HIGHER ORDER PHASE TRANSITIONS

R. KENNA

MOTIVATION

EHRENFEST CLASSIFICATION: Order of transition is order of derivative of free energy where there first appears a discontinuity (or divergence)

Ex: 1^{st} Order: Solid - Liquid - Vapour

2^{nd} Order: Spontaneous magnetization

1999: Goodrich (Louisiana) + Hall (Tallahassee) experiment on "quirky" superconductor $Ba_{0.6}K_{0.4}BiO_3$ which has $T_c = 32K$

- No singularity in C_v or χ

Kumar (Florida) explained results as a fourth order transition

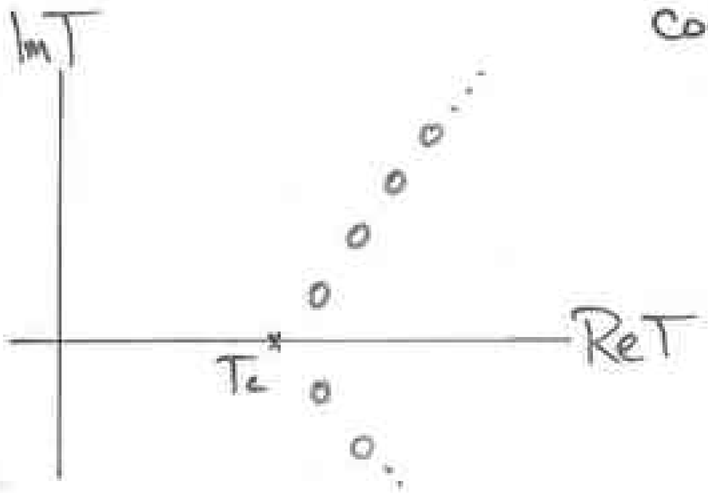
OUR APPROACH: Examine higher order transitions from zeroes point of view.

$$F = \ln Z$$

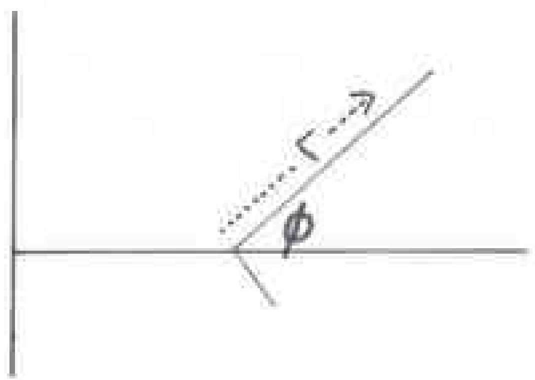
↑ Non-analytic if $Z(T) = 0$

↑ Can happen for complex T

$L < \infty$



$L = \infty$



$T(r) = T_c + r e^{i\phi}$

⇝ 2 things { Impact angle ϕ ; Density $g(r)$ }

FISHER ZEROES (Partition function is $Z(T)$)

$$f = \frac{1}{V} \ln Z(T)$$

$$Z(T) = \prod_j (T - T_j)$$

$$\Rightarrow \quad f = \frac{1}{V} \sum_j \ln (T - T_j)$$

$$= \frac{1}{V} \int_0^R \sum_j \ln (T - T(r))\, \delta(r - r_j)\, dr$$

$L \to \infty$

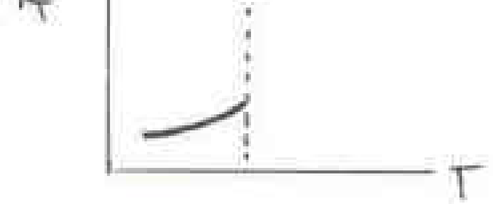
$$= \int_0^R \ln (T - T(r))\, g(r)\, dr \quad + cc$$

For an m^{th} order transition,

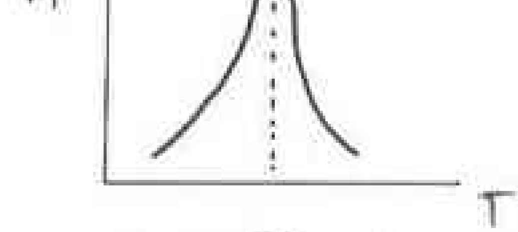
$f, \; f_T^{(1)}, \; f_T^{(2)}, \dots \; f_T^{(m-1)}$ are continuous

but $f_T^{(m)}$ has singularity

consider



DISCONTINUOUS



DIVERGING

Discontinuous m^{th} Order Transitions

$f(T)$ is continuous and

$$f_+(T) - f_-(T) = \sum_{n=1}^{\infty} c_n (T-T_c)^n$$

If $c_1 = c_2 = \ldots = c_{m-1} = 0$ and $c_m \neq 0$

transition is m^{th} order with $\Delta f^{(m)} = m!\, c_m$

Now,
$$\mathrm{Re}\, f_+(T) = \mathrm{Re}\, f_-(T)$$

$$\Rightarrow \quad \mathrm{Re} \sum_{n=1}^{\infty} c_n (T-T_c)^n = 0$$

Along transition, $T = T_c + r e^{i\phi}$

$$\Rightarrow \quad \mathrm{Re} \sum_{n=1}^{\infty} c_n r^n e^{in\phi} = 0$$

But $c_1 = c_2 = \ldots = c_{m-1} = 0$ already

$$\Rightarrow \quad c_m r^m \cos m\phi = 0$$

~~Ket~~

$$\Rightarrow \quad \phi = \frac{(2l+1)\pi}{2m} \qquad l = 0, 1, \ldots 2m-1$$

m	Permitted values of impact angle, ϕ																
1									$\frac{\pi}{2}$								
2					$\frac{\pi}{4}$								$\frac{3\pi}{4}$				
3				$\frac{\pi}{6}$					$\frac{\pi}{2}$					$\frac{5\pi}{6}$			
4			$\frac{\pi}{8}$				$\frac{3\pi}{8}$				$\frac{5\pi}{8}$				$\frac{7\pi}{8}$		
5		$\frac{\pi}{10}$				$\frac{3\pi}{10}$			$\frac{\pi}{2}$			$\frac{7\pi}{10}$				$\frac{9\pi}{10}$	
6	$\frac{\pi}{12}$				$\frac{\pi}{4}$			$\frac{5\pi}{12}$		$\frac{7\pi}{12}$			$\frac{3\pi}{4}$				$\frac{11\pi}{12}$

Table 1: Impact angles permitted at a discontinuous phase transition of order m. For a given m-value, $2m$ different impact angles are allowed. Those impacting from the upper half plane are listed here.

Vertical impact at 2nd order discontinuous trans^{n} is not allowed

Vertical impact only allowed at trans^{n} of odd order

DENSITY: It turns out that also need

$$g(r) = G\, r^{m-1} \quad \text{for } m^{th} \text{ order transition.}$$

This gives $\Delta f^{(m)} = (-1)^{l}\, 2\pi G\, (m-1)!$

(if $m=1$ this recovers $\Delta f^{(1)} = 2\pi G$)

DIVERGENT TRANSITIONS

Recall in 2^{nd} order case $\quad C(t) = C_{\pm}|t|^{-\alpha_{\pm}}$

$$\alpha_+ = \alpha_- \qquad C_+ \neq C_-$$

It is known that $\quad g(r) \propto r^{1-\alpha}$

For an m^{th} order transition, $\quad f_t^{(m)} \sim |t|^{-\mu}$

$$f(t) \sim \int_0^R \ln(T - T(r))\, g(r)\, dr$$

$$\Rightarrow \quad f_t^{(m)} \sim \int_0^R \frac{g(r)}{(T - T_c - re^{i\phi})^m}\, dr$$

$$\Rightarrow \quad |t|^{-\mu} \sim \int_0^{\frac{R}{t}\sim\infty} \frac{g(r't)}{t^m (1 - r'e^{i\phi})^m}\, t\, dr'$$

$$\Rightarrow \quad g(r) \sim r^{m-1-\mu}$$

In fact the integral depends on whether

$$t \gtrless 0.$$

One finds that if $f_{\pm}^{(m)}(t) = M_{\pm} |t|^{-\mu_{\pm}}$

$$\mu_{+} = \mu_{-}$$

$$\frac{M_{+}}{M_{-}} = \frac{\cos((m-\mu)\phi + \mu\pi)}{\cos((m-\mu)\phi)}$$

~ This recovers a known result in 2^{nd} order case $(m = 2,\ \mu = \alpha)$

~ Critical amplitudes coincide if

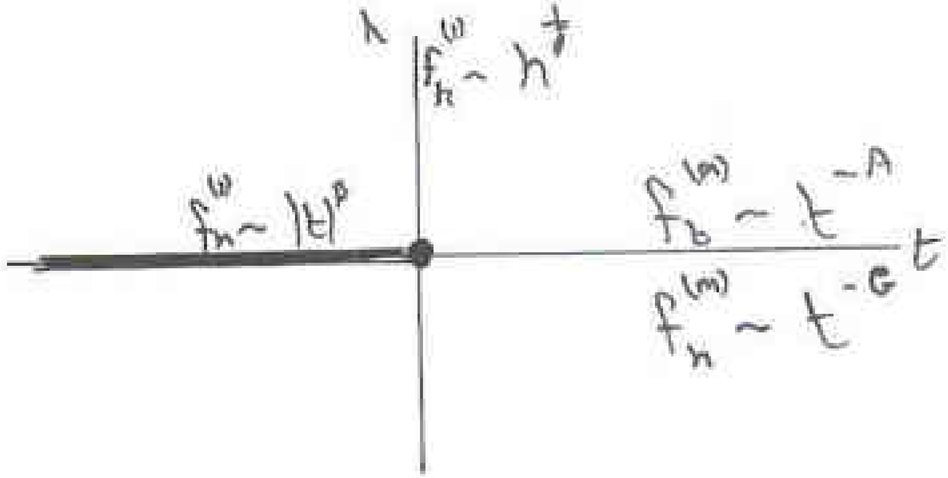
$$\phi = \frac{\pi}{2} \frac{2n-\mu}{2m-\mu} \qquad n \in \mathbb{Z}$$

~ If $m = 2n$ (even), impact angle of $\frac{\pi}{2}$ results in symmetry of amplitudes

(this was known for $m = 2$

We see it extends to any even m)

Lee-Yang Zeroes ($Z = Z(H,T)$ or $Z(h,t)$)



In 2^{nd} order case

$$f_t^{(2)} = C_v \sim t^{-\alpha} \qquad A = \alpha$$

$$f_{th}^{(2)} = \chi \sim t^{-\gamma} \qquad G = \gamma$$

Kumar et al $\Rightarrow$ $(m-1)A + m\beta + G = m(m-1)$

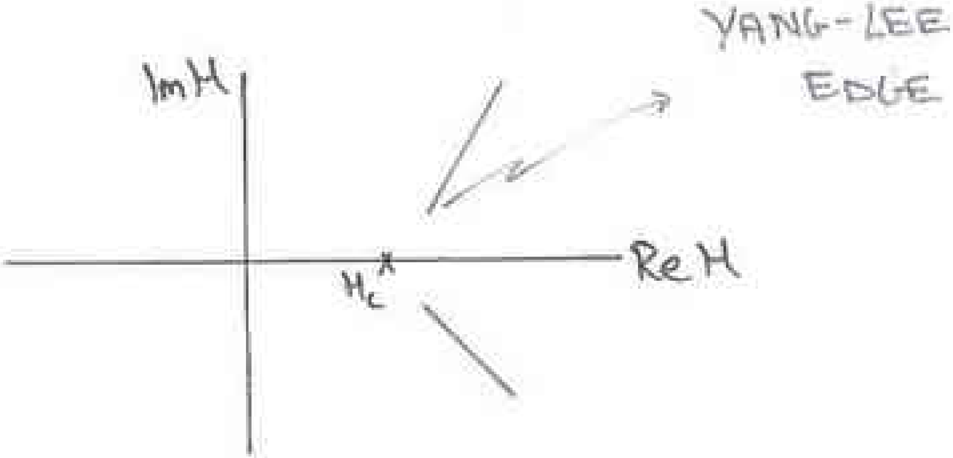
$$G = \beta\left[\delta(m-1) - 1\right]$$

cf

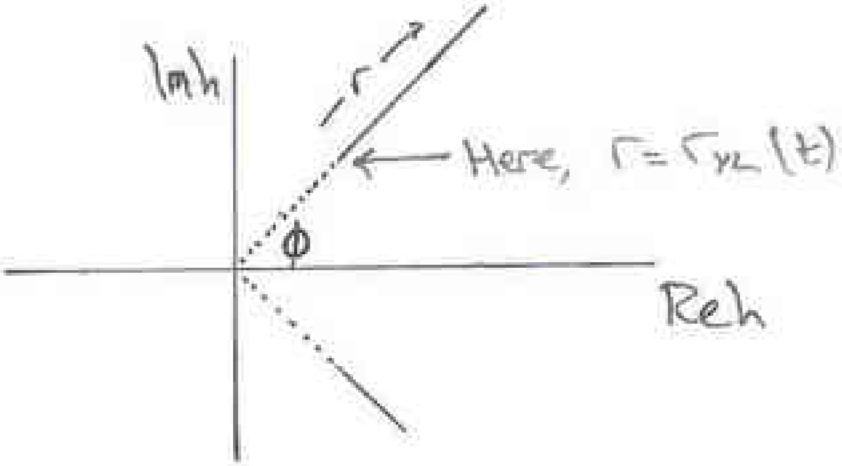
$$\alpha + 2\beta + \gamma = 2 \qquad \text{Rushbrooke}$$

$$\gamma = \beta(\delta - 1) \qquad \text{Griffiths}$$

FOR LEE-YANG ZEROS, THERE IS AN EDGE TO THEIR DISTRIBUTION



In the h-plane,



$$f(h,t) = \int_{r_{YL}(t)}^{R} \ln(h - h(r,t))\, g(r,t)\, dr$$

where $h(r,t) = r e^{i\phi(r,t)}$

$g(r,t) = 0$ if $r < r_{YL}(t)$

$$\Rightarrow \quad f_h^{(m)}(h,t) \sim \int_{r_{yL}}^{R} \frac{g(r,t)}{(h-h(r,t))^m}\, dr \; + cc$$

Input $\quad f_h^{(m)}(h=0,t) \sim t^{-G}$

$$\Rightarrow \quad g(r,t) = t^{-G}\, r_{yL}(t)^{m-1}\, \Phi\left(\frac{r}{r_{yL}(t)}\right)$$

Input $\quad f_h^{(1)}(h,t=0) \sim h^{\frac{1}{\delta}}$

$$\Rightarrow \quad r_{yL}(t) \sim t^{\frac{G\delta}{(m-1)\delta-1}}$$

Input $\quad f_h^{(1)}(h=0,t) \sim |t|^{\beta}$

and $\quad f_h^{(m)}(h=0,t) \sim t^{-A}$

$$\Rightarrow \quad G = \beta[\delta(m-1)-1] \qquad \text{(Griffiths)}$$

$$(m-1)A + m\beta + G = m(m-1) \qquad \text{(Rushbrooke)}$$

So we have

- recovered Kumar et al's scaling laws

- Found

$$r_{yc}(t) \sim t^{\frac{G\delta}{(m-1)\delta - 1}}$$

$$g(r,t) \sim r^{\frac{G}{(m-1)\delta - 1}} \Phi\left(\frac{r}{r_{yc}(t)}\right)$$

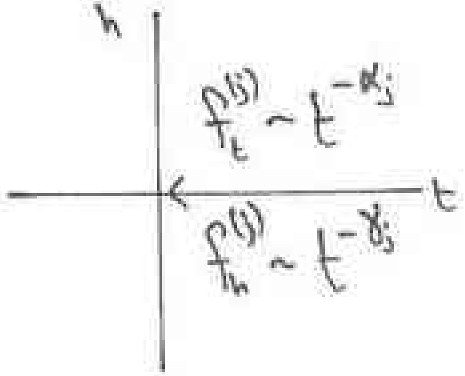
$$\text{probably } r^{\frac{G}{(m-1)\delta - 1}} \left(1 - \frac{r}{r_{yc}}\right)^{x}$$

- Also find

$$f(h,t) = t^{G\frac{\delta+1}{(m-1)\delta - 1}} \psi\left(\frac{h}{r_{yc}(t)}\right)$$

From $f(h,t) = t^{G\frac{\delta+1}{(m-1)\delta-1}} \mathcal{Y}_\phi\left(\frac{h}{r_{YL}(t)}\right)$

can recover usual scaling laws at m^{th} order trans^{n}



Suppose $f_h^{(j)} \sim t^{-\gamma_j}$

(γ_j can be negative)

$$\Rightarrow \quad f_h^{(j)} = t^{G\frac{\delta+1}{(m-1)\delta-1}} \frac{1}{r_{YL}^{j}} \mathcal{Y}^{(j)}\left(\frac{h}{r_{YL}}\right)$$

$$\sim t^{\beta-\beta\delta(j-1)}$$

$$\Rightarrow \quad \gamma_j = \beta\delta(j-1)-\beta$$

e.g. $\gamma_2 \equiv \gamma = \beta\delta - \beta$ (Griffiths)

Similarly $(j-1)\alpha_j + j\beta + \gamma_j = j(j-1)$

e.g. $\alpha + 2\beta + \gamma = 2$

Ex of Scaling Laws

Ising + 2D Gravity or 3D Spherical model has

$\alpha = -1 \qquad \beta = \frac{1}{2} \qquad \gamma = 2$

↓

$C_v \sim t^1$ is continuous, Trans^{n} = 3rd order

This fits into usual Rushbrooke law:

$$\alpha + 2\beta + \gamma = 2$$

On other hand, $f_t^{(2)} = \frac{\partial C_v}{\partial t} \sim t^0 \Rightarrow A = 0$

From $f(h,t) = t^{G\frac{\delta+1}{(m-1)\delta-1}} \mathcal{F}_\phi\left(\frac{h}{r_{hc}}\right)$ can show $G = \frac{9}{2}$

$A = 0 \qquad \beta = \frac{1}{2} \qquad G = \frac{9}{2}$

fits into

$$(m-1)A + m\beta + G = m(m-1)$$

SUMMARY

- For Fisher zeroes at discontinuous m^{th} order transition,

$$\phi = \frac{(2l+1)\pi}{2m}, \quad g(r) = G r^{m-1}, \quad \Delta f^{(m)} = 2\pi G (m-1)!$$

- For Fisher zeroes at divergent m^{th} order transition

If $f_{\pm}^{(m)}(t) = M_{\pm} |t|^{-\mu_{\pm}}$

$$\mu_+ = \mu_-, \quad \frac{M_+}{M_-} = \frac{\cos((m-\mu)\phi + \mu\pi)}{\cos((m-\mu)\phi)},$$

$$g(r) \sim r^{m-1-\mu}$$

- For LY zeroes at m^{th} order transition various scaling laws are recovered.