Lattice supersymmetry and supersymmetry breaking: The Wess-Zumino model

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Supersymmetry on the lattice: Where and how ?

- Aproximate
  - N=1 Super-Yang Mills
    (with Wilson fermions – Curci and Veneziano formulation)
- Exact
  - N=1 Super-Yang Mills
    (with domain wall fermions – Kaplan-Schmaltz formulation)
    (Pinsky and coll. – Light-Cone discretization)
  - Extended supersymmetric theories (Kaplan and coll., Giedt (hep-lat/0405021), Sugino (hep-lat/0311021), ...)
  - Four and two dimensional Wess-Zumino model (Golterman and Petcher, Catterall and coll., Fujikawa and coll., Bonini, Beccaria, Campostrini, ...)

Recent reviews

1. D. B. Kaplan, hep-lat/0309099, (Plenary talk at lattice 2003), "Recent developments in lattice supersymmetry"
2. A. F., hep-lat/0210015, Nucl.Phys.Proc.Suppl.119:198-209,2003, (Plenary talk at Lattice 2002), "Supersymmetry on the lattice"
3. I. Montvay, hep-lat/0112007, Int.J.Mod.Phys. A17 (2002) 2377, "Supersymmetric Yang-Mills theory on the lattice"
4. P. M. Vranas, hep-lat/0011066, Nucl.Phys.Proc.Suppl. 94 (2001) 177, (Plenary talk at Lattice 2000), "Domain wall fermions and applications"
5. I. Montvay, hep-lat/9709080, Nucl.Phys.Proc.Suppl. 63 (1998) 108, (Plenary talk at Lattice 1997), "SUSY on the lattice"

Outline

- Lattice formulation of supersymmetry
  - Difficulties
  - Wilson and domain wall fermions
- Exact supersymmetry on the lattice?
  - Four dimensional $N = 1$ Wess-Zumino using Ginsparg-Wilson fermions
    Bonini and A.F. hep-lat/0402034
- Outlook

Fermions on the Lattice

- Recently, following the rediscovery of the Ginsparg-Wilson relation (1982), it has emerged that chiral gauge theories can be put on the lattice in a consistent way:
  - The overlap (Narayanan-Neuberger 1993,1995,1998)
  - Domain wall fermions (Kaplan-Shamir 1992, 1993, 1994)
  - Perfect action (Hasenfratz-Niedermayer 1994, 1998).

  This was beleived to be impossible for a long time (Nielsen-Ninomiya, 1981, no-go theorem).

- A naive formulation of fermions on the lattice fails.

$$S_F = \frac{1}{2}\sum_x \sum_\mu \bar{\psi}(x)(\gamma_\mu \Delta_\mu + m)\psi(x) + h.c.$$

  and the resulting propagator is

$$\tilde{\Delta}(k) = \frac{-i\sum_\mu \gamma_\mu \sin k_\mu + m}{\sum_\mu \sin^2 k_\mu + m^2}$$

- There is a pole for small k_μ representing the physical particle, but additional poles near $k_\mu = \pm\pi$ appears. S_F describes 16 instead of 1 particle. $\rightarrow$ Doubling problem.

- Two popular choices introduced in order to deal with this problem:
  - Wilson fermions: Get rids of the doubling species but breaks chiral symmetry explicitily by the Wilson term.
  - Staggered fermions (Kogut-Susskind): Reduce from 16 to 4 fermions and for massless fermions a chiral $U(1) \oplus U(1)$ symmetry remains.

- In the Wilson formulation the bare mass m is hidden in the hopping parameter by the relation $k = \frac{1}{8r+2m_0}$.

- Take Wilson's or staggered fermions for the quarks fields $\psi^f_{\alpha c}(x)$, the complete action is $S = S_W + S_F$. And for an observable we write down

$$\langle \Omega \rangle = \frac{1}{Z} \int \Pi_{x,\mu} dU_\mu(x) \int \Pi_x \bar{d}\psi(x) d\psi(x) \Omega e^{-S_W - S_F},$$

- After integrating out the quarks fields the expectation value reads

$$\langle \Omega \rangle = \frac{1}{Z} \int \Pi_{x,\mu} dU_\mu(x) \Pi_f \mathrm{det}(D + m_f) \Omega e^{-S_W},$$

Where D is the Dirac operator.

Lattice formulation of SYM theory

The question of whether it is possible to formulate supersymmetric theories on the lattice has been addressed in the past by several authors

- The lattice regularized theory is not supersymmetric as the Poincaré invariance (a sector of the superalgebra) is lost.

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu$$

  Not a severe problem. Calculating at several lattice spacing a and then take the limit $a \to 0$. No fine tuning is needed.

- If there are scalar mass terms in the SUSY theory that break SUSY. Since this operators are relevant fine tuning is necessary to cancel their contributions.

- A naive regularization of fermions results in the doubling problem

Nielsen & Ninomiya '81

$\rightarrow$ wrong number of fermions and violation of the balance between bosons and fermions

  – The problem can be treated as in QCD. This is the case of $N = 1$ SYM.

Wilson fermions

Propose to give up manifest SUSY on the lattice and restore it in the continuum limit.

Curci & Veneziano '87

SUSY is broken by the lattice, by the Wilson term and a a soft breaking due to the gluino mass is present.

- SUSY is recovered in the continuum limit by tuning the bare parameters g and gluino mass $m_{\tilde{g}}$ to the SUSY point.

- The chiral and SUSY limit can be recovered simultaneously at $m_{\tilde{g}} = 0$.

Domain wall fermions

A new lattice fermion regulator. Very nice innovation. Application of DWF in SUSY theories

Neuberger '98
Kaplan & Schmaltz '00

Monte Carlo simulation for $N = 1$ $SU(2)$ SYM with DWF

Fleming, Kogut & Vranas '01

DWF were introduced in

Kaplan '92,'93

with further developed in

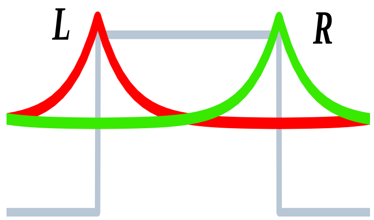
Narayanan & Neuberger '93,'94,'95
Shamir '93
Furman & Shamir '95

Difficulties in using Wilson fermions.

- Need to fine tuning. The Wilson term breaks chiral symmetry
- The Pfaffian. Is not positive definite at finite lattice spacing.

DWF are defined extending space-time to five dimensions.

L_s is the size of the fifth dimension.



In the limit $L_s \to \infty$ chiral symmetry is exact, even at finite lattice spacing.

- There is not need for fine tuning.

The domain wall action is

$$S = S_G(U) + S_F(\Psi, U) + S_{PV}(\Phi, U)$$

$$S_F = -\sum_{x,x',s,s'} \bar{\Psi}_{x,s}(D_F)_{x,s;x',s'}\Psi_{x',s'}$$

The effective action is

$$S_{KS} = \beta \sum_{pl} \left(1 - \frac{1}{2}\mathrm{Tr}U_{pl}\right) - \frac{1}{2}\log\det D_F[U] + \frac{1}{2}\log\det D_F[m_f = 1; U] \, .$$

Difficulties in using DWF.

- 2 extra parameters in DWF: L_s and m_0 (m_0 is the domain wall height or five-dimensional mass that controls the number of flavors).

$$m_{eff} = m_0(2 - m_0)[m_f + (1 - m_0)^{L_s}]\,, \quad 0 < m_0 < 2$$

- The two chiralities do not decouple $\rightarrow$ no restoration of chiral symmetry. (Need large values of L_s)

- Harder to simulate than QCD (with Wilson fermions SYM easier to simulate than QCD)

Exact supersymmetry on the lattice

The lattice action is not unique. Improve the action in order to approach the continuum limit faster and/or have less symmetry breaking.

Improving lattice SUSY seems to be a difficult task for gauge theories because on the lattice the gauge field and the fermions are treated in a very different way.

It is possible to obtain perfect SUSY respect to the SUSY transformations. Nice examples:

Golterman & Petcher, Nucl.Phys.B319:307-341,1989
Bietenholz, Mod.Phys.Lett.A14:51-62,1999
Catterall & Karamov, Phys.Rev.D65:094501,2002,
Phys.Rev.D68:014503,2003
Fujikawa & Ishibashi, Phys.Lett.B528:295-300,2002
Fujikawa, Nucl.Phys.B636:80-98,2002
Beccaria, Campostrini & A. F., Phys.Rev.D69:095010,2004
(hep-lat/0402007) and hep-lat/0405016
Bonini & A. F., hep-lat/0402034

Four dimensional lattice Wess-Zumino model with GW fermions

Our starting point is the paper by Fujikawa ’02.

We show that it is actually possible to formulate the theory in such a way that the full action is invariant under a lattice superymmetry transformation at a fixed lattice spacing.

The action and the transformation are written in terms of the Ginsparg-Wilson operator and reduce to their continuum expression in the naive continuum limit $a \to 0$.

The lattice supersymmetry transformation is non-linear in the scalar fields and depends on the parameters m and g entering in the superpotential.

We also show that the lattice supersymmetry transformation close the algebra, which is a necessary ingredient to guarantee the request of supersymmetry.

The Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

implies a continuum symmetry of the fermion action which may be regarded as a lattice form of the chiral symmetry (Lüscher 98).

The fermion lagrangian with a Yukawa interaction

$$\mathcal{L} = \bar{\psi}D\psi + g\bar{\psi}(P_+\phi\hat{P}_+ + P_-\phi^\dagger\hat{P}_-)\psi\,,$$

where

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5)\,, \qquad \hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5)$$

are the lattice chiral projection operators and $\hat{\gamma}_5 = \gamma_5(1 - aD)$, is invariant under the lattice chiral transformation

$$\delta\psi = i\varepsilon\hat{\gamma}_5\psi\,, \qquad \delta\bar{\psi} = i\bar{\psi}\gamma_5\varepsilon\,, \qquad \delta\phi = -2i\varepsilon\phi\,.$$

By writing ψ in terms of two Majorana fermions

$$\psi = \chi + i\eta\,,$$

it can be seen that the interaction term couples the two Majorana fermions and therefore there is a conflict between lattice chiral symmetry and the Majorana condition (Fujikawa '02).

This is due to the fact that the projection operators $\hat{P}_\pm$ depend on D. By making the following field redefinition

$$\psi' = (1 - \frac{a}{2}D)\psi\,, \qquad \bar{\psi}' = \bar{\psi}\,,$$

the Yukawa interaction becomes

$$g\bar{\psi}'(P_+\phi P_+ + P_-\phi^\dagger P_-)\psi'$$

and the two Majorana components of ψ' decouple.

Taking advantage of this property, one can define the four dimensional Wess-Zumino on the lattice with Majorana fermions.

We start with a lagrangian defined in terms of the Ginsparg-Wilson fermions on the $d = 4$ euclidean lattice.

$$D = \frac{1}{a}\big(1 - \frac{X}{\sqrt{X^\dagger X}}\big)\,, \qquad X = 1 - aD_w\,,$$

where

$$D_w = \frac{1}{2}\gamma_\mu(\nabla^\star_\mu + \nabla_\mu) - \frac{a}{2}\nabla^\star_\mu\nabla_\mu$$

and

$$\nabla_\mu\phi(x) = \frac{1}{a}(\phi(x + a\hat{\mu}) - \phi(x))\,, \qquad \nabla^\star_\mu\phi(x) = \frac{1}{a}(\phi(x) - \phi(x - a\hat{\mu}))$$

It is convenient to write

$$D = D_1 + D_2$$

where

$$D_1 = \frac{1}{a}\left(1 - \frac{1 + \frac{a^2}{2}\nabla_\mu^\star \nabla_\mu}{\sqrt{X^\dagger X}}\right), \qquad D_2 = \frac{1}{2}\gamma_\mu \frac{\nabla_\mu^\star + \nabla_\mu}{\sqrt{X^\dagger X}} \equiv \gamma_\mu D_{2\mu}\,.$$

In terms of D_1 and D_2 the Ginsparg-Wilson relation becomes

$$D_1^2 - D_2^2 = \frac{2}{a} D_1\,.$$

The action of the 4-dimensional Wess-Zumino model on the lattice

$$S_{WZ} = \sum_x \{\frac{1}{2}\bar{\chi}(1-\frac{a}{2}D_1)^{-1}D_2\chi - \frac{2}{a}\phi^\dagger D_1\phi + F^\dagger(1-\frac{a}{2}D_1)^{-1}F + \frac{1}{2}m\bar{\chi}\chi$$
$$+m(F\phi + (F\phi)^\dagger) + g\bar{\chi}(P_+\phi P_+ + P_-\phi^\dagger P_-)\chi + g(F\phi^2 + (F\phi^2)^\dagger)\}\,,$$

where ϕ and F are scalar fields and χ is a Majorana fermion which satisfies the Majorana condition

$$\bar{\chi} = \chi^T C$$

and C is the charge conjugation matrix which satisfies

$$C^T = -C\,, \qquad CC^\dagger = 1\,.$$

Moreover, our conventions are

$$C\gamma_\mu C^{-1} = -(\gamma_\mu)^T$$
$$C\gamma_5 C^{-1} = (\gamma_5)^T\,.$$

In the continuum limit reduces to the continuum Wess-Zumino action

$$S = \int \{\frac{1}{2}\bar{\chi}(\partial\!\!\!/ + m)\chi + \phi^\dagger\partial^2\phi + F^\dagger F + m(F\phi + (F\phi)^\dagger) + g\bar{\chi}(P_+\phi P_+ + P_-\phi^\dagger P_-)\chi + g(F\phi^2 + (F\phi^2)^\dagger)\}.$$

The supersymmetric transformation

If one defines the real components by

$$\phi \to \frac{1}{\sqrt{2}}(A + iB)\,, \quad F \to \frac{1}{\sqrt{2}}(F - iG)$$

the WZ model $S_{WZ} = S_0 + S_{int}$

$$S_0 = \sum_x \{\frac{1}{2}\bar{\chi}(1 - \frac{a}{2}D_1)^{-1}D_2\chi - \frac{1}{a}(AD_1A + BD_1B) + \frac{1}{2}F(1 - \frac{a}{2}D_1)^{-1}F + \frac{1}{2}G(1 - \frac{a}{2}D_1)^{-1}G\}\,,$$

$$S_{int} = \sum_x \{\frac{1}{2}m\bar{\chi}\chi + m(FA + GB) + \frac{1}{\sqrt{2}}g\bar{\chi}(A + i\gamma_5 B)\chi + \frac{1}{\sqrt{2}}g[F(A^2 - B^2) + 2G(AB)]\}\,.$$

The free part of the action, S_0, is invariant under the lattice supersymmetry transformation

$$\begin{aligned}
\delta A &= \bar{\varepsilon}\chi = \bar{\chi}\varepsilon \\
\delta B &= -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon \\
\delta \chi &= -D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon \\
\delta F &= \bar{\varepsilon} D_2 \chi \\
\delta G &= i\bar{\varepsilon} D_2 \gamma_5 \chi\,.
\end{aligned}$$

In fact, the variation of S_0 under the this transformation is

$$\begin{aligned}
\delta S_0 &= \\
&= \sum_x \{\bar{\chi}(1 - \frac{a}{2}D_1)^{-1} D_2[-D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon] - \frac{2}{a}\bar{\chi}\varepsilon D_1 A \\
&\qquad + \frac{2i}{a}\bar{\chi}\gamma_5\varepsilon D_1 B + (\bar{\varepsilon} D_2 \chi)(1 - \frac{a}{2}D_1)^{-1} F + i(\bar{\varepsilon} D_2 \gamma_5 \chi)(1 - \frac{a}{2}D_1)^{-1} G\}\,.
\end{aligned}$$

and integrating by part *, this variation becomes

$$\sum_x \{ -\bar{\chi}\varepsilon\big[(1-\frac{a}{2}D_1)^{-1}D_2^2+\frac{2}{a}D_1\big]A + i\bar{\chi}\gamma_5\varepsilon\big[(1-\frac{a}{2}D_1)^{-1}D_2^2+\frac{2}{a}D_1\big]B$$
$$-\bar{\chi}(1-\frac{a}{2}D_1)^{-1}D_2(F-i\gamma_5 G)\varepsilon + \bar{\chi}D_2\varepsilon(1-\frac{a}{2}D_1)^{-1}F + i\bar{\chi}D_2\gamma_5\varepsilon(1-\frac{a}{2}D_1)^{-1}G\}$$
$$= 0\,,$$

where we used the Ginsparg-Wilson relation, which implies

$$(1-\frac{a}{2}D_1)^{-1}D_2^2 = -\frac{2}{a}D_1\,.$$

*For instance, for any scalar function $\mathcal{F}$ one has $\mathcal{F}\bar{\varepsilon}D_2\chi = \bar{\chi}D_2\mathcal{F}\varepsilon$.

Failure of the Leibniz rule

The variation of S_{int} under the susy transformation does not vanish because of the failure of the Leibniz rule at finite lattice spacing (Fujikawa '02 and Dondi '77)

$$
\begin{aligned}
&\frac{1}{a}(f(x+a)g(x+a)-f(x)g(x)) = \\
&= \frac{1}{a}(f(x+a)-f(x))g(x)+\frac{1}{a}f(x)(g(x+a)-g(x)) \\
&+a\frac{1}{a}(f(x+a)-f(x))\frac{1}{a}(g(x+a)-g(x)) \\
&= (\nabla f(x))g(x)+f(x)(\nabla g(x))+a(\nabla f(x))(\nabla g(x))
\end{aligned}
$$

breaking of supersymmetry is of order $O(a)$.

- In order to discuss the symmetry properties of the lattice Wess-Zumino model one possibility is to modify the action by adding irrelevant terms which make invariant the full action.

- Alternatively, one can modify the supersymmetry transformation in such a way that the action has an exact symmetry for a different from zero

Since the transformation leaves invariant the free part of the action, this modification must vanish for $g = 0$.

$$\begin{aligned}
\delta A &= \bar{\varepsilon}\chi = \bar{\chi}\varepsilon \\
\delta B &= -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon \\
\delta\chi &= -D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon + gR\varepsilon \\
\delta F &= \bar{\varepsilon}D_2\chi \\
\delta G &= i\bar{\varepsilon}D_2\gamma_5\chi
\end{aligned}$$

- R to be determined by requiring that the variation of the action vanishes.
  - We assume that R depends on the scalar and auxiliary fields and their derivatives and not on χ.

Exact susy transformation for the full action

The variation of the Wess-Zumino action under the transformation is

$$\begin{aligned}\delta S_{WZ} &= \sum_x \{g\bar{\chi}(1-\frac{a}{2}D_1)^{-1}D_2R\varepsilon - m\bar{\chi}[D_2(A-i\gamma_5B)\varepsilon + (F-i\gamma_5G)\varepsilon - gR\varepsilon] \\ &+m(A\bar{\varepsilon}D_2\chi + F\bar{\chi}\varepsilon + iB\bar{\varepsilon}D_2\gamma_5\chi - iG\bar{\chi}\gamma_5\varepsilon) + \frac{g}{\sqrt{2}}\bar{\chi}(\bar{\varepsilon}\chi + \gamma_5(\bar{\varepsilon}\gamma_5\chi))\chi \\ &-\sqrt{2}g\bar{\chi}(A+i\gamma_5B)[D_2(A-i\gamma_5B)\varepsilon + (F-i\gamma_5G)\varepsilon - gR\varepsilon] \\ &+\frac{g}{\sqrt{2}}[(A^2-B^2)\bar{\varepsilon}D_2\chi + 2FA\bar{\chi}\varepsilon + 2iFB\bar{\chi}\gamma_5\varepsilon \\ &+2iAB\bar{\varepsilon}D_2\gamma_5\chi + 2GB\bar{\chi}\varepsilon - 2iGA(\bar{\chi}\gamma_5\varepsilon)]\}\,.\end{aligned}$$

By using the Fierz identity, terms with four fermions cancel as in the continuum.

Moreover, g independent terms cancel out after an integration by part, and one is left with

$$\begin{aligned}\delta S_{WZ} &= \sum_x \{g\bar{\chi}[(1-\frac{a}{2}D_1)^{-1}D_2R + mR]\varepsilon - \frac{g}{\sqrt{2}}[2\bar{\chi}(A+i\gamma_5B)D_2(A-i\gamma_5B)\varepsilon \\ &\qquad - \bar{\chi}D_2(A-i\gamma_5B)^2\varepsilon] + \sqrt{2}g^2\bar{\chi}(A+i\gamma_5B)R\varepsilon\}\,.\end{aligned}$$

The function R is determined by imposing the vanishing of δS_{WZ}. By expanding R in powers of g

$$R = R^{(1)} + gR^{(2)} + \cdots$$

and imposing the symmetry condition order by order in perturbation theory, we find

$$R^{(1)} = ((1 - \frac{a}{2} D_1)^{-1} D_2 + m)^{-1} \Delta L$$

with

$$\begin{aligned} \Delta L \equiv & \frac{1}{\sqrt{2}} (2(A + i\gamma_5 B) D_2 (A - i\gamma_5 B) - D_2 (A - i\gamma_5 B)^2) \\ = & \frac{1}{\sqrt{2}} \{ 2(AD_2A - BD_2B) - D_2(A^2 - B^2) \\ & + 2i\gamma_5 [(AD_2B + BD_2A) - D_2(AB)] \} . \end{aligned}$$

To order g^2 one has

$$R^{(2)} = -\sqrt{2}((1-\frac{a}{2}D_1)^{-1}D_2+m)^{-1}(A+i\gamma_5 B)((1-\frac{a}{2}D_1)^{-1}D_2+m)^{-1}\Delta L\,,$$

and for $n \geq 2$

$$R^{(n)} = -\sqrt{2}((1-\frac{a}{2}D_1)^{-1}D_2+m)^{-1}(A+i\gamma_5 B)R^{(n-1)}\,.$$

The formal solution is

$$[(1-\frac{a}{2}D_1)^{-1}D_2+m+\sqrt{2}g(A+i\gamma_5 B)]R = \Delta L\,.$$

- $R \to 0$ for $a \to 0$, since ΔL vanishes in this limit.
- ΔL is different from zero because of the breaking of the Leibniz rule for a finite lattice spacing.

The algebra

By the commutator of two supersymmetries one find a transformation which is still a symmetry of the Wess-Zumino action, i.e. the transformations of the fields form a closed algebra, order by order in g.

Up to order g^1, (the rest can be generalized!)

Two supersymmetry transformations on the scalar field A give

$$\begin{aligned}\delta_1\delta_2 A &= \delta_1(\bar{\varepsilon}_2\chi) \\ &= -\bar{\varepsilon}_2[D_2(A - i\gamma_5 B)\varepsilon_1 + (F - i\gamma_5 G)\varepsilon_1 - gR\varepsilon_1]\end{aligned}$$

and their commutator yields

$$[\delta_2, \delta_1]A = -2\bar{\varepsilon}_1 D_2 \varepsilon_2 A + g(\bar{\varepsilon}_1 R \varepsilon_2 - \bar{\varepsilon}_2 R \varepsilon_1)\,.$$

The order g^1 of the second term on the r.h.s. reads

$$g(\bar{\varepsilon}_1 R^{(1)}\varepsilon_2 - \bar{\varepsilon}_2 R^{(1)}\varepsilon_1) = \\ \sqrt{2}g\bar{\varepsilon}_2 \frac{m(1 - \frac{a}{2}D_1)}{m^2(1 - \frac{a}{2}D_1) + \frac{2}{a}D_1}[D_2(A^2 - B^2) - 2(AD_2A - BD_2B)]\varepsilon_1$$

Finally, the commutator of two supersymmetries on the scalar field A is

$$[\delta_2,\delta_1]A = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}A + \frac{g}{\sqrt{2}}\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(A^2-B^2)-2(AD_{2\mu}A-BD_{2\mu}B)]\}\,.$$

Similarly, the commutators of two supersymmetries on the other fields, up to terms of order g^1, are

$$[\delta_2,\delta_1]B = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}B + \sqrt{2}g\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(AB)-(AD_{2\mu}B+BD_{2\mu}A)]\}\,,$$

$$[\delta_2,\delta_1]F = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}F - \frac{g}{\sqrt{2}}\frac{D_2^2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(A^2-B^2)-2(AD_{2\mu}A-BD_{2\mu}B)]\}\,,$$

$$[\delta_2,\delta_1]G = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}G - \sqrt{2}g\frac{D_2^2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(AB)-(AD_{2\mu}B+BD_{2\mu}A)]\}$$

and

$$[\delta_2,\delta_1]\chi = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}\chi - \frac{g}{\sqrt{2}}\frac{m(1-\frac{a}{2}D_1)-D_2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}\big(D_2(A-i\gamma_5 B)\gamma_\mu\chi + (A+i\gamma_5 B)D_2\gamma_\mu\chi - D_2[(A-i\gamma_5 B)\gamma_\mu\chi]\big)\}\,.$$

Therefore, the general expression of these commutators is

$$[\delta_1,\delta_2]\Phi = \alpha^\mu P^\Phi_\mu(\Phi)\,, \qquad \Phi = (A,B,F,G,\chi)\,,$$

where $\alpha^\mu = -2\bar{\varepsilon}_2\gamma^\mu\varepsilon_2$ and $P^\Phi_\mu(\Phi)$ are polynomials in Φ defined as

$$P^\Phi_\mu(\Phi) = D_{2\mu}\Phi + O(g)$$

. We have verified that the closure works, i.e. the action is invariant under the transformation

$$\Phi \to \Phi + \alpha^\mu P^\Phi_\mu(\Phi)$$

up to terms of order g^1. Notice that, in the continuum limit $D_{2\mu} \to \partial_\mu$ and the transformation reduces to

$$\Phi \to \Phi + \alpha^\mu\partial_\mu\Phi$$

Conclusions and Outlook

- WZ model is an interesting model to understand how to put GW fermions with exact lattice supersymmetry.
- Study of the Ward identities.
- numerical simulations of this model (at leat in two dimensions)
- The forward step would be to apply to $N = 1$ SYM (more tricky!)
- ...