QUANTUM PHASE TRANSITIONS & THE NONLINEAR σ-MODEL

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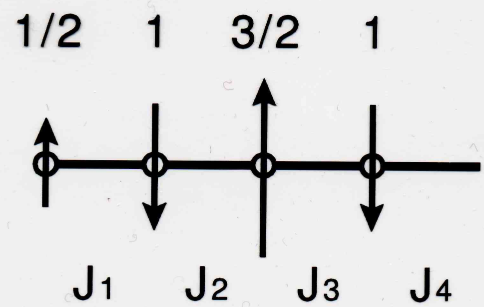
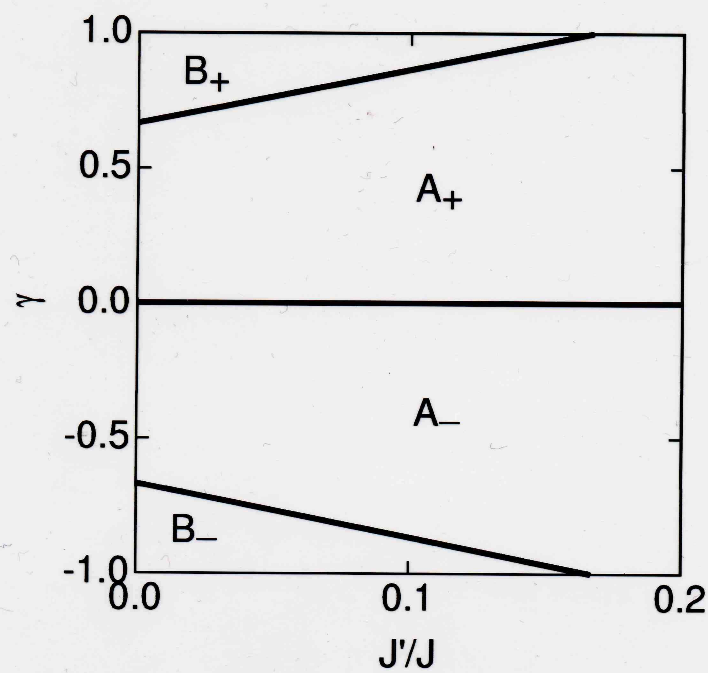
HALDANE'S CONJECTURE : SPIN EXCITATION SPECTRUM HAS A GAP FOR INTEGER SPIN SYSTEM

HALDANE'S DERIVATION FOLLOWS FROM MAPPING TO NONLINEAR σ-MODEL



$$H = \sum_j \left(J_1\, S_{4j+1} \cdot S_{4j+2} + J_2\, S_{4j+2} \cdot S_{4j+3} + J_3\, S_{4j+3} \cdot S_{4j+4} + J_4\, S_{4j+4} \cdot S_{4j+5} \right)$$



QUANTUM PHASE DIAGRAM

$$J = J_1 = J_4$$

$$J' = J_2 (1-\gamma) = J_3 (1+\gamma)$$

NONLINEAR σ-MODEL FOR MIXED-SPIN

$$H = \sum_j \big(J_1\, S_{4j+1} \cdot S_{4j+2} + J_2\, S_{4j+2} \cdot S_{4j+3} + J_3\, S_{4j+3} \cdot S_{4j+4} + J_4\, S_{4j+4} \cdot S_{4j+5} \big)$$

REWRITE IN TERMS OF UNIT VECTOR n

RESTRICTIONS FOR SINGLETS (LIEB-MATTIS THEOREM)

$$\langle S_j \rangle = (-1)^j s_j n_j \qquad \sum_j^{\overbrace{2b}^{\text{PERIOD}}} (-1)^j s_j = 0$$

CAN THEN DEFINE PARTITION FUNCTION IN TERMS OF n

$$\Omega = \int D[n_j] \prod_j \delta(n_j^2 - 1)\, e^{-S}$$

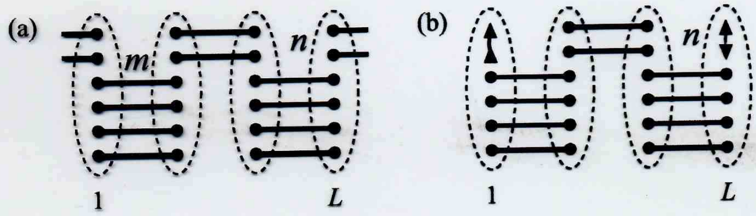
$$S = i \underbrace{\sum_{j=1}^{N} (-1)^j s_j\, w[n_j]}_{\text{BERRY PHASE - BOUNDARY CONDITION}} + \int_0^\beta d\tau \sum_{j=1}^{N} J_j s_j s_{j+1}\, n_j \cdot n_{j+1}$$

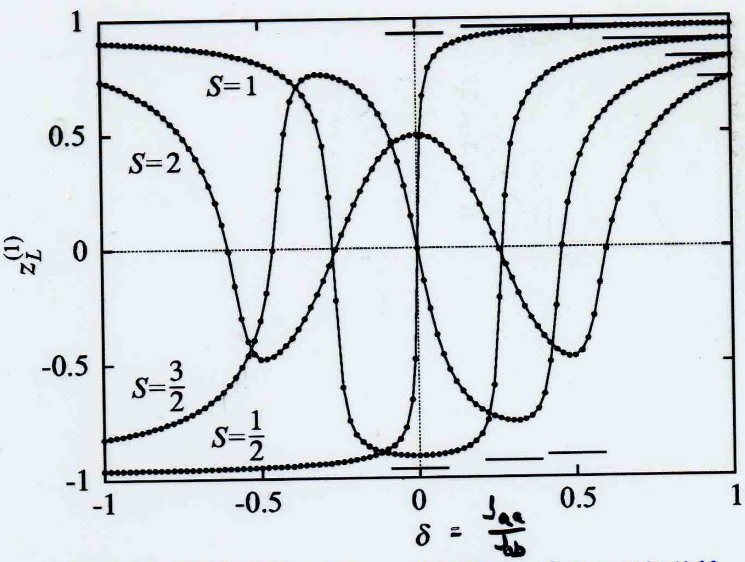
CAN NOW REXPRESS n AS SLOWLY VARYING m + FLUCTUATION AND THEN PERFORM INTEGRATION AND LARGE VOLUME LIMIT

$$S_{EFF} = \int_0^\beta \int_0^L dx\, d\tau \left\{ -i \frac{J^{(0)}}{J^{(1)}}\, m \cdot (\partial_\tau m \times \partial_x m) + \frac{1}{2a\, J^{(1)}} \left(\frac{J^{(1)}}{J^{(2)}} - \frac{J^{(0)}}{J^{(1)}} \right) (\partial_\tau m)^2 + \frac{a}{2} (\partial_x m)^2 \right\}$$

$$\frac{1}{J^{(a)}} = \frac{1}{2b} \sum_{q=1}^{2b} \frac{\left(\sum_{k=1}^{q} (-1)^{k+1} s_k \right)^a}{J_q\, s_q s_{q+1}}$$

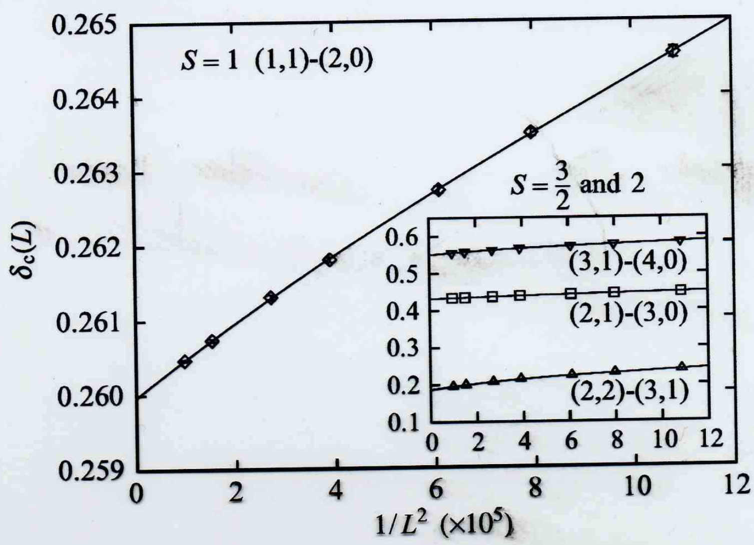
INTERFERENCE GIVES DIMERISED PAIRS





$$z_L^{(1)} = \sum_{i,j} \langle S_i \ \exp\left[i \frac{2\pi}{L} \sum_{j=1}^{L} j S_j \right] S_j \rangle$$

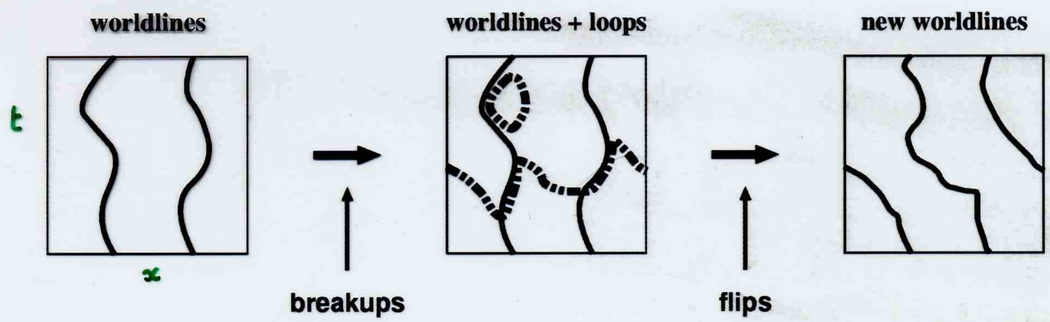
A LATTICE MEASUREMENT TO FIND QUANTUM TRANSITION POINTS USING A TWIST ORDER PARAMETER



TRANSITION VALUE OF ANISOTROPY PARAMETER CALCULATED AS A FUNCTION OF LATTICE VOLUME^{1}, DETERMINED FROM TWIST ORDER PARAMETER $z_L^{(1)}$

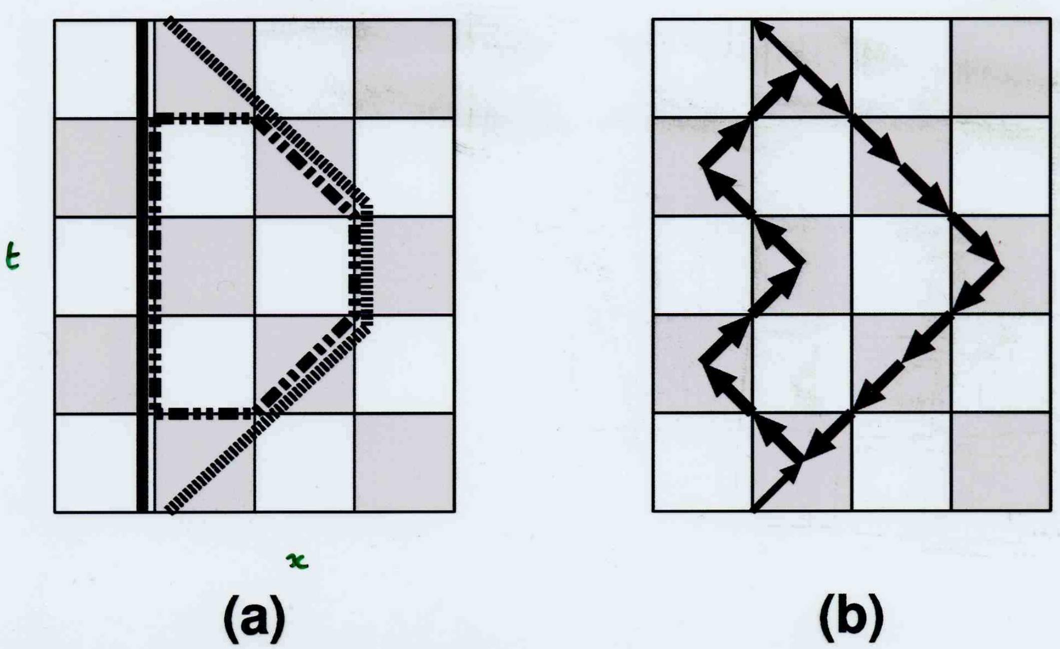
HOW TO SIMULATE NONLINEAR σ-MODEL ON LATTICE

LOOP CLUSTER METHOD - GLOBAL CHANGE FROM LOCAL UPDATE



LOOP GENERATION STEP IS STOCHASTIC

GLOBAL CHANGE IS METROPOLIS DECISION ACCEPTANCE OF CONFIGURATION WEIGHT (SUM OF LOCAL WEIGHTS)



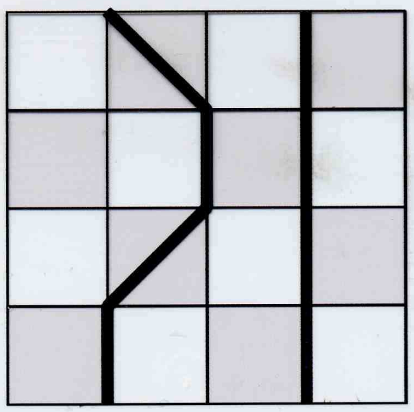
WORLDLINE $\longrightarrow$ NEW WORLDLINE

MUST TAKE ARBITRARLY LARGE TIME EXTENT AS

$$\Omega = \mathrm{Tr}\, e^{-\beta H} = \lim_{N_t \to \infty} \mathrm{Tr}\left(e^{-\frac{\beta}{N_t} H}\right)^{N_t}$$

CONTINUOUS TIME LOOP CLUSTER METHOD

ALLOW THE LOOPS TO PROPAGATE FOR VARIABLE N_t UNTIL THEY CLOSE

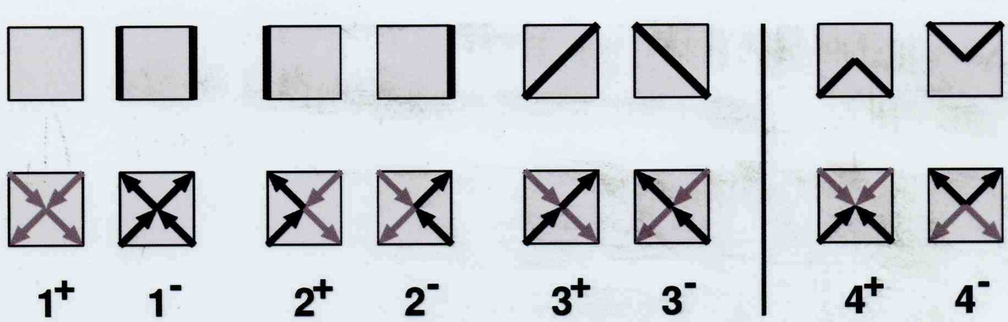


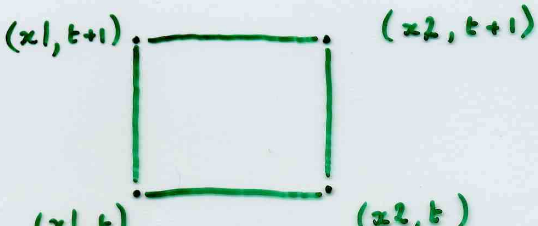
(a) (b)

LOCAL WEIGHTS ARE DEFINED USING $N_t \to \infty$ LIMIT AND USED TO MAKE LOCAL METROPOLIS DECISIONS

LOOP GENERATION STEP IS <u>NOT</u> STOCHASTIC (METROPOLIS)

GLOBAL CHANGE IS ACHIEVED BY FINAL OVERRELAXATION STEP (FLIP ALL SPINS)

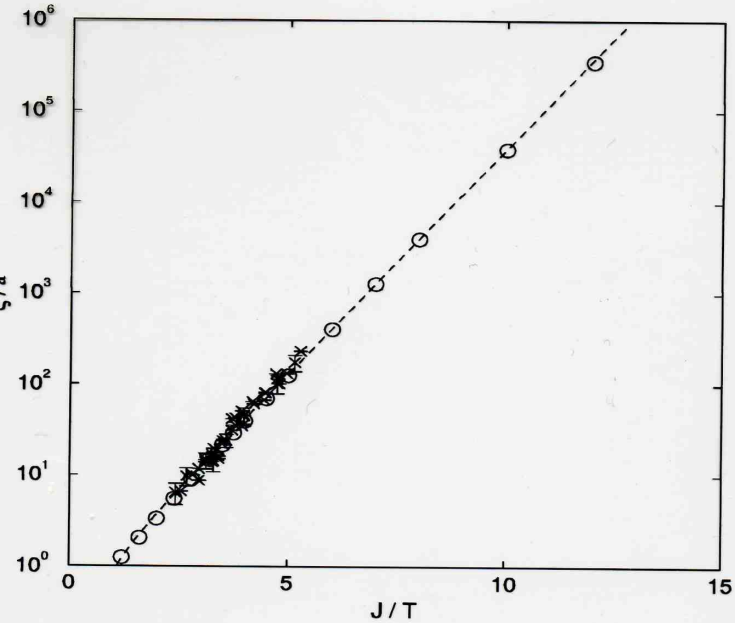




$$T = \begin{pmatrix} (x1|x1) & (x1|x2) \\ (x2|x1) & (x2|x2) \end{pmatrix}$$

TRANSFER MATRIX

COMARISON OF CORRELATION LENGTH LATTICE MEASUREMENT WITH CHIRAL PERTURBATION THEORY (HASENFRATZ + NIEDERMAYER '02)



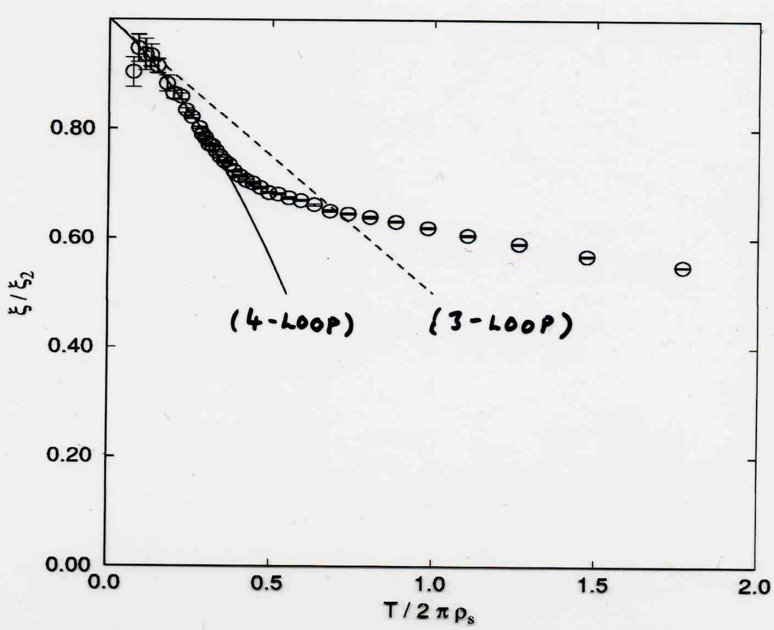
CORRELATION LENGTH (2-LOOP)

$$\xi_{CPT} = \frac{e \hbar c}{16\pi \rho_s} \exp\left(\frac{1}{t}\right)\left[1 - \frac{t}{2} + O(t^2)\right]$$

$$t = \frac{kT}{2\pi \rho_s}$$

$[H, \Sigma S_i] = 0$

$\Rightarrow$ O(3)

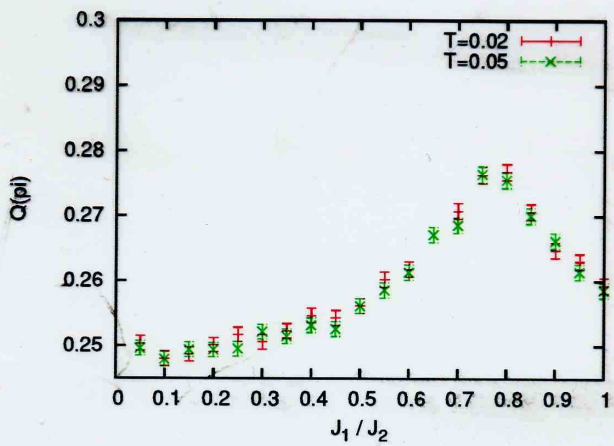


CORRECTIONS - NORMALISED CORRELATION LENGTH BY 2-LOOP RESULT

DEVIATION FROM CHIRAL PERTURBATION THEORY AT SMALL T

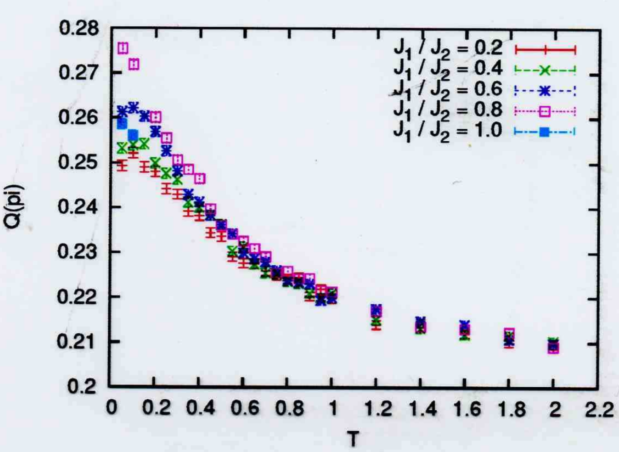
INTERNAL ENERGY SUSCEPTIBILITY

1-1-½-½   L = 512   # = 100,000



COUPLING ANISOTROPY PREDICTION OF NONLINEAR σ-MODEL $J_1/J_2 \approx 0.76$ [T. TAKANA '02]

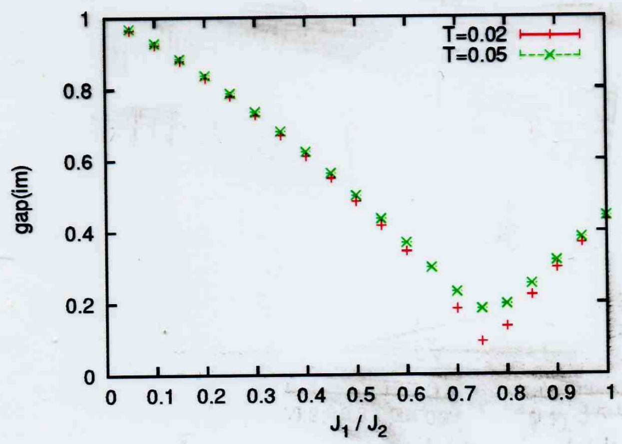
RELATIVELY INSENSITIVE TO TEMPERATURE



NO CRITICAL SLOWING DOWN OBSERVED IN CONTINUOUS-TIME QUANTUM MONTE CARLO METHOD THROUGH LOCAL-UPDATING METHOD

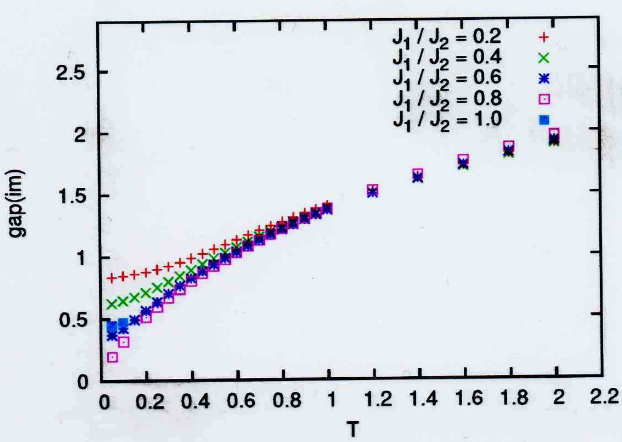
GAP (INVERSE CORRELATION LENGTH) VIA SECOND MOMENT METHOD IN IMAGINARY TIME

1-1-½-½   L = 512   # = 100,000



CORRELATION LENGTH

$$\xi_\tau = \frac{\beta}{2\pi} \sqrt{\frac{C(\pi,0)}{C(\pi, 2\pi/\beta)} - 1.}$$



GAP CONVERGES TO PHYSICAL VALUE FOR

$L \geq 256$ IN SINGLE SPIN CHAINS

$S = 1, 3/2, 2$ ... [S. TODO, M. TROYER '01]

CAN INVESTIGATE CORRECTIONS TO CORRELATION LENGTH (COMPARED WITH CHIRAL PERTURBATION THEORY) BY UNDERSTANDING OF WHICH ARE TEMPERATURE DEPENDENT CORRECTIONS AND WHICH ARE VOLUME DEPENDENT

FOR LOCAL-UPDATE CONTINUOUS TIME METHOD KNOW TRANSFER MATRIX OF ALL SITES EXPLICITLY

$$T = \begin{pmatrix} (x1|x1) & (x2|x1) \\ (x1|x2) & (x2|x2) \end{pmatrix}$$

TAKING THE CONTINUOUS TIME LIMIT $N_t \to \infty$

$$T = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix} \qquad \lambda = \frac{\beta J}{2}$$

WHICH CAN THEN EVALUATE NUMERICALLY FOR THE SPECIFIC VALUES OF λ' THAT APPEAR AS TRANSITION PROBABILITIES IN THE MONTE CARLO SIMULATION

PARTITION FUNCTION

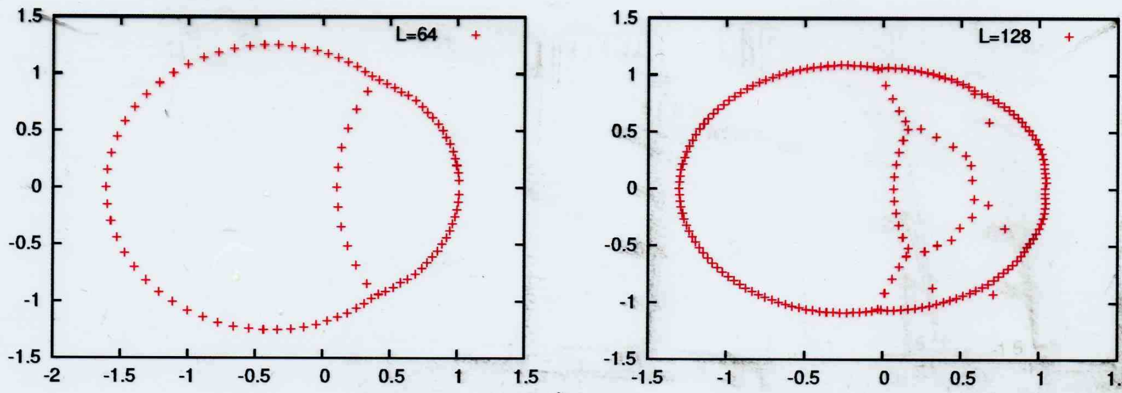
$$\begin{aligned} \Omega &= \mathrm{Tr}\ T^N \\ &= \sum_n c_n \lambda^n \\ &= \sum_n \Omega_n \end{aligned}$$

HAVE WRITTEN PARTITION FUNCTION AS A SEMI-ANALYTIC POLYNOMIAL IN λ

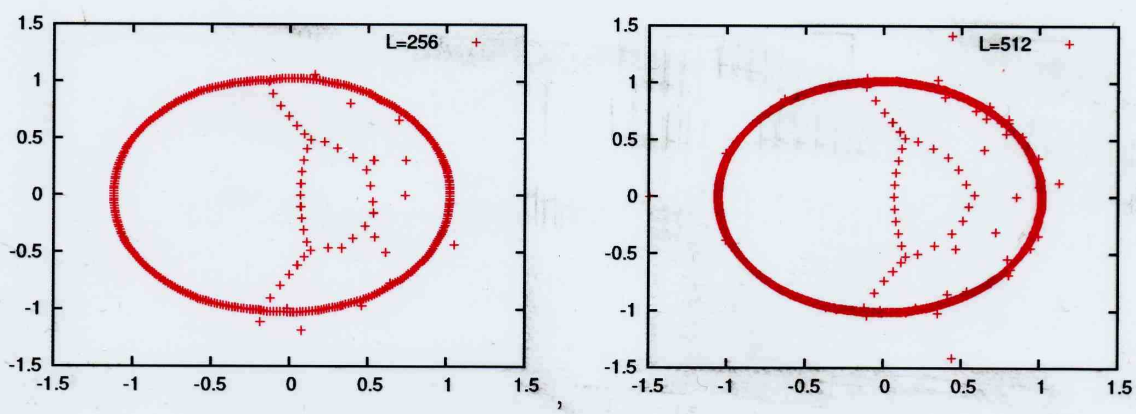
PROBABILITY DENSITY FUNCTION FOR n LOOPS ON THE LATTICE GIVEN BY Ω_n

FISHER ZEROES OF PARTITION FUNCTION IN $\lfloor J^2$

(LEE - YANG THEOREM) LOCI CONVERGENCE TO REAL AXIS CORRESPONDS TO CRITICAL BEHAVIOR



OUTER RING CORRESPONDS TO ISING - LIKE SOLUTION, BUT WITH OBVIOUS CONVERGENCE OF NEGATIVE ZEROES TO UNIT CIRCLE (FINITE VOLUME EFFECT)



QUANTUM PHASE TRANSITION INTERNAL FEATURE

REPEAT ROOT FROM POLYNOMIAL DEFINITION $\rightarrow$

$1-\lambda, \lambda$

AGAIN $J_1/J_2 \sim 0{\cdot}76$ FOR QUANTUM TRANSITION

CALCULATION IN PROGRESS TO EVALUATE CORRELATION LENGTH CRITICAL EXPONENT VIA FINITE SIZE SCALING $L = 16, 32, 64, 128, 256, 512, 1024$

COMPARE THERMAL (QUANTUM) CORRECTIONS TO SCALING OF GAP WITH SCALING OF REAL & IMAGINARY PARTS OF FISHER ZEROES WHICH SCALE AS PSUEDOCRITICAL AND CRITICAL POINTS - DISTINGUISH FINITE VOLUME EFFECTS WITH COMPARISON OF SCALING OF GAP

DIMENSIONAL REDUCTION $1+1 \rightarrow 1$ QUANTUM INTRODUCES A SELF-ENERGY VACUUM CONTRIBUTION

$$H = J \sum_{i}^{N} S_i S_{i+1} + \sum_{i}^{N} S_i^2$$

PARTITION FUNCTION : TENSOR PRODUCT

$$\Omega = \mathrm{Tr}\ T^N$$

METROPOLIS WEIGHT STRICTLY DEFINED ONLY IN TROTTER LIMIT $N \rightarrow \infty$, AT SMALL N THIS LEADS TO A CORRECTION TO THE FREE ENERGY IN THE UPDATE

→ DIFFERENCE BETWEEN SPATIAL PROPAGATION OF LOOPS AND PROPAGATION FROM PERIODIC B.C.

→ PLANAR PHASE TRANSITION, SPACE → TIME AT $J=1$

PROBABILITY DENSITY FUNCTION FOR n-LOOPS ON LATTICE FOR LATTICES EVALUATED AT DIFFERENT β AND J

LARGE T

$\beta = \frac{1}{T}$

1D $S = \frac{1}{2}$ AHFM

SMALL T

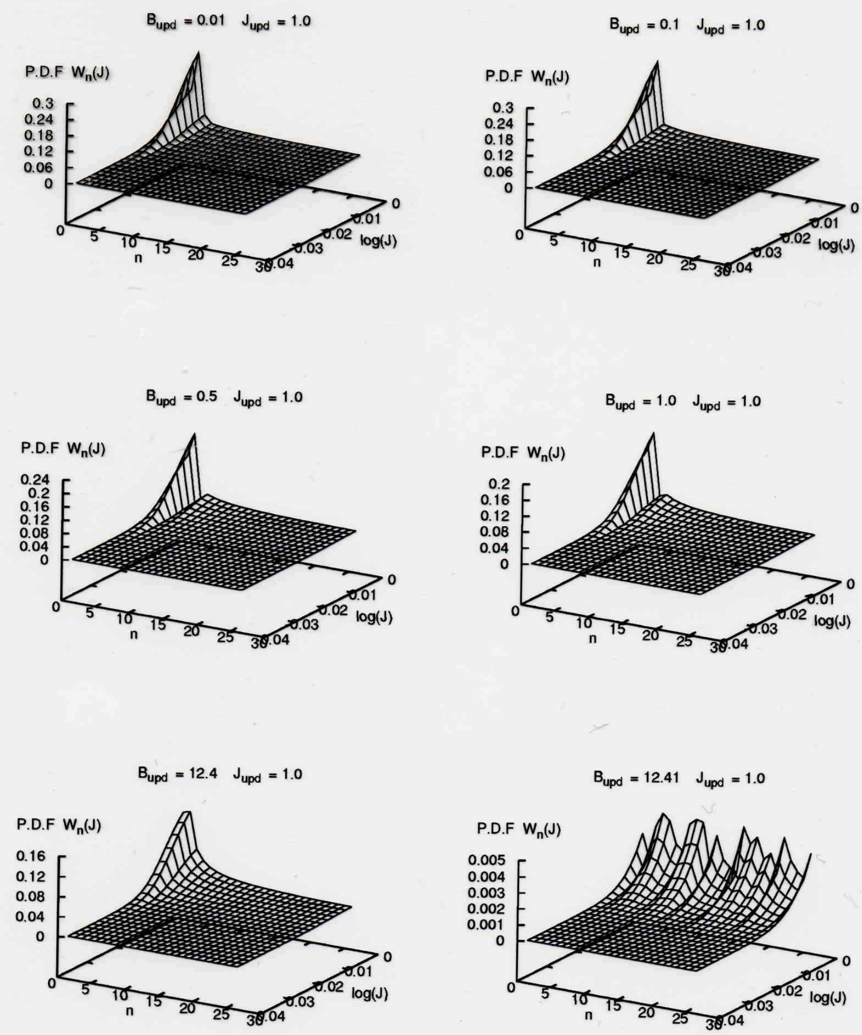
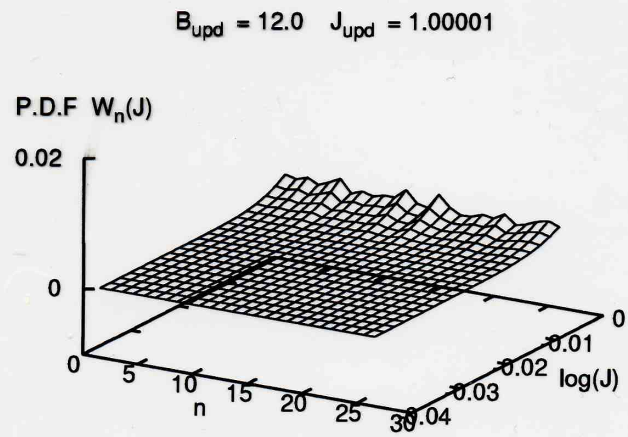


Figure 5: Plot of PDFs $W_n(J)$ as a function of n and $log(J)$ for the low-lying modes. Varying β_{upd} at fixed J_{upd} the peak of the PDF in the pure phase can be seen to slowly decrease (note the varying scales) prior to the sharp change at some critical $\beta_{upd} \sim 12.4$ to an essentially flat distribution.

UNDERGO TEMPERATURE PHASE TRANSITION AT $\beta = 12 \cdot 41$ WHERE PEAKED PROBABILITY DENSITY FUNCTION (PURE PHASE) $\longrightarrow$ FLAT DISTRIBUTION (DISORDERED PHASE)

AT T_c VARYING COUPLING J SLIGHTLY GIVES TRANSITION - INDICATES THAT QUANTUM EFFECTS (DEPENDENT ON J) CAN BE DISTINGUISHED FROM THERMAL EFFECTS WITH THIS APPROACH



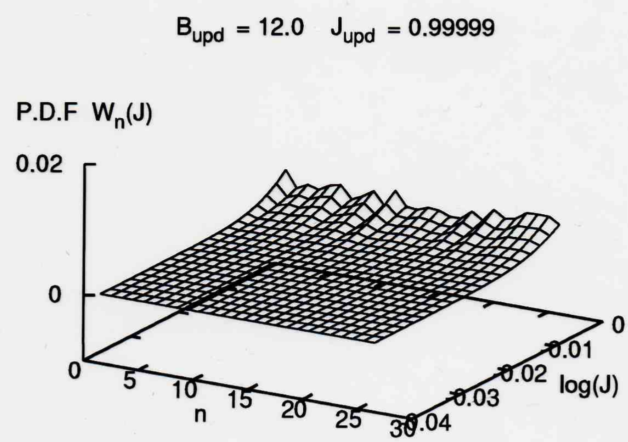


Figure 6: Plot of PDFs $W_n(J)$ as a function of n and $log(J)$ for the low-lying modes. At large β_{upd}, and J_{upd} infinitesimally away from the transition at $J = 1$, the distribution is essentially flat. This should be compared with Figure 4 which has essentially a peaked distribution for an ensemble at a larger fixed value of βJ generated at $J_{upd} = 1$.

VARYING J GIVES DIFFERENT INTERFERENCE TERM IN NONLINEAR σ-MODEL $\rightarrow$ DIFFERENT QUANTUM PHASE ABOVE T_c

EFFECTIVE THEORIES SHARING THE SAME NONLINEAR σ-MODEL AS SPIN CHAINS, AS FOR CHIRAL PERTURBATION THEORY AND 2D AFHM CAN COMPARE 1-, 2-, 3-LOOP CALLULATIONS OF CORRELATION LENGTH, SPIN WAVE VELOCITY, SUSCEPTIBILITY ... TO SEE NONPERTURBATIVE EFFECTS

CAN SHOW GENERALLY $S = 1/2$ XXX SPIN MODEL LIEB-MATTIS CONSTRAINT ON HAMILTONIAN $[H, S_i] = 0$ DEFINES AN $SL(2,R)$ ALGEBRA, SIMILARLY HIGHER SPINS $S = 1, 3/2 \ldots$ & ANISOTROPIC SPIN COUPLINGS LEAD TO $SU(2)$, $SU(3)$ ALGEBRAS XXZ ...

1/ $N = 4$ SUPER YANG-MILLS

(BEISERT, HEP-TH/0310252
HERNANDEZ, HEP-TH/0403139)

SPIN CHAIN NONLINEAR σ-MODEL $\leftrightarrow$ DILATION OPERATOR OF ANOMALOUS DIMENSION

INTEGRABILITY OF NONLINEAR σ-MODEL FOR $SU(3)$ IN $N = 4$ SYM CAN ONLY BE ESTABLISHED BY CONSIDERING THE MIXED REPRESENTATION $SU(2|3)$

2/ TWIST-3 PARTON DISTRIBUTIONS

(DERKACHOV, HEP-PH/9909539)

HAS BEEN STUDIED (XXX) THROUGH ANALYTIC SOLUTION OF TRANSFER MATRIX, LIGHT CONE GAUGE QCD $\rightarrow SL(2,R)$, PREDICT EXCITED GAP STATES