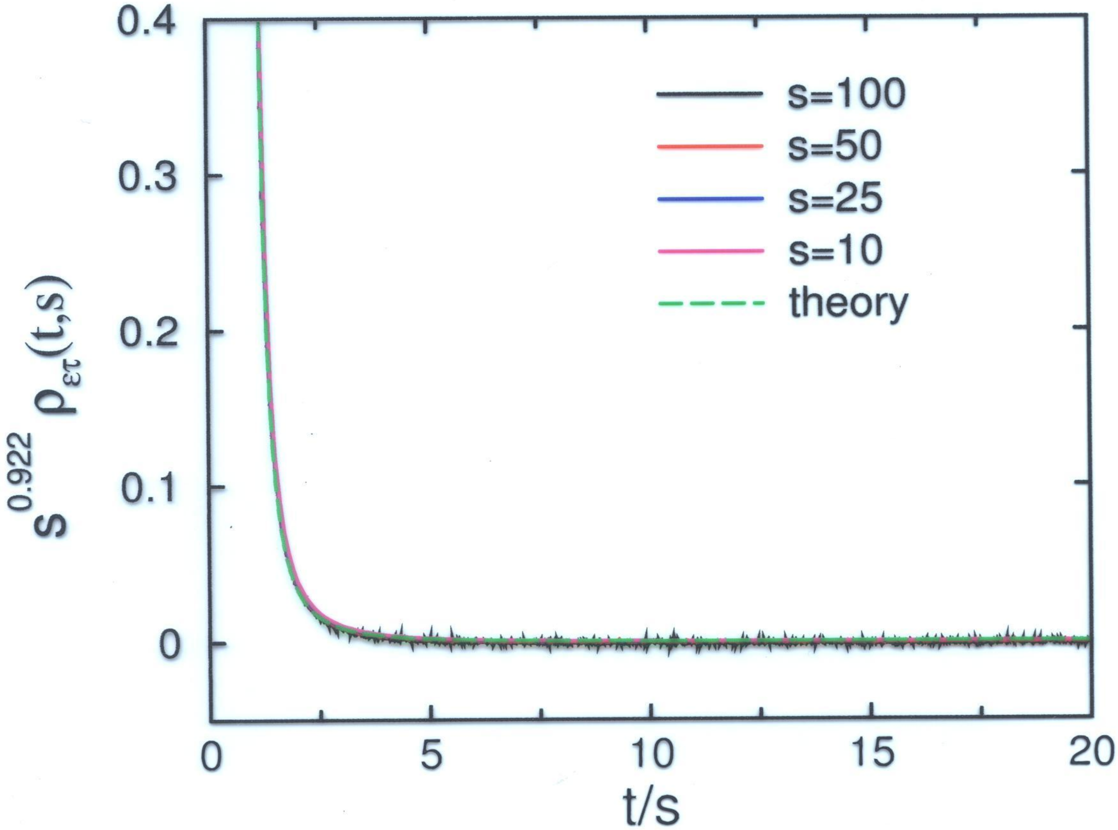
2d Ising



Response of the energy to a temperature fluctuation

$a = 2\,(1 - \alpha)/(\nu\, z)$

2d Ising, $a = 0.922$, $\lambda_R/z = 2.4$



HOWEVER:

Field-theoretical calculations yield for the $O(N)$ model at two-loops corrections to the prediction of local scale invariance (Calabrese/Gambassi '02)
($c = \frac{2-\eta}{z} - 1$ and $\theta = \frac{d}{z} - \frac{\lambda_R}{z} - c$)

$$R_{q=0}(t,s) = A_R\,(t-s)^c \left(\frac{t}{s}\right)^\theta F_R(s/t)$$

with

$$F_R(v) = 1 + \varepsilon^2 \frac{3(N+2)}{8(N+8)^2}\,[f(v) - f(0)]$$

Two remarks:

1) The correction term is extremely small for the Ising model ($N = 1$)

$$0 \geq F_R(v) - 1 \geq -0.07\,\varepsilon^2$$

2) In our work we studied the autoresponse $R_{x=0}(t,s)$, whereas Calabrese and Gambassi studied the long-wave-limit of the Fourier transform of the space-time response

3) The simulations are based on a master equation, whereas Calabrese and Gambassi start from a Langevin equation. But is is known (Bray '90, Bray/Derrida '95) for the one-dimensional Ising model at $T = 0$ that the time-dependent Ginzburg-Landau equation yields results which differ from the exact ones

$T < T_c$

Phys. Rev. Lett. **87**, 265701 (2001)
Europhys. Lett. **62**, 664 (2003)
Phys. Rev. Lett. **90**, 099602 (2003)
Phys. Rev. E **69**, 056109 (2004)

Thermoremanent magnetization

$$M_{TRM}(t,s)/h = \int_0^s du\, R(t,u) = r_0\, s^{-a}\, f_M(t/s)$$

with $a = 1/z = 1/2$

More complete scaling form for the thermoremanent magnetization
(W. Zippold, R. Kühn, and H. Horner '00)

$$\begin{aligned} M_{TRM}(t,s)/h &= \int_0^s du\, R(t,u) \\ &= r_0\, s^{-a}\, f_M(t/s) + r_1\, s^{-\lambda_R/z}\, g_M(t/s) \end{aligned}$$

This scaling form describes the cross-over between two power-law regimes governed by the exponents a and λ_R/z, respectively

Example:
2d Ising model with $T < T_c$: $a = 1/2$, $\lambda_R/z = 0.63$

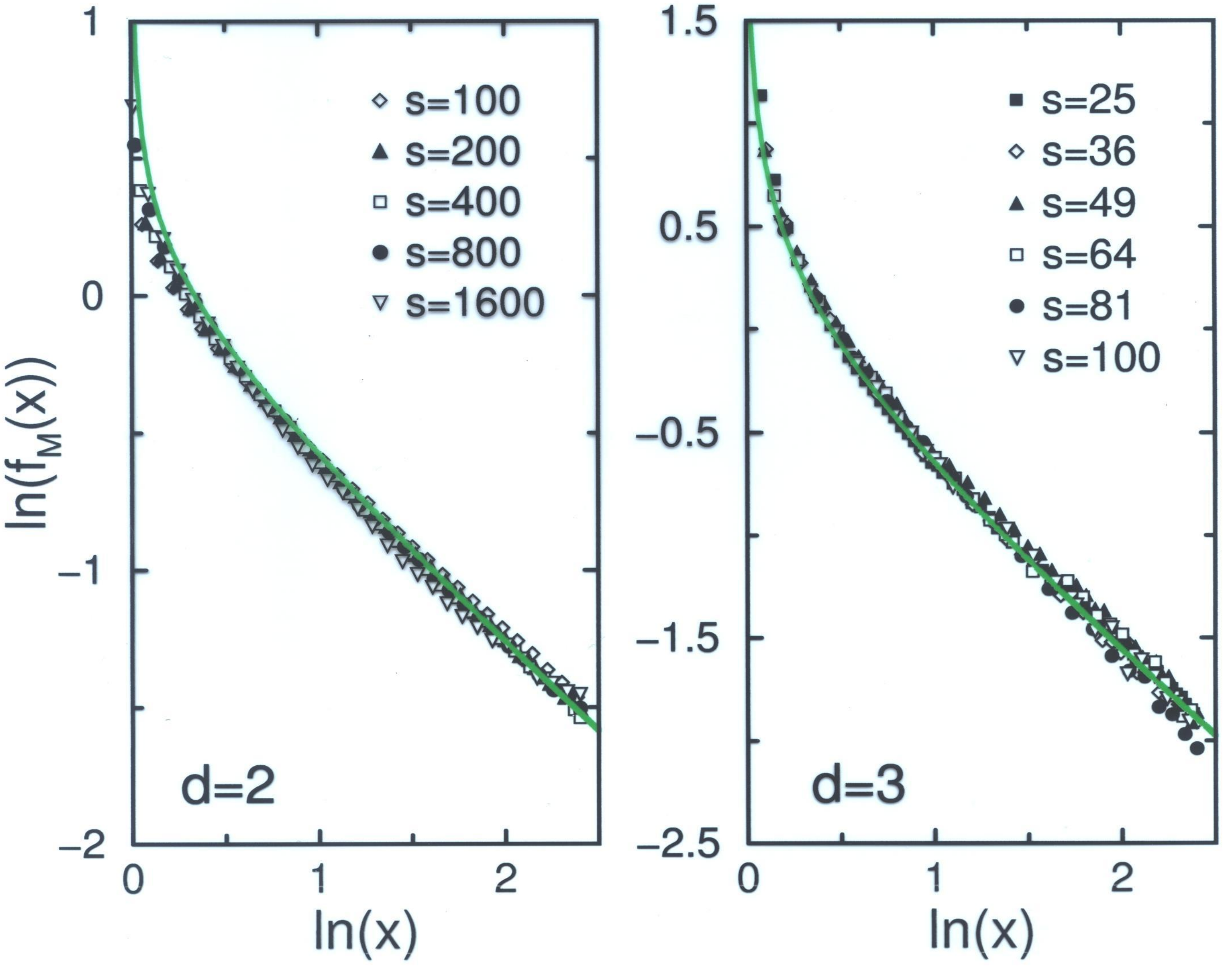
If local scale invariance holds, the scaling functions read

$$f_M(x) = x^{-\lambda_R/z}\, {}_2F_1\left(1+a, \frac{\lambda_R}{z} - a; \frac{\lambda_R}{z} - a + 1; x^{-1}\right)$$

$$g_M(x) \approx x^{-\lambda_R/z}$$

$$M_{TRM}(t,s)/h = r_0\, s^{-a}\, f_M(t/s) + r_1\, s^{-\lambda_R/z}\, g_M(t/s)$$

Determination of the scaling function $f_M(t/s)$ and comparison with the theoretical prediction $(T = 0.66\, T_c)$



The space-time response

Phys. Rev. E **68**, 065101(R) (2003)

$$R(t,s;\mathbf{r}-\mathbf{r}') = \left.\frac{\delta\langle\sigma_{\mathbf{r}}(t)\rangle}{\delta h_{\mathbf{r}'}(s)}\right|_{h=0}$$

Prediction coming from local scale invariance for the case $z = 2$

$$R(t,s;\mathbf{r}-\mathbf{r}') = R(t,s)\,\exp\left(-\frac{\mathcal{M}}{2}\frac{(\mathbf{r}-\mathbf{r}')^2}{t-s}\right)$$

where $R(t,s)$ is the autoresponse function and $\mathcal{M}$ is a direction-dependent non-universal constant

Scaling form of the temporally integrated response function:

$$M_{TRM}(t,s;\mathbf{r}) = r_0 s^{-a} F_0\left(\frac{t}{s},\mathcal{M}\frac{r^2}{s}\right) + r_1 s^{-\lambda_R/z} G_0\left(\frac{t}{s},\mathcal{M}\frac{r^2}{s}\right)$$

The scaling functions F_0 and G_0 are again known explicitly:

$$F_0(x,y) = \int_0^1 \mathrm{d}v\,\exp[-y/2(x-1+v)]\,h_R(x,v)$$

$$G_0(x,y) \simeq x^{-\lambda_R/z} e^{-y/2x}$$

with

$$h_R(x,v) := f_R\left(\frac{x}{1-v}\right)(1-v)^{-1-a}$$

Scaling of the spatially and temporally integrated response function

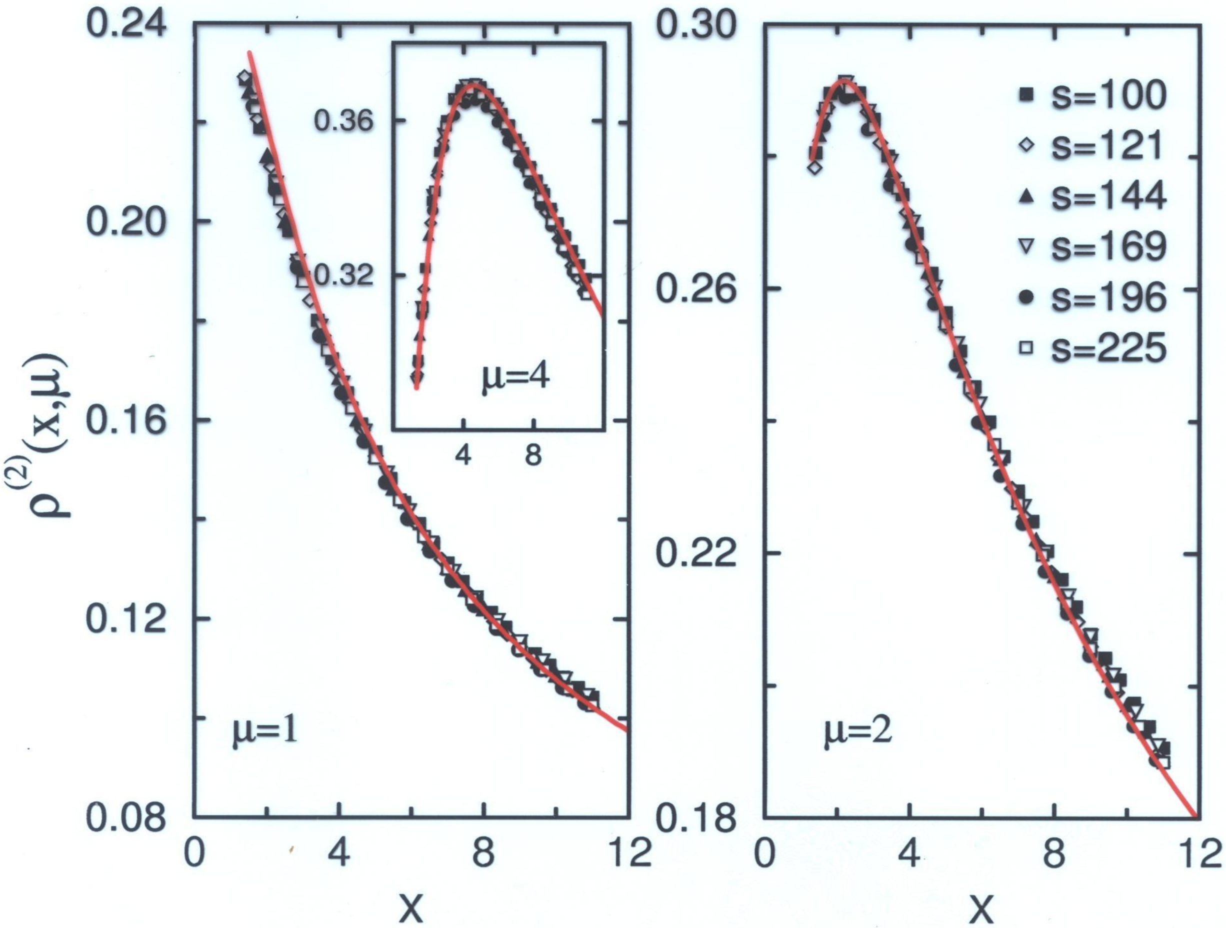
$$\varrho(t,s;\mu)/h = T\int_0^s \mathrm{d}u \int_0^{\sqrt{\mu s}} \mathrm{d}r\, r^{d-1} R(t,u;\mathbf{r})$$

$$= r_0\, s^{d/2-a}\, \varrho^{(2)}(t/s,\mu) + r_1\, s^{d/2-\lambda_R/z}\, \varrho^{(3)}(t/s,\mu)$$

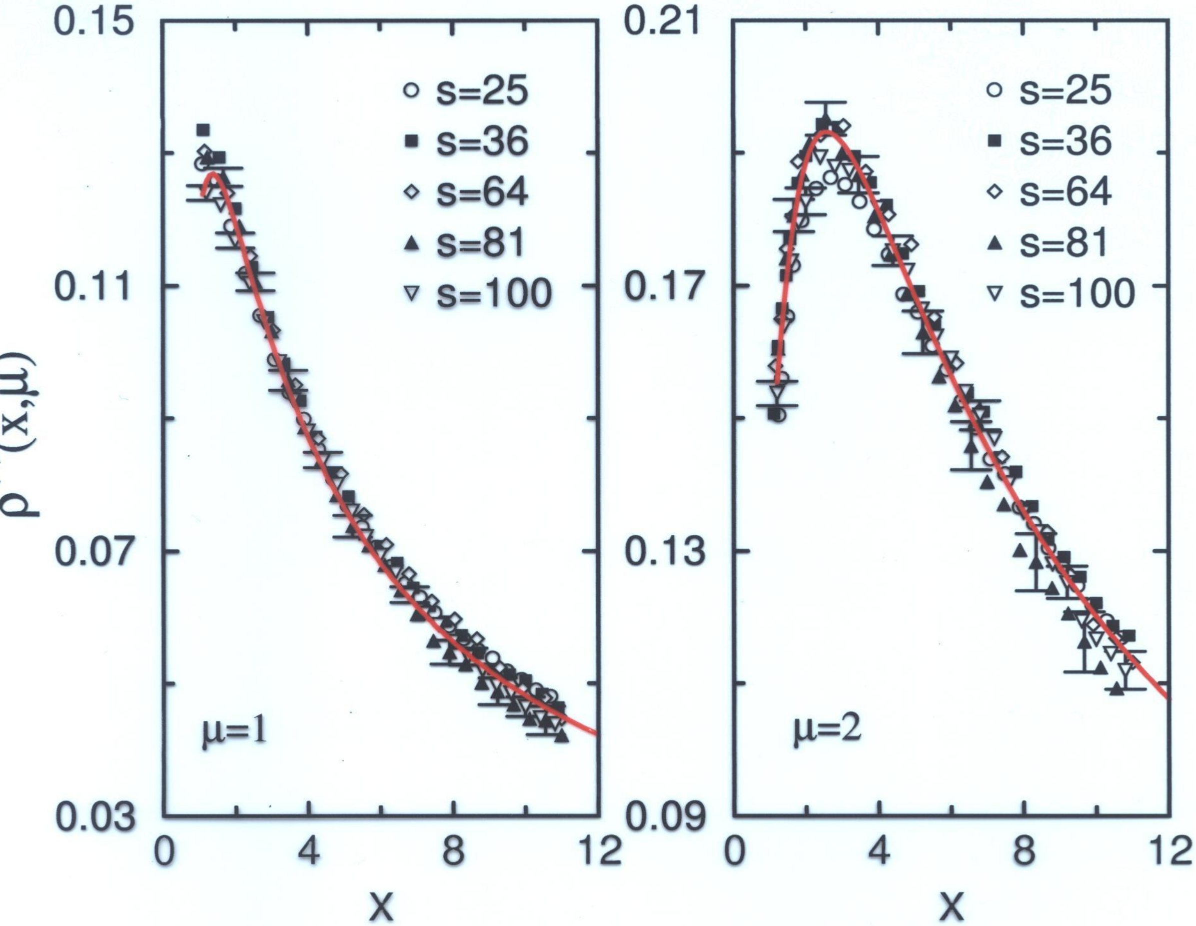
where the space integral is along a straight line of length $\Lambda = \sqrt{\mu s}$

The scaling functions $\varrho^{(2)}(t/s,\mu)$ and $\varrho^{(3)}(t/s,\mu)$ are again known explicitly

2d Ising model and $T = 0.66\,T_c$



3d Ising model and $T = 0.66\, T_c$



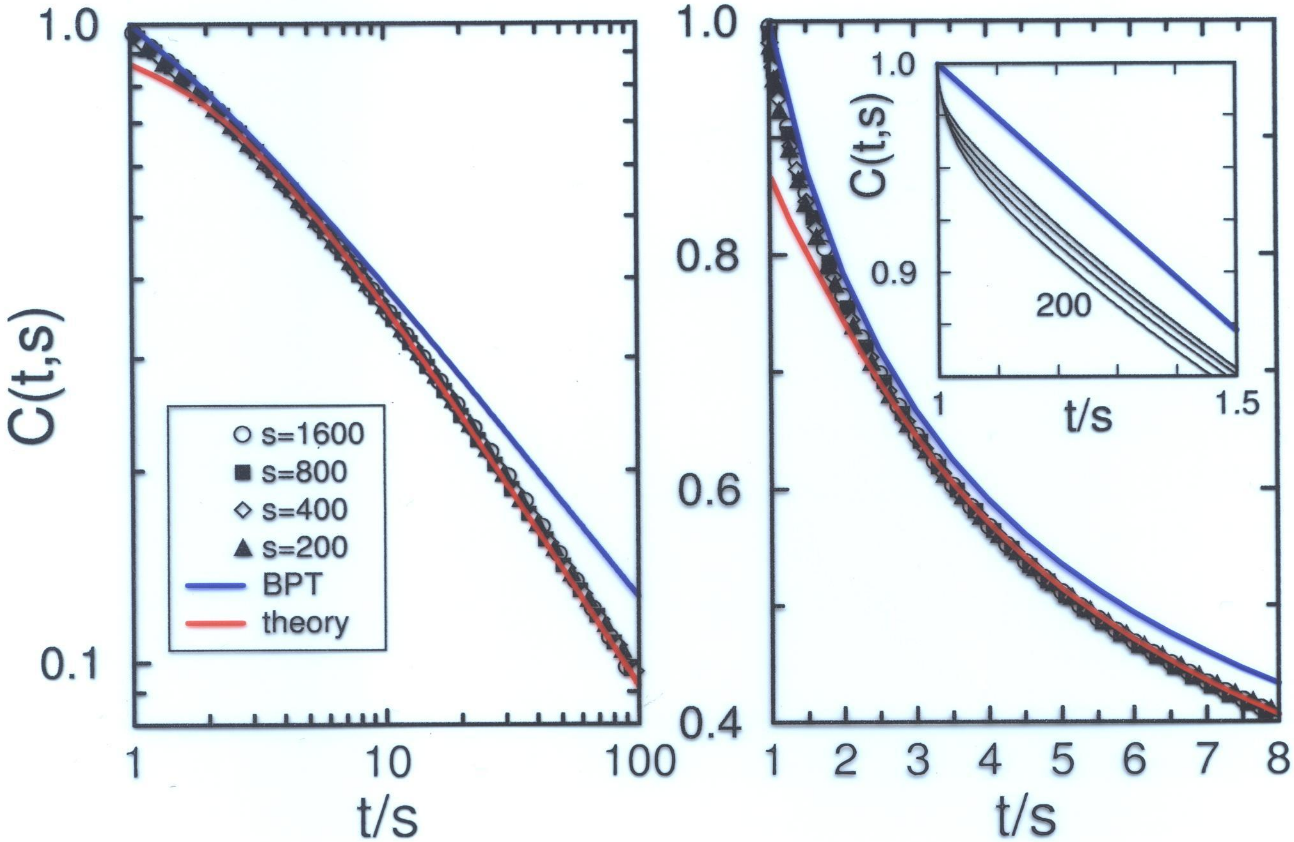
Local scale invariance is a true space-time symmetry of statistical systems undergoing phase ordering kinetics

Autocorrelation with $z = 2$

submitted to Europhys. Lett., cond-mat/0404464

$$C(t,s) = M_{\rm eq}^2\, f_C(t/s)$$

two-dimensional Ising model



Conclusions

Ageing phenomena taking place in nonequilibrium systems far from equilibrium

Here: study of two-time functions

- autoresponse
- spatial-temporal response
- autocorrelation

The dynamical scale invariance realized in nonequilibrium ageing phenomena can be generalized towards local scale invariance

$\Longrightarrow$ explicit predictions for the corresponding scaling functions

Numerical tests in various models (spherical models, Ising models with non-conserved and conserved order parameter, three-dimensional XY model, ...) at $T \leqslant T_c$ are in complete agreement with the theoretical predictions

Local scale invariance seems to be a true space-time symmetry of statistical systems undergoing ageing