# The nucleon in a finite volume and in chiral perturbation theory

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goal:
understand the quark mass (pion mass) dependence
and
the volume dependence of the nucleon mass

- Introduction
- Calculations in chiral effective field theory
  dependence of the nucleon mass on the pion mass and the volume
- Lüscher's formula
- Monte Carlo data ($N_f = 2$)
- Comparison with chiral perturbation theory
- Side remarks
- Comparison with quenched data
- Summary and outlook

see hep-lat/0312030
to appear in Nucl. Phys. B

# Introduction

QCD at low energies: non-perturbative

- models
- lattice regularisation & Monte Carlo simulations
- chiral effective field theories (chEFT)
  low-energy theories incorporating the constraints from (spontaneously broken) chiral symmetry

## lattice regularisation & Monte Carlo simulations

no free parameters, except for the quark masses

In general, results from lattice simulations must be extrapolated in several variables before they can be compared with reality:

- lattice spacing $a \to 0$ (continuum limit)
  $a \searrow$ simulation costs $\nearrow$
- box size $L \to \infty$ (thermodynamic limit)
  $L \nearrow$ simulation costs $\nearrow$
- pseudoscalar mass $m_\pi \to m_\pi^{\rm phys}$ ("chiral limit")
  $m_\pi \searrow$ simulation costs $\nearrow$

## chiral effective field theories

(complicated) parametrisation close to certain limits:

- lattice spacing $a \to 0$ (continuum limit)
  $a \neq 0$ only recently implemented, usually considered in the continuum ($a = 0$)
- box size $L \to \infty$ (thermodynamic limit)
- pseudoscalar mass $m_\pi \to 0$ (chiral limit)

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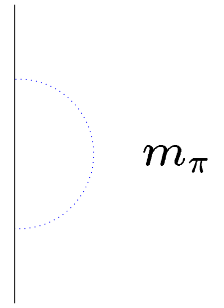
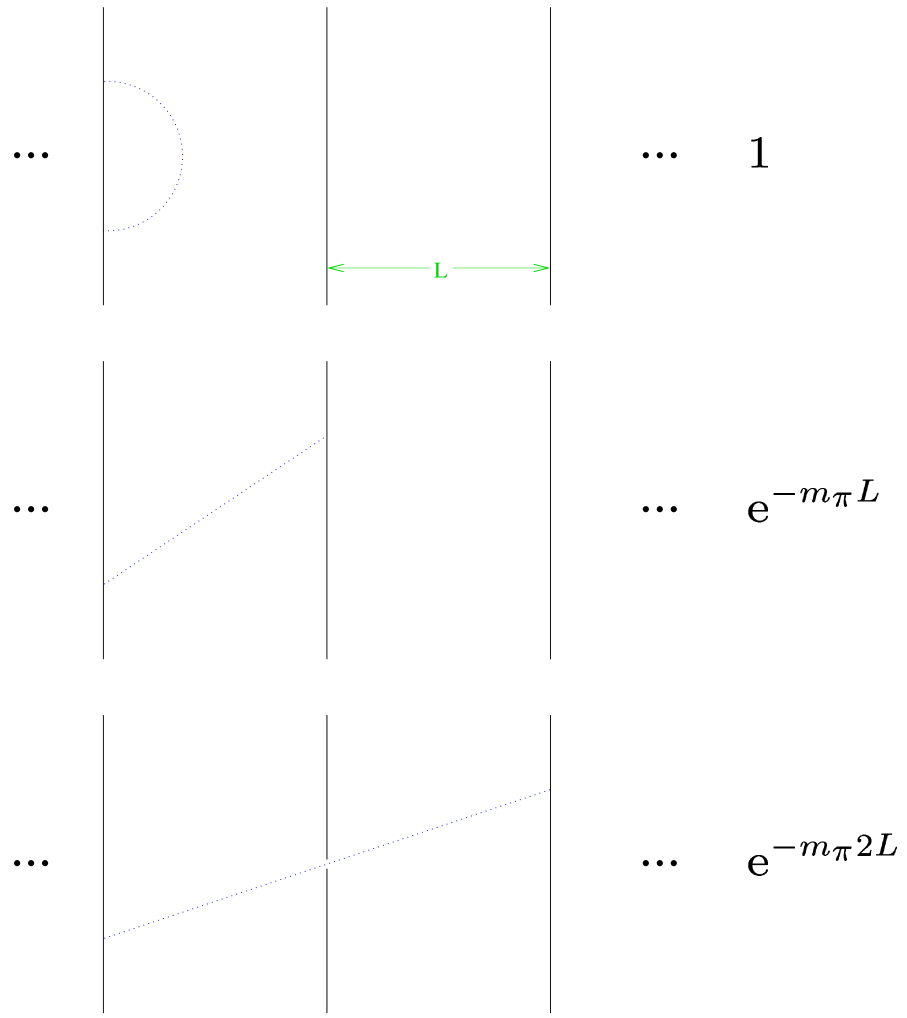
Why do chiral effective field theories contain information about finite size effects?

originally used in the infinite volume for $m_\pi \to 0$

nucleon propagator in the infinite volume:




finite volume with periodic boundary conditions:




The exchanged particle (pion) travels once, twice, . . . around the "world".

# Calculations in chiral effective field theory

relativistic $SU(2)_f$ baryon chiral perturbation theory
(Becher and Leutwyler, Eur. Phys. J. C9 (1999) 643)

pion field $U(x) \in SU(2)$ nucleon Dirac field $\Psi(x)$

effective Lagrangian up to $O(p^4)$:

$$\mathcal{L} = \mathcal{L}_N^{(1)} + \mathcal{L}_N^{(2)} + \mathcal{L}_N^{(3)} + \mathcal{L}_N^{(4)} + \mathcal{L}_\pi^{(2)}$$

$$\begin{aligned}
\mathcal{L}_N^{(1)} &= \bar{\Psi}\left(i\gamma_\mu D^\mu - m_0\right)\Psi + \tfrac{1}{2} g_A \bar{\Psi}\gamma_\mu\gamma_5 u^\mu \Psi \\
\mathcal{L}_N^{(2)} &= c_1 \mathrm{Tr}(\chi_+)\bar{\Psi}\Psi - \frac{c_2}{4m_0^2}\mathrm{Tr}(u_\mu u_\nu)(\bar{\Psi} D^\mu D^\nu \Psi + \mathrm{h.c.}) \\
&\quad + \frac{c_3}{2}\mathrm{Tr}(u_\mu u^\mu)\bar{\Psi}\Psi + \cdots \\
\mathcal{L}_N^{(4)} &= -\frac{e_1}{16}\left(\mathrm{Tr}(\chi_+)\right)^2 \bar{\Psi}\Psi + \cdots \\
\mathcal{L}_\pi^{(2)} &= \frac{f_\pi^2}{4}\mathrm{Tr}\left[(\partial_\mu U^\dagger)(\partial^\mu U) + \chi U^\dagger + \chi^\dagger U\right]
\end{aligned}$$

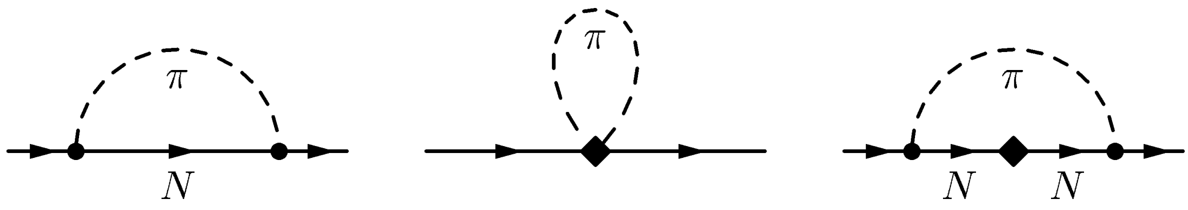
$u^2 = U$, $u_\mu = \mathrm{i} u^\dagger \partial_\mu U u^\dagger$, $\Gamma_\mu = \frac{1}{2}[u^\dagger, \partial_\mu u]$, $D_\mu = \partial_\mu + \Gamma_\mu$,

$\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u$, $\chi = 2B\,\mathrm{diag}(m_u, m_d)$, $B = -\langle \bar{q} q \rangle / f_\pi^2$

all constants in the chiral limit, isospin breaking will be neglected:
$m_u = m_d = m_q$

nucleon propagator

$$\mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px}\langle 0|T\Psi(x)\bar{\Psi}(0)|0\rangle = \frac{1}{m_0+\Sigma(p)-\not{p}}$$




$(a)$ $(b)$ $(c)$

order $p^3$ order $p^4$

solid circle: vertex from $\mathcal{L}_N^{(1)}$, diamond: vertex from $\mathcal{L}_N^{(2)}$

self energy:

$$\Sigma = \underbrace{-4c_1 m_\pi^2}_{p^2} + \underbrace{\Sigma_a}_{p^3} + \underbrace{\Sigma_b+\Sigma_c+e_1 m_\pi^4}_{p^4} + O(p^5)$$

infrared regularisation (Becher, Leutwyler) for one-loop integrals

mass formula at $O(p^3)$:

$$m_N = m_0 - 4c_1 m_\pi^2 + \left[e_1^r(\lambda) + \frac{3g_A^2}{64\pi^2 f_\pi^2 m_0}\left(1-2\ln\frac{m_\pi}{\lambda}\right)\right] m_\pi^4$$
$$-\frac{3g_A^2}{16\pi^2 f_\pi^2}\, m_\pi^3 \sqrt{1-\frac{m_\pi^2}{4m_0^2}}\left[\frac{\pi}{2}+\arctan\frac{m_\pi^2}{\sqrt{4m_0^2 m_\pi^2 - m_\pi^4}}\right]$$

$\lambda$: renormalisation scale, $m_\pi^2 = 2Bm_q$

Procura, Hemmert, Weise, hep-lat/0309020 evaluation of the theory in a finite (spatial) volume: integral over the (spatial components of the) loop momenta replaced by a sum over the discrete set of momenta allowed by the boundary conditions

periodic boundary conditions for box length $L$: $p_i = \frac{2\pi}{L}\ell_i \, , \ell_i \in \mathbb{Z}$

example for a one-loop integral in $d$ Euclidean dimensions:

$$\Gamma(r) \int \frac{\mathrm{d}^d p}{(2\pi)^d} (p^2 + M^2)^{-r} \to \frac{\Gamma(r)}{L^d} \sum_p (p^2 + M^2)^{-r}$$

$\sum_p$ can be rewritten with the help of the Poisson summation formula:

$$\frac{1}{L^d} \sum_p H(p) = \sum_{n \in \mathbb{Z}^d} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, \mathrm{e}^{\mathrm{i}p \cdot nL} H(p)$$

example

scalar propagator in the infinite volume:

$$G(x) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{p^2 + M^2}$$

scalar propagator in a finite volume with periodic boundary conditions:

$$G_L(x) = \sum_{n \in \mathbb{Z}^d} G(x + nL)$$

$$\frac{1}{L^d} \sum_p \frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{p^2 + M^2} = \sum_n \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, \mathrm{e}^{\mathrm{i}p\cdot nL} \frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{p^2 + M^2}$$

$n_i$: number of times the particle (pion) crosses the “boundary” of the box in the $i$ direction

Poisson summation formula:

$$\frac{1}{L^d}\sum_p H(p) = \sum_{n\in\mathbb{Z}^d}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}p\cdot nL}H(p)$$

$$= \int\frac{\mathrm{d}^d p}{(2\pi)^d}\,H(p) + \sum_n{}'\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}p\cdot nL}H(p)$$

$\sum_n{}'$: omit $n=0$

$\Rightarrow$ "finite volume - infinite volume" finite and unambiguous (in a large class of regularisations)

$$\frac{1}{L^d}\sum_p H(p) - \int\frac{\mathrm{d}^d p}{(2\pi)^d}\,H(p) = \sum_n{}'\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}p\cdot nL}H(p)$$

Fourier transform of $H$

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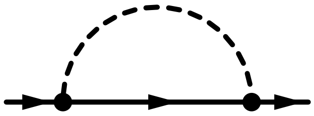
$\rightarrow$ $p$ expansion: $m_\pi = O(p)$, $L^{-1} = O(p)$ $\quad\Rightarrow m_\pi L = O(p^0)$

see: Gasser and Leutwyler, Phys. Lett. B184 (1987) 83
Hasenfratz and Leutwyler, Nucl. Phys. B343 (1990) 241

contribution of graph (a)  to the nucleon mass in Euclidean notation:

infrared regularisation

$$m_a = D \int_0^{\infty} \mathrm{d}x \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left[p^2 + m_0^2 x^2 + m_\pi^2 (1-x)\right]^{-2}$$

with $D = \dfrac{3 g_A^2 m_0 m_\pi^2}{2 f_\pi^2}$

spatial box of length $L$, infinite extent in time direction
$\rightarrow$ finite size effect of the nucleon mass at $O(p^3)$:

$$m_N(L) - m_N(\infty) = \Delta_a(L)$$

$$= D \int_0^{\infty} \mathrm{d}x \int \frac{\mathrm{d}p_4}{2\pi} \left[ \frac{1}{L^3} \sum_{\vec{p}} \left(\vec{p}^{\,2} + p_4^2 + m_0^2 x^2 + m_\pi^2 (1-x)\right)^{-2} - \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\vec{p}^{\,2} + p_4^2 + m_0^2 x^2 + m_\pi^2 (1-x)\right)^{-2} \right]$$

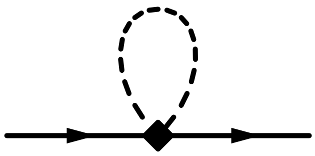
work out the Fourier integrals:

$$\Delta_a(L) = \frac{3 g_A^2 m_0 m_\pi^2}{16 \pi^2 f_\pi^2} \int_0^{\infty} \mathrm{d}x \sum_{\vec{n}}{}' K_0\left(L |\vec{n}| \sqrt{m_0^2 x^2 + m_\pi^2 (1-x)}\right)$$

at $O(p^4)$ two new coupling constants $c_2$ and $c_3$
nucleon mass in the infinite volume:

$$m_N = m_0 - 4c_1 m_\pi^2 - \frac{3g_A^2}{32\pi f_\pi^2} m_\pi^3 + \left[ e_1^r(\lambda) - \frac{3}{64\pi^2 f_\pi^2}\left(\frac{g_A^2}{m_0} - \frac{c_2}{2}\right) - \frac{3}{32\pi^2 f_\pi^2}\left(\frac{g_A^2}{m_0} - 8c_1 + c_2 + 4c_3\right) \ln\frac{m_\pi}{\lambda} \right] m_\pi^4 + \frac{3g_A^2}{256\pi f_\pi^2 m_0^2} m_\pi^5 + O(m_\pi^6)$$

Becher and Leutwyler; Procura, Hemmert and Weise

only graph (b)  contributes

$m_N$ determined by

$$m_0 + \Sigma|_{\not p = m_N} = m_0 - 4c_1 m_\pi^2 + \Sigma_a + \Sigma_b + \Sigma_c + e_1 m_\pi^4|_{\not p = m_N} = m_N$$

finite size effect:

$$\Delta_b(L) = \frac{3m_\pi^4}{4\pi^2 f_\pi^2} \sum_{\vec{n}}{}' \left[ (2c_1 - c_3)\frac{K_1(L|\vec{n}|m_\pi)}{L|\vec{n}|m_\pi} + c_2 \frac{K_2(L|\vec{n}|m_\pi)}{(L|\vec{n}|m_\pi)^2} \right]$$

total finite size effect:

$$m_N(L) - m_N(\infty) = \Delta_a(L) + \Delta_b(L) + O(p^5)$$

# Lüscher's formula (Cargèse 1983)

consider the relative finite size effect

$$\delta_N = \left(m_N(L) - m_N(\infty)\right)/m_N(\infty)$$

leading contribution in terms of the pion-nucleon coupling constant $g_{\pi N}$ and the pion-proton forward elastic scattering amplitude $F_{\pi p}(\nu)$
$\nu = (s-u)/(4m_N)$ with $m_N = m_N(\infty)$ ("crossing variable")

$$\delta_N^{\text{Lüscher}} = \frac{9}{16\pi m_N L}\left(\frac{m_\pi}{m_N}\right)^2 g_{\pi N}^2 \exp\left(-m_\pi L\sqrt{1-\frac{m_\pi^2}{4m_N^2}}\right)$$
$$- \frac{3}{8\pi m_\pi L}\left(\frac{m_\pi}{m_N}\right)^2 \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\exp\left(-m_\pi L\sqrt{1+p^2}\right)F_{\pi p}(\mathrm{i}m_\pi p)$$
$$+ O\left(\mathrm{e}^{-\alpha m_\pi L}\right) \qquad \alpha \geq \sqrt{3/2}$$

decompose $F_{\pi p}(\nu)$ into the pseudovector Born term and a remainder, which can be expressed with the help of the so-called subthreshold expansion:

$$F_{\pi p}(\nu) = \frac{6g_{\pi N}^2}{1-4m_N^2\nu^2/m_\pi^4} + 6m_N\sum_{k=0}^{\infty} d_{k0}^{+}\nu^{2k}$$

Lüscher's formula becomes

$$\delta_N^{\text{Lüscher}} = \frac{9}{8\pi^2}\left(\frac{m_\pi}{m_N}\right)^2 g_{\pi N}^2\left\{\frac{\pi}{2m_N L}\exp\left(-m_\pi L\sqrt{1-\frac{m_\pi^2}{4m_N^2}}\right) - \frac{1}{m_\pi L}\int_{-\infty}^{\infty}\mathrm{d}p\,\frac{\exp\left(-m_\pi L\sqrt{1+p^2}\right)}{1+4m_N^2\,p^2/m_\pi^2}\right\}$$
$$- \frac{9}{4\pi}\frac{m_\pi}{m_N L}\sum_{k=0}^{\infty} m_\pi^{2k} d_{k0}^{+}(-1)^k \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\exp\left(-m_\pi L\sqrt{1+p^2}\right)p^{2k}$$

Lüscher's formula

sums the chiral expansion for the scattering amplitude, . . .
but takes into account only pions which travel around the lattice exactly once

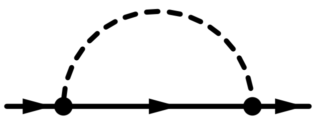
for a comparison with our results

- insert the chiral expansion of $g_{\pi N}$, $m_N$, $d^+_{k0}$ in Lüscher's formula
- restrict the sums over $\vec{n}$ in our formulae to $|\vec{n}| = 1$

$$m_N = m_0 + O(p^2) \quad , \quad g_{\pi N} = \frac{g_A m_0}{f_\pi} + O(p^2)$$

(Goldberger, Treiman)

we get from graph (a)



$$\Delta_a(L) = \frac{3 g_A^2 m_0 m_\pi^2}{16\pi^2 f_\pi^2} \int_0^\infty \mathrm{d}x \sum_{\vec{n}}{}' K_0\left(L|\vec{n}|\sqrt{m_0^2 x^2 + m_\pi^2(1-x)}\right)$$

$$\Downarrow$$

$$\delta_a = \frac{3 g_A^2 m_\pi^2}{16\pi^2 f_\pi^2} \int_0^\infty \mathrm{d}x \cdot 6 \cdot K_0\left(L\sqrt{m_0^2 x^2 + m_\pi^2(1-x)}\right)$$

one can show:

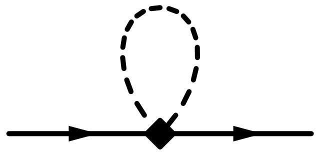
$$\begin{aligned}
&\int_0^\infty \mathrm{d}x\, K_0\left(L\sqrt{m_0^2x^2+m_\pi^2(1-x)}\right)\\
&=\int_{-\infty}^\infty \mathrm{d}x\, K_0\left(L\sqrt{m_0^2x^2+m_\pi^2(1-x)}\right)\\
&\qquad\qquad -\int_{-\infty}^0 \mathrm{d}x\, K_0\left(L\sqrt{m_0^2x^2+m_\pi^2(1-x)}\right)\\
&=\frac{\pi}{m_0L}\exp\left(-m_\pi L\sqrt{1-\frac{m_\pi^2}{4m_0^2}}\right)\\
&\qquad\qquad -\frac{1}{m_\pi L}\int_{-\infty}^\infty \mathrm{d}p\,\frac{\exp\left(-m_\pi L\sqrt{1+p^2}\right)}{1+4m_0^2\,p^2/m_\pi^2}
\end{aligned}$$

$$\boxed{\begin{aligned}
\delta_a = \frac{9}{8\pi^2}\left(\frac{m_\pi}{m_0}\right)^2 g_{\pi N}^2\Bigg\{ &2\frac{\pi}{2m_0L}\exp\left(-m_\pi L\sqrt{1-\frac{m_\pi^2}{4m_0^2}}\right)\\
&-\frac{1}{m_\pi L}\int_{-\infty}^\infty \mathrm{d}p\,\frac{\exp\left(-m_\pi L\sqrt{1+p^2}\right)}{1+4m_0^2\,p^2/m_\pi^2}\Bigg\}
\end{aligned}}$$

to be compared with

$$\begin{aligned}
\delta_N^{\text{Lüscher}} = \frac{9}{8\pi^2}\left(\frac{m_\pi}{m_N}\right)^2 g_{\pi N}^2\Bigg\{ &\frac{\pi}{2m_NL}\exp\left(-m_\pi L\sqrt{1-\frac{m_\pi^2}{4m_N^2}}\right)\\
&-\frac{1}{m_\pi L}\int_{-\infty}^\infty \mathrm{d}p\,\frac{\exp\left(-m_\pi L\sqrt{1+p^2}\right)}{1+4m_N^2\,p^2/m_\pi^2}\Bigg\}\\
-\frac{9}{4\pi}\frac{m_\pi}{m_NL}\sum_{k=0}^\infty m_\pi^{2k}d_{k0}^+(-1)^k&\int_{-\infty}^\infty\frac{\mathrm{d}p}{2\pi}\exp\left(-m_\pi L\sqrt{1+p^2}\right)p^{2k}
\end{aligned}$$

What about the contribution from graph (b)?



chiral perturbation theory:

$$d_{00}^{+} = -\frac{2m_\pi^2}{f_\pi^2}(2c_1 - c_3) + O(m_\pi^3)$$
$$d_{10}^{+} = \frac{2}{f_\pi^2}c_2 + O(m_\pi)$$

(see, e.g., Becher and Leutwyler)

With the help of these equations we get

$$-\frac{9}{4\pi}\frac{m_\pi}{m_0 L}\sum_{k=0}^{\infty} m_\pi^{2k} d_{k0}^{+}(-1)^k \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\,\exp\left(-m_\pi L\sqrt{1+p^2}\right)p^{2k}$$
$$= \frac{9}{2\pi^2 f_\pi^2}\frac{m_\pi^4}{m_0}\left[(2c_1 - c_3)\frac{K_1(m_\pi L)}{m_\pi L} + c_2\frac{K_2(m_\pi L)}{(m_\pi L)^2} + \cdots\right]$$

in agreement with our formula for $\Delta_b$

$|\vec{n}| = 1$ contributions of our finite size formula in Lüscher's form
(within the accuracy of our chiral expansion)

$$\delta_N = 2\frac{9}{16\pi m_N L}\left(\frac{m_\pi}{m_N}\right)^2 g_{\pi N}^2 \exp\left(-m_\pi L\sqrt{1-\frac{m_\pi^2}{4m_N^2}}\right)$$
$$-\frac{3}{8\pi m_\pi L}\left(\frac{m_\pi}{m_N}\right)^2\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\,\exp\left(-m_\pi L\sqrt{1+p^2}\right)F_{\pi p}(\mathrm{i}m_\pi p)$$

???

# Monte Carlo data

$N_f = 2$ dynamical quarks (pion cloud is expected to be substantially modified by the quenched approximation)

comparison with chEFT: continuum results needed (in physical units)

how to set the scale? $\rightarrow r_0 = 0.5\,\mathrm{fm}$ reliable?

(preferable) alternative: consider dimensionless ratios like $m_N/f_\pi$

| | gauge field action | fermion action |
|---|---|---|
| UKQCD + QCDSF | standard Wilson | non-pert. improved clover |
| JLQCD | standard Wilson | non-pert. improved clover |
| CP-PACS | RG improved | mean field improved clover |

| | Coll. | $\beta$ | $\kappa_{\mathrm{sea}}$ | volume |
|---|---|---|---|---|
| 1 | QCDSF | 5.20 | 0.1342 | $16^3 \times 32$ |
| 2 | UKQCD | 5.20 | 0.1350 | $16^3 \times 32$ |
| 3 | UKQCD | 5.20 | 0.1355 | $16^3 \times 32$ |
| 4 | UKQCD | 5.20 | 0.13565 | $16^3 \times 32$ |
| 5 | UKQCD | 5.20 | 0.1358 | $16^3 \times 32$ |
| 6 | QCDSF | 5.25 | 0.1346 | $16^3 \times 32$ |
| 7 | UKQCD | 5.25 | 0.1352 | $16^3 \times 32$ |
| 8 | QCDSF | 5.25 | 0.13575 | $24^3 \times 48$ |
| 9 | UKQCD | 5.26 | 0.1345 | $16^3 \times 32$ |
| 10 | UKQCD | 5.29 | 0.1340 | $16^3 \times 32$ |
| 11 | QCDSF | 5.29 | 0.1350 | $16^3 \times 32$ |
| 12 | QCDSF | 5.29 | 0.1355 | $12^3 \times 32$ |
| 13 | QCDSF | 5.29 | 0.1355 | $16^3 \times 32$ |
| 14 | QCDSF | 5.29 | 0.1355 | $24^3 \times 48$ |

UKQCD: Allton et al., Phys. Rev. D65 (2002) 054502

| | Coll. | $\beta$ | $\kappa_{sea}$ | volume |
|---|---|---|---|---|
| 15 | CP-PACS | 1.95 | 0.1410 | $16^3 \times 32$ |
| 16 | CP-PACS | 1.95 | 0.1400 | $16^3 \times 32$ |
| 17 | CP-PACS | 1.95 | 0.1390 | $16^3 \times 32$ |
| 18 | CP-PACS | 1.95 | 0.1375 | $16^3 \times 32$ |
| 19 | CP-PACS | 2.10 | 0.1382 | $24^3 \times 48$ |
| 20 | CP-PACS | 2.10 | 0.1374 | $24^3 \times 48$ |
| 21 | CP-PACS | 2.10 | 0.1367 | $24^3 \times 48$ |
| 22 | CP-PACS | 2.10 | 0.1357 | $24^3 \times 48$ |
| 23 | CP-PACS | 2.20 | 0.1368 | $24^3 \times 48$ |
| 24 | CP-PACS | 2.20 | 0.1363 | $24^3 \times 48$ |
| 25 | CP-PACS | 2.20 | 0.1358 | $24^3 \times 48$ |
| 26 | CP-PACS | 2.20 | 0.1351 | $24^3 \times 48$ |
| 27 | JLQCD | 5.20 | 0.1340 | $12^3 \times 48$ |
| 28 | JLQCD | 5.20 | 0.1343 | $12^3 \times 48$ |
| 29 | JLQCD | 5.20 | 0.1346 | $12^3 \times 48$ |
| 30 | JLQCD | 5.20 | 0.1350 | $12^3 \times 48$ |
| 31 | JLQCD | 5.20 | 0.1355 | $12^3 \times 48$ |
| 32 | JLQCD | 5.20 | 0.1340 | $16^3 \times 48$ |
| 33 | JLQCD | 5.20 | 0.1343 | $16^3 \times 48$ |
| 34 | JLQCD | 5.20 | 0.1346 | $16^3 \times 48$ |
| 35 | JLQCD | 5.20 | 0.1350 | $16^3 \times 48$ |
| 36 | JLQCD | 5.20 | 0.1355 | $16^3 \times 48$ |
| 37 | JLQCD | 5.20 | 0.1340 | $20^3 \times 48$ |
| 38 | JLQCD | 5.20 | 0.1343 | $20^3 \times 48$ |
| 39 | JLQCD | 5.20 | 0.1346 | $20^3 \times 48$ |
| 40 | JLQCD | 5.20 | 0.1350 | $20^3 \times 48$ |
| 41 | JLQCD | 5.20 | 0.1355 | $20^3 \times 48$ |

CP-PACS: Ali Khan et al., Phys. Rev. D65 (2002) 054505
JLQCD: Aoki et al., Phys. Rev. D68 (2003) 054502

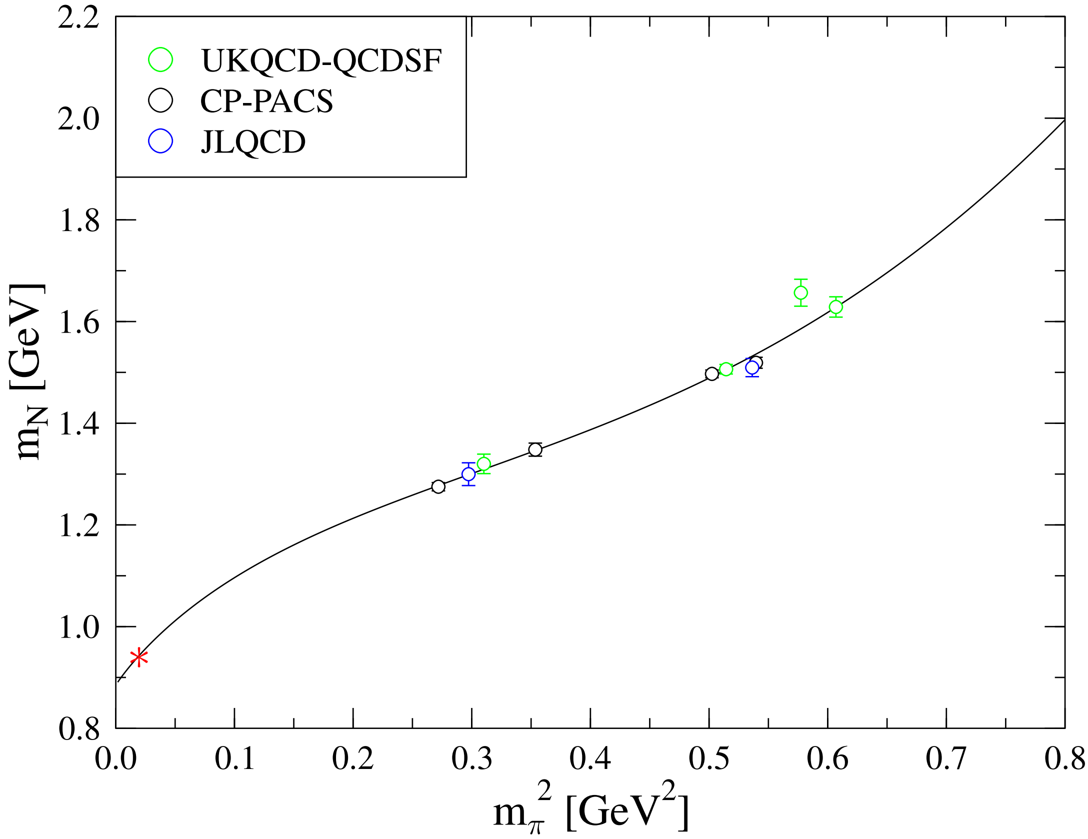
# Comparison with chiral perturbation theory

consider ten data points with $a < 0.15\,\text{fm}$, $m_\pi L > 5$, $m_\pi < 800\,\text{MeV}$ ("infinite" volume)

$$m_N = m_0 - 4c_1 m_\pi^2 - \frac{3g_A^2}{32\pi f_\pi^2} m_\pi^3 + \left[ e_1^r(\lambda) - \frac{3}{64\pi^2 f_\pi^2}\left(\frac{g_A^2}{m_0} - \frac{c_2}{2}\right) - \frac{3}{32\pi^2 f_\pi^2}\left(\frac{g_A^2}{m_0} - 8c_1 + c_2 + 4c_3\right)\ln\frac{m_\pi}{\lambda}\right] m_\pi^4 + \frac{3g_A^2}{256\pi f_\pi^2 m_0^2} m_\pi^5 + O(m_\pi^6)$$

| fix: | $g_A$ | $f_\pi$ | $c_2$ | $c_3$ |
|---|---|---|---|---|
| | 1.267 | 92.4 MeV | $3.2\,\text{GeV}^{-1}$ | $-3.4\,\text{GeV}^{-1}$ |
| | 1.2 | 88 MeV | | $-4.7\,\text{GeV}^{-1}$ |

fit: $m_0$ $\quad c_1 \quad$ $e_1^r(\lambda = 1\,\text{GeV})$




Fit1

chEFT applicable at $m_\pi \approx 750\,\mathrm{MeV}$ ?

force the fit curve through the physical point by adapting $m_0$ and restrict the fit to the four smallest masses $\rightarrow$ Fit 3, 4

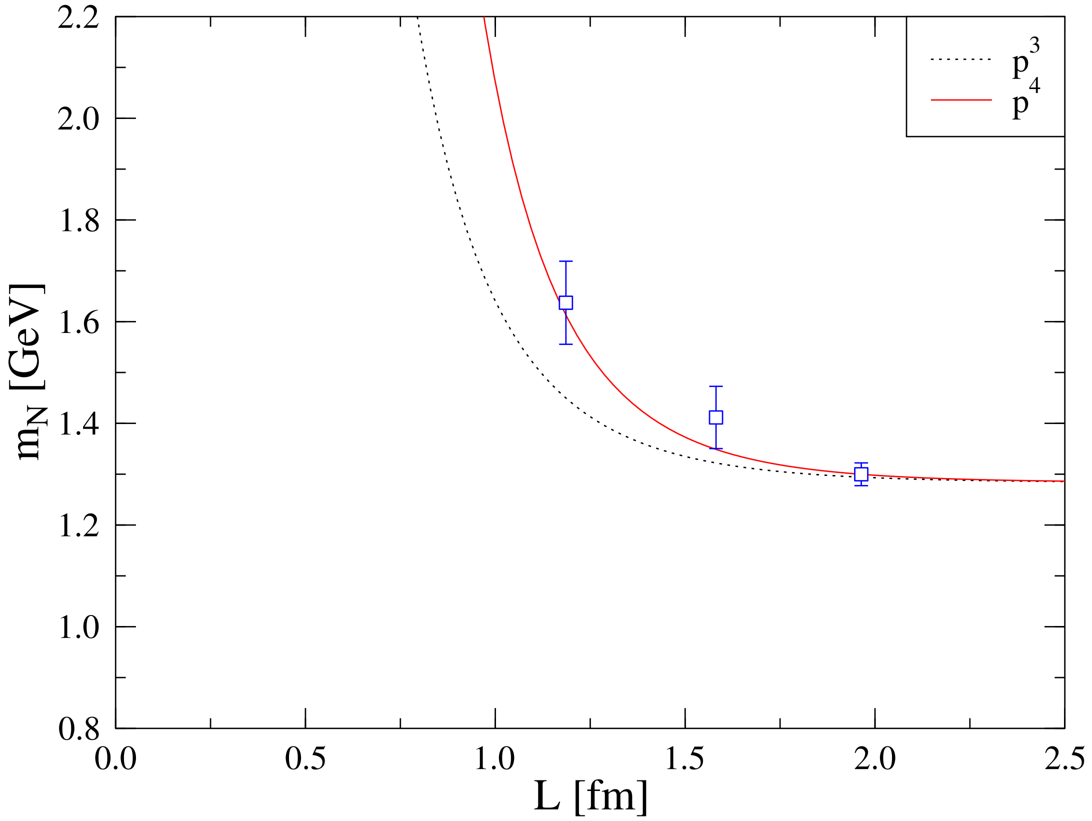
results for $c_1$ consistent with phenomenological determinations

| | $c_3$ [GeV$^{-1}$] | $m_0$ [GeV] | $c_1$ [GeV$^{-1}$] | $e_1^r$ [GeV$^{-3}$] | $\chi^2$ |
|---|---|---|---|---|---|
| Fit 1 | $-3.4$ | 0.89(6) | $-0.93(5)$ | 2.8(4) | 12.18 |
| Fit 2 | $-4.7$ | 0.76(6) | $-1.25(5)$ | 1.7(5) | 11.85 |
| Fit 3 | $-3.4$ | 0.88(-) | $-0.93(4)$ | 3.0(6) | 0.29 |
| Fit 4 | $-4.7$ | 0.87(-) | $-1.11(4)$ | 3.2(6) | 0.39 |

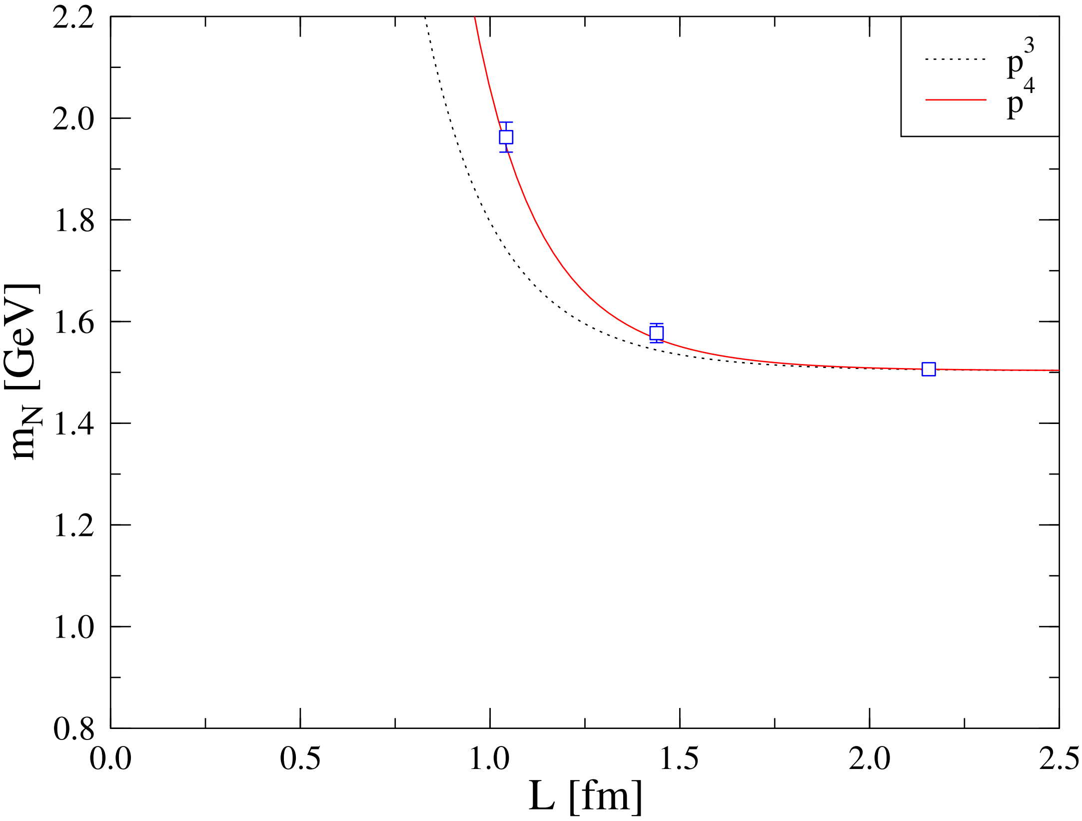
parameters from Fit 1 $\rightarrow$ evaluate the finite size corrections

$$m_N(L) = m_N(\infty) + \underbrace{\overbrace{\Delta_a(L)}^{p^3} + \Delta_b(L)}_{p^4}$$

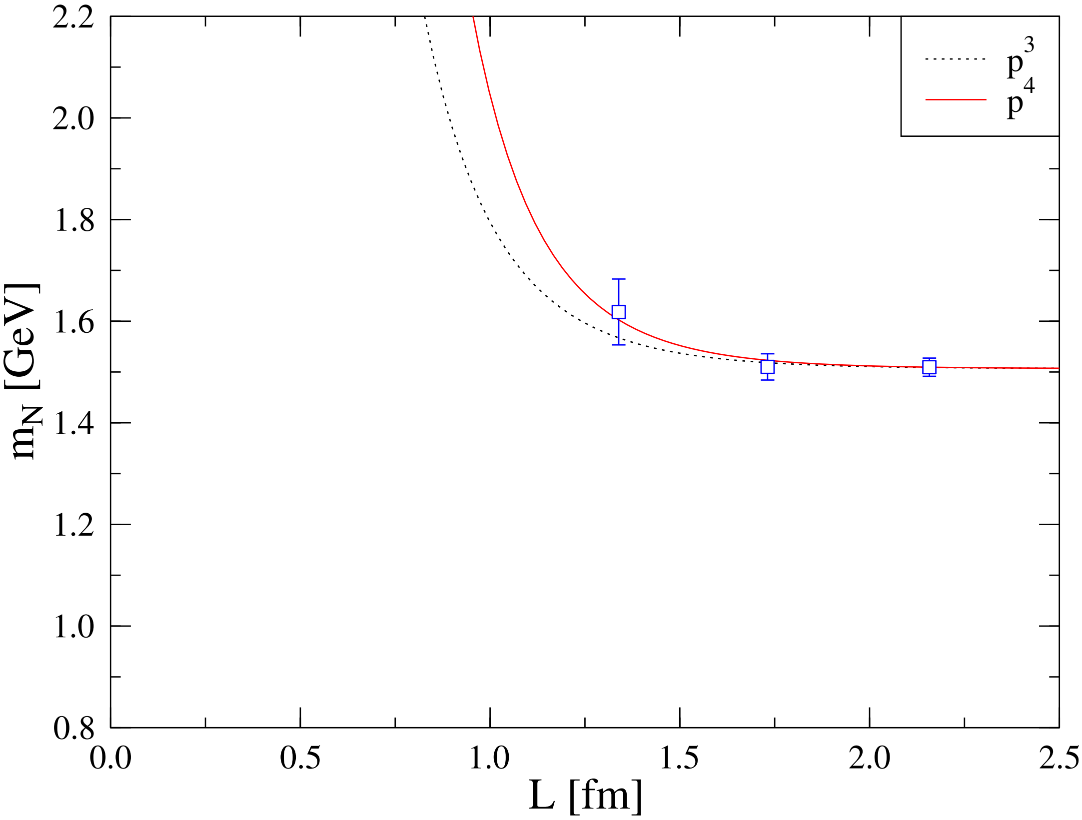
$m_N(\infty)$: $m_N(L)$ on the largest lattice agrees with the Monte Carlo value
$m_\pi$: take the value from the largest lattice




$m_\pi = 545$ MeV



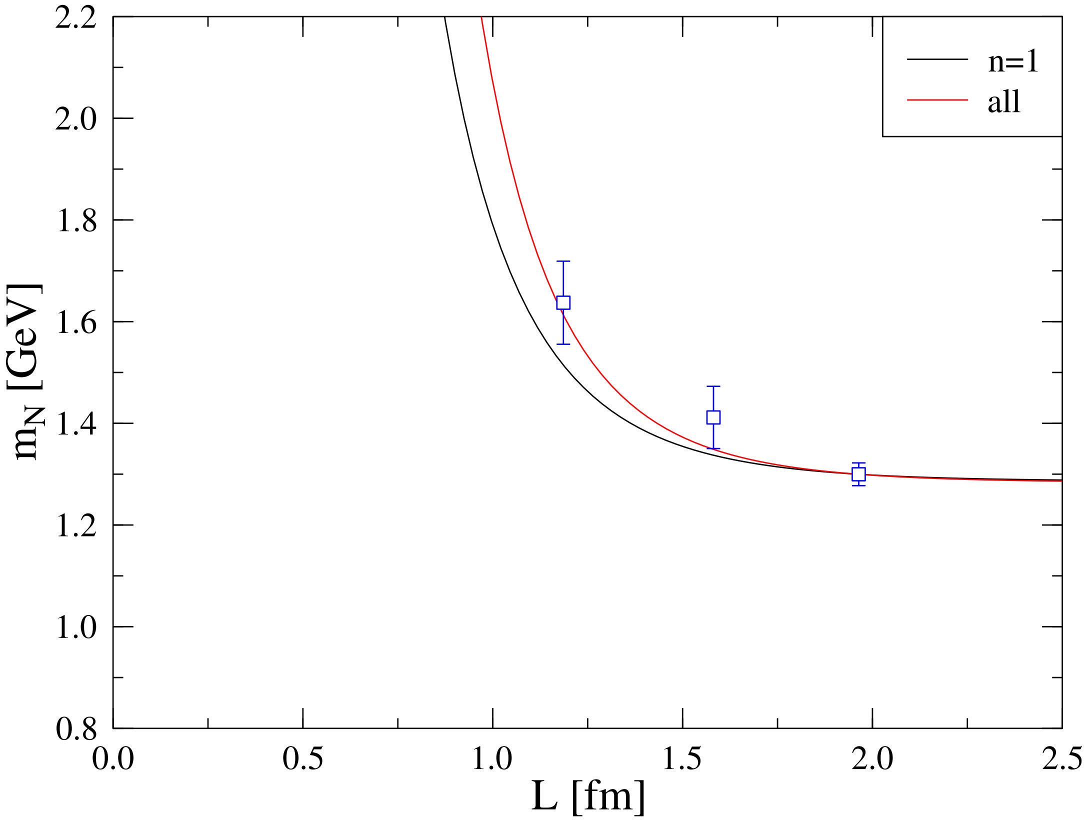

$m_\pi = 717\,\mathrm{MeV}$




$m_\pi = 732\,\mathrm{MeV}$

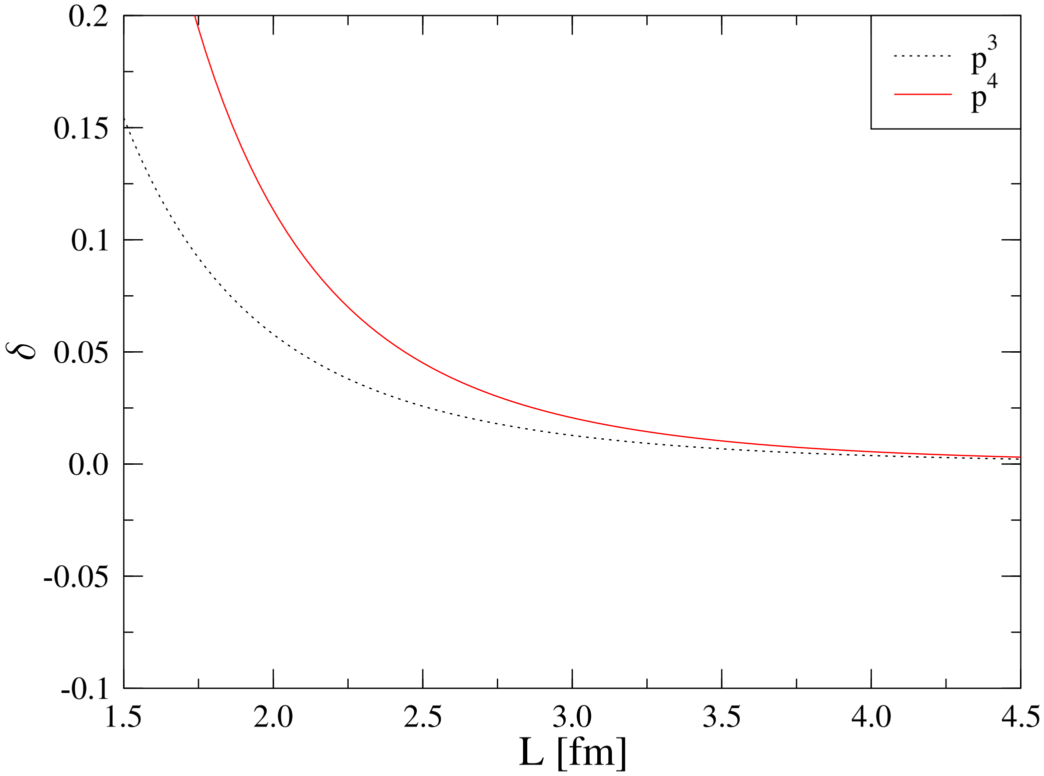
# Side remarks

Pions which travel around the lattice more than once $(|\vec{n}| > 1)$ contribute sizably:

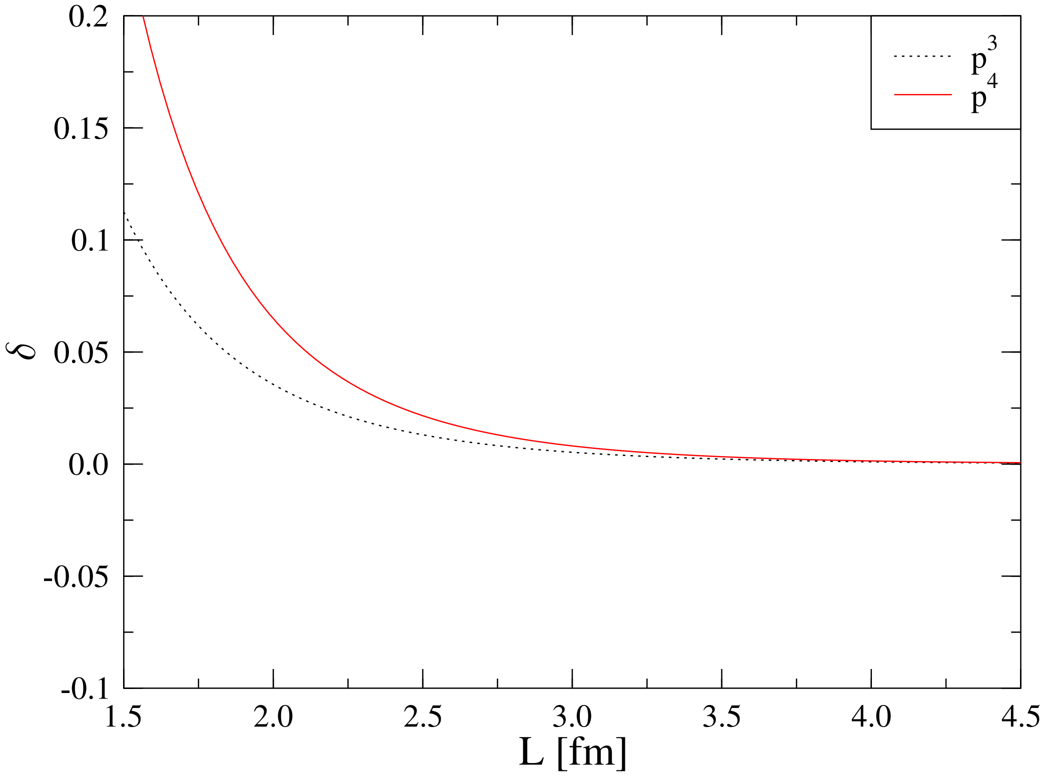



$m_\pi = 545$ MeV

# What happens at smaller pion masses?




$m_\pi = 140$ MeV



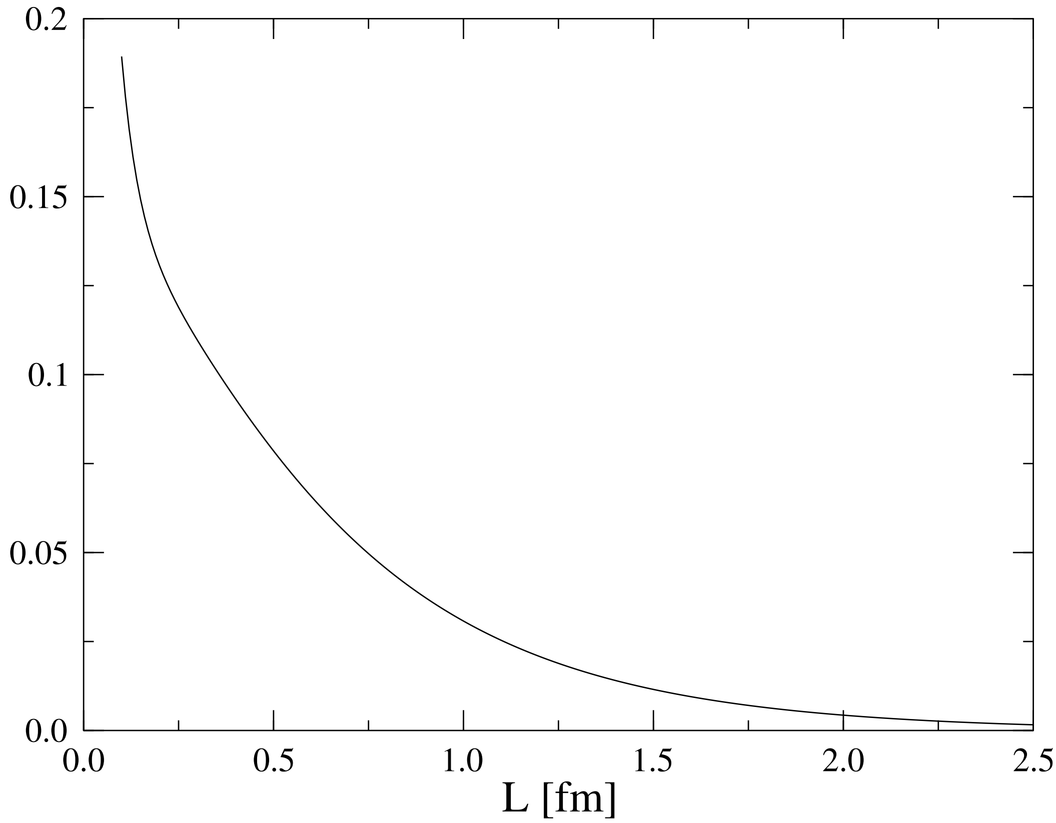

$m_\pi = 250$ MeV

relative finite size effect $\delta$

Consider the ratio

$$\frac{\int_1^{\infty} \mathrm{d}x \, \sum_{\vec{n}}{}' \, K_0\left(L|\vec{n}|\sqrt{m_0^2 x^2 + m_\pi^2(1-x)}\right)}{\int_0^{\infty} \mathrm{d}x \, \sum_{\vec{n}}{}' \, K_0\left(L|\vec{n}|\sqrt{m_0^2 x^2 + m_\pi^2(1-x)}\right)}$$

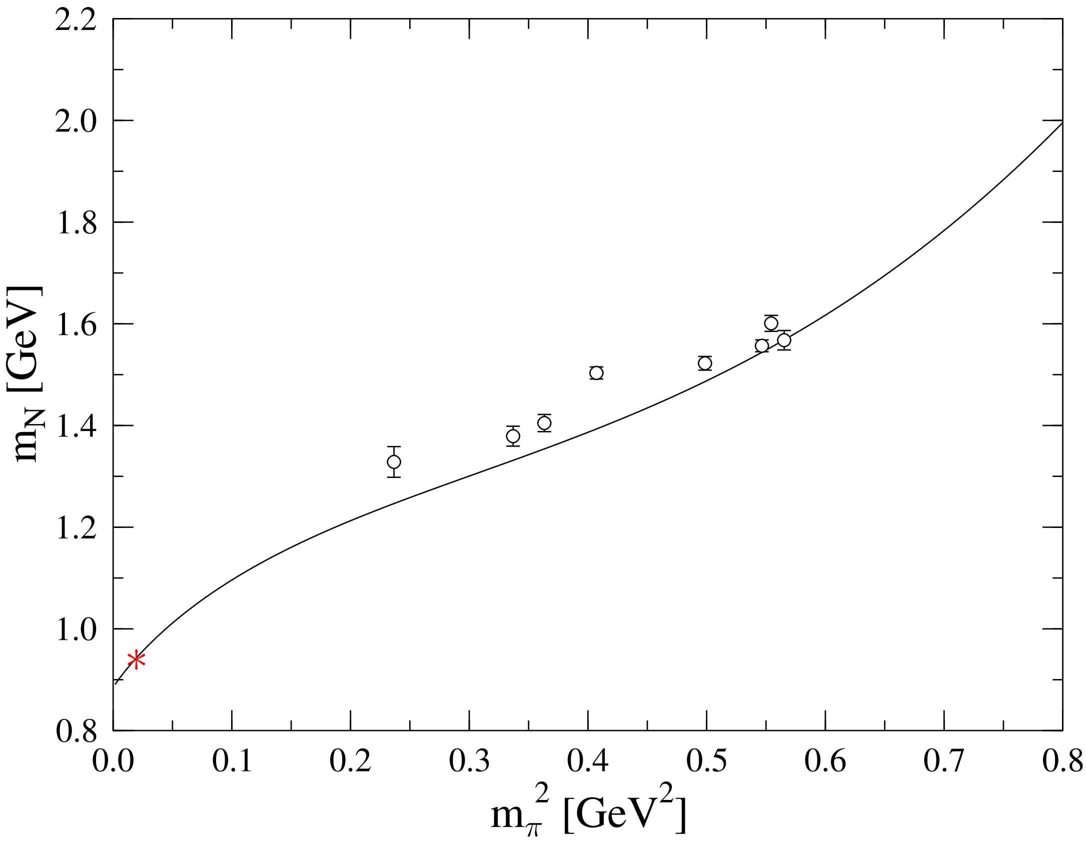
fraction of $\Delta_a(L)$ which the infrared regularisation treats as a short-distance contribution (beyond the reach of chiral perturbation theory)



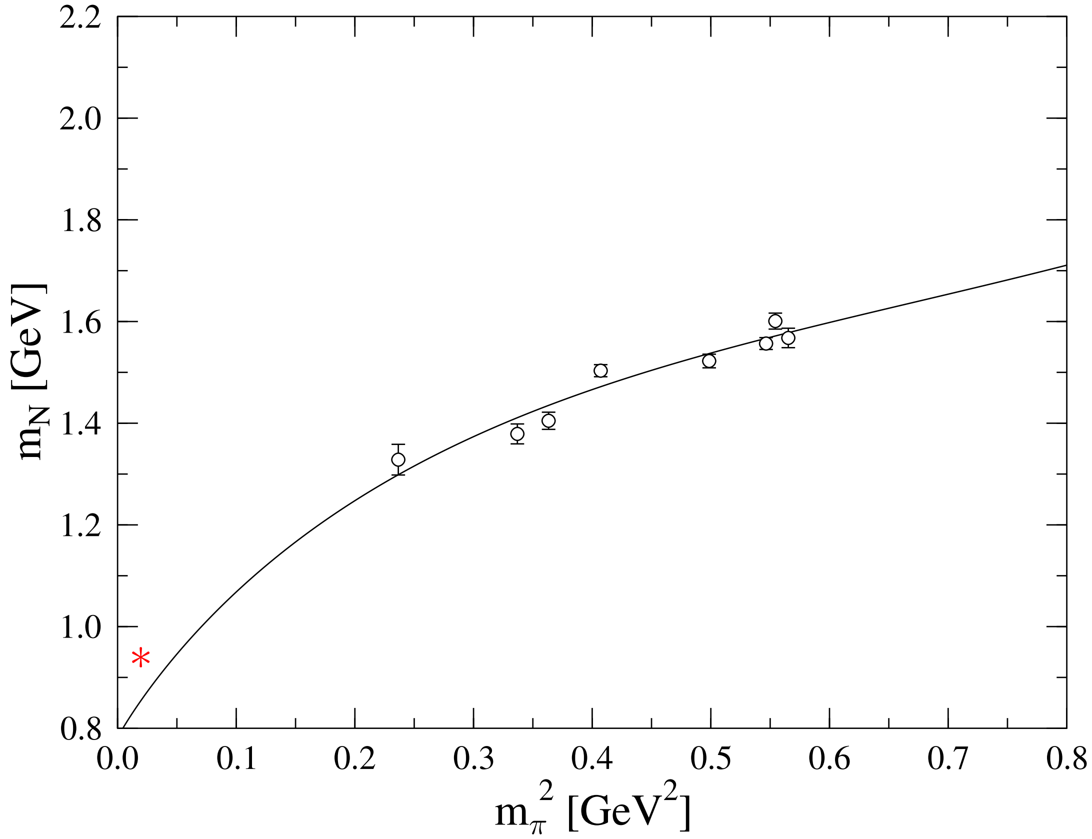

$m_\pi = 545$ MeV, $m_0 = 880$ MeV

# Comparison with quenched data

quenched data from JLQCD and QCDSF on reasonably fine ($a < 0.1$ fm) and reasonably large ($m_\pi L > 4.5$) lattices for $m_\pi < 800$ MeV compared with the curve from Fit 1



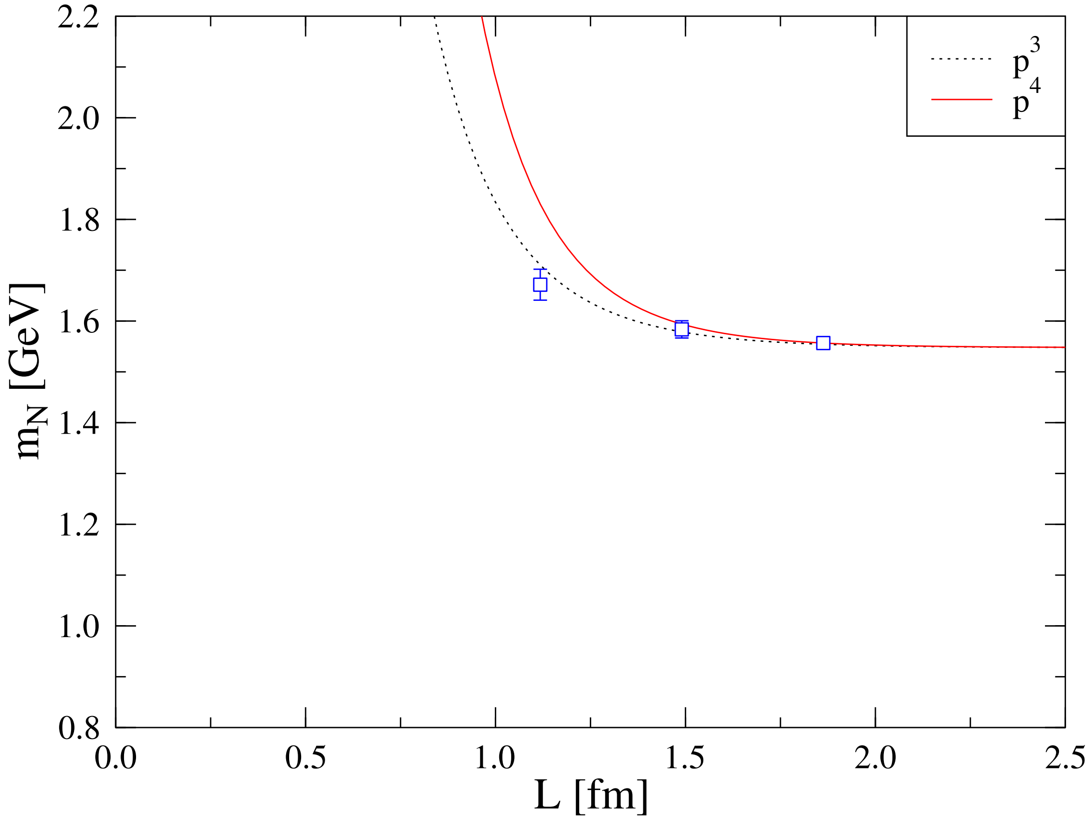
fit with the unquenched formula



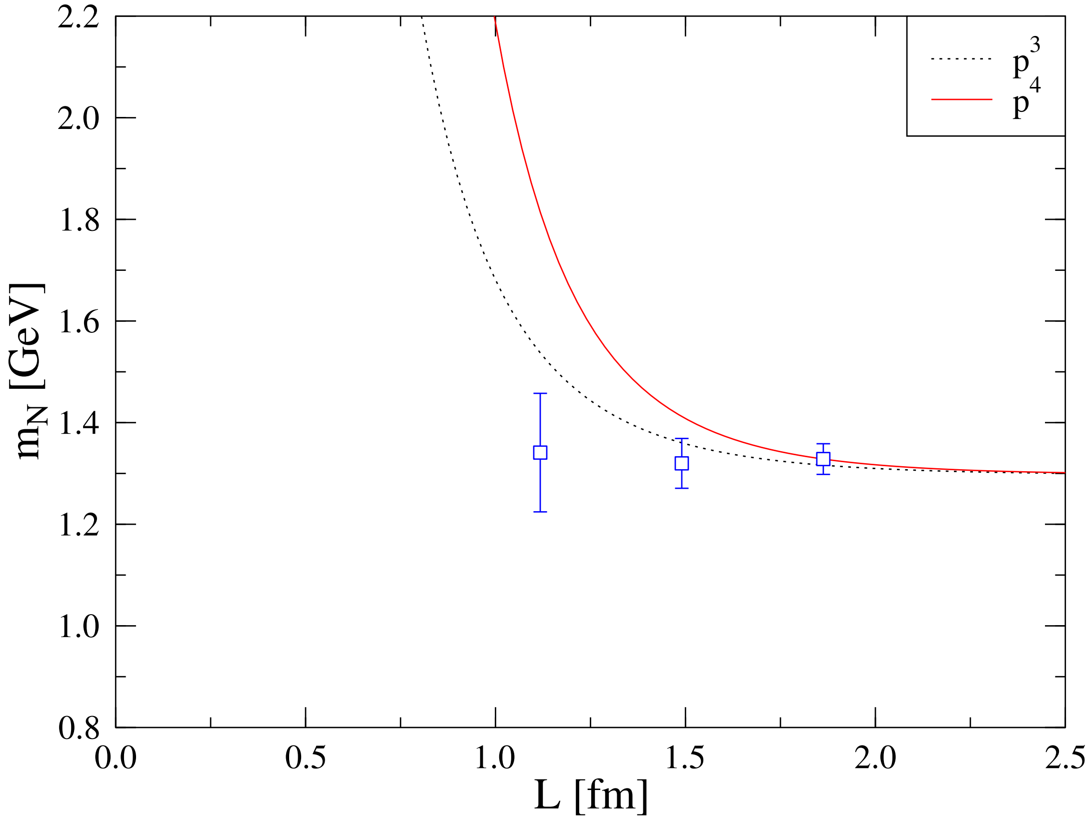

$$c_3 = -3.4\,\text{GeV}^{-1}$$

| data | $m_0$ [GeV] | $c_1$ [GeV$^{-1}$] | $e_1^r$ [GeV$^{-3}$] | $\chi^2$ |
|---|---|---|---|---|
| unquenched | 0.89(6) | $-0.93(5)$ | 2.8(4) | 12.18 |
| quenched | 0.78(11) | $-1.11(8)$ | 1.4(6) | 19.56 |

quenched masses from JLQCD $\leftrightarrow$ unquenched finite size formula




$m_\pi = 739$ MeV




$m_\pi = 486$ MeV

parameters from Fit 1 to the unquenched masses

# Summary and outlook

- chEFT in a finite volume $\rightarrow$ expansion in powers of $m_\pi$ and $L^{-1}$ with coefficients depending on $m_\pi L$
- formalism applied to the nucleon mass
  using relativistic SU(2)$_\mathrm{f}$ baryon chiral perturbation theory up to $O(p^4)$
- $N_f = 2$ Monte Carlo data for $m_N$, scale set with $r_0 = 0.5\,$fm
  may be problematic
  - large volume data (at not too large pion masses) well described with phenomenologically reasonable values of the coupling constants
    no free parameters left in the volume dependence
  - finite size effects reproduced by the $O(p^4)$ formulae
- finite size effects in the quenched approximation considerably smaller at a given pion mass
  pion cloud reduced by quenching
- For the future:
  - avoid the scale problem by considering ratios such as $m_N/f_\pi$
  - smaller quark masses (and lattice spacings) in the simulations
  - include the volume dependence of $m_\pi$ in the analysis