Complex matrix models for QCD-like theories with chemical potential

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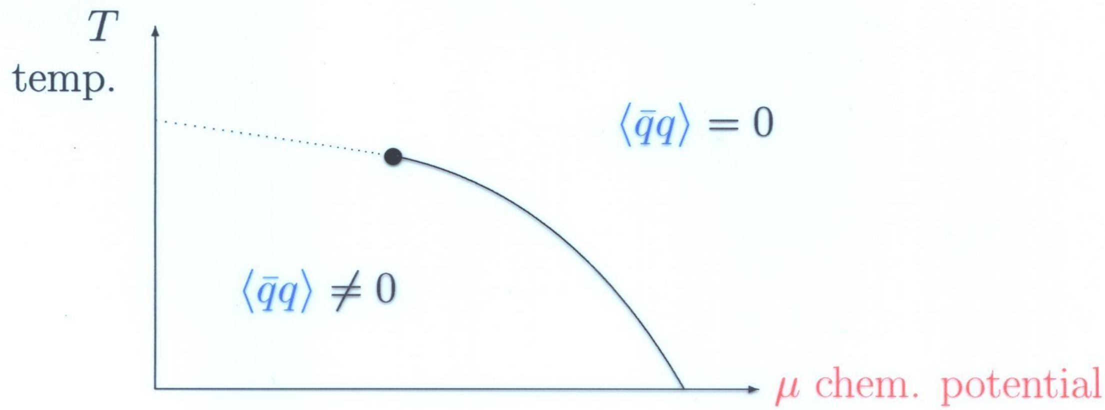
Outline

- χSB Patterns and Symmetry Classes
- Effective Theory: 0-mode χPT + μ
- Complex Random Matrix Models for $\beta_D = 2$ & 4

Motivation



- **QCD Lattice** simulations at $\mu \neq 0, T \approx 0$ **remain difficult**
  $\longrightarrow$ **different theory** $SU(2)$ **fund. / adj.** (or: quenching)

- alternative **Matrix Model approach** (or χPT): **Symmetry!**

  – **analytically solvable**

  – descibe **detailed flavour** (N_f, m_f)**, topology** (ν)**, chemical potential dependence**

  $\longrightarrow$ compare to Lattice simulations

- patterns of **complex Dirac eigenvalue distribution very different !**

Chiral Symmetry Breaking in QCD

$$\mathcal{Z}_{\text{QCD}} \equiv \int [dA\, dq]\ \exp[-\int \bar{q}(\not{D} + M)q + \mu\bar{q}\gamma_0 q\ - \int \mathrm{Tr} F^2]$$

helicity basis $q_{R,L} \equiv \frac{1}{2}(1 \pm \gamma_5)q$

chiral symmetry (global):

$U_{R,L} \in\ SU(N_f)$

$$q_L \rightarrow U_L q_L$$
$$q_R \rightarrow U_R q_R$$

breaking

- $\bar{q}Mq = m(\bar{q}_R q_L + \bar{q}_L q_R)$ explicit
- $\langle\bar{q}q\rangle = \langle\bar{q}_R q_L + \bar{q}_L q_R\rangle \neq 0$ **spontaneous**

$$SU_L(N_f) \times SU_R(N_f) \rightarrow SU(N_f)$$

$\exists$ 2 other χSB patterns $(\mu = 0)$**:**

$SU(N_c = 2)$ fund.: $SU(2N_F) \rightarrow Sp(N_f)$

$SU(N_c \geq 2)$ adj.: $SU(N_F) \rightarrow SO(N_f)$

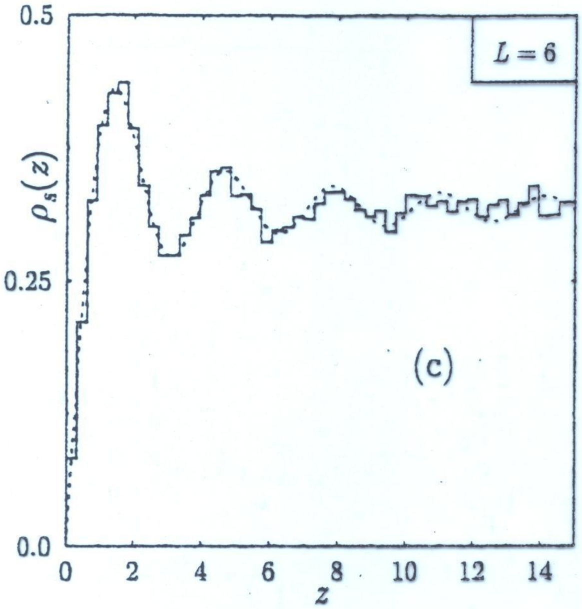
χSB patterns & symmetry classes ($\mu = 0$)

gauge group and representation	χSB patterns	MM
$SU(N_c \geq 3)$, fundamental:	$SU(N_f) \times SU(N_f)$	$\beta_D = 2$
$\not{D} = \begin{pmatrix} 0 & i\Phi \\ i\Phi^\dagger & 0 \end{pmatrix}$ **complex**	$\longrightarrow SU(N_f)$	chGUE
$SU(N_c = 2)$, fundamental:	$SU(2N_f) \longrightarrow Sp(2N_f)$	$\beta_D = 1$
$\not{D} = \begin{pmatrix} 0 & \Phi \\ -\Phi^T & 0 \end{pmatrix}$ **real** from $[C\sigma_2 K, \not{D}] = 0$		chGOE
$SU(N_c)$, adjoint:	$SU(N_f) \longrightarrow SO(N_f)$	$\beta_D = 4$
$\not{D} = \begin{pmatrix} 0 & \Phi \\ -\Phi^\dagger & 0 \end{pmatrix}$ **real quatern.** from $[CK, \not{D}] = 0$		chGSE

$\beta_D =$ **d.o.f.**

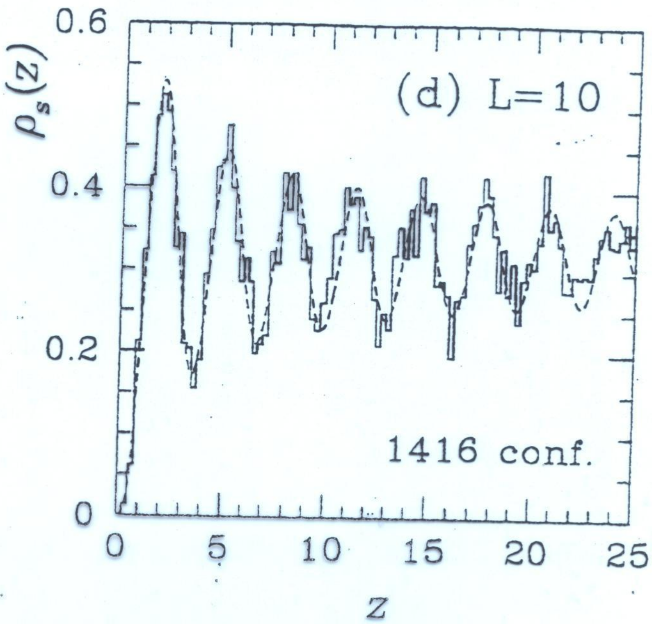
- lattice **staggered fermions: exchange** $\beta_D = 1 \longleftrightarrow 4$

Comparison with QCD lattice data



quenched SU(3)

Göckeler et al. 99



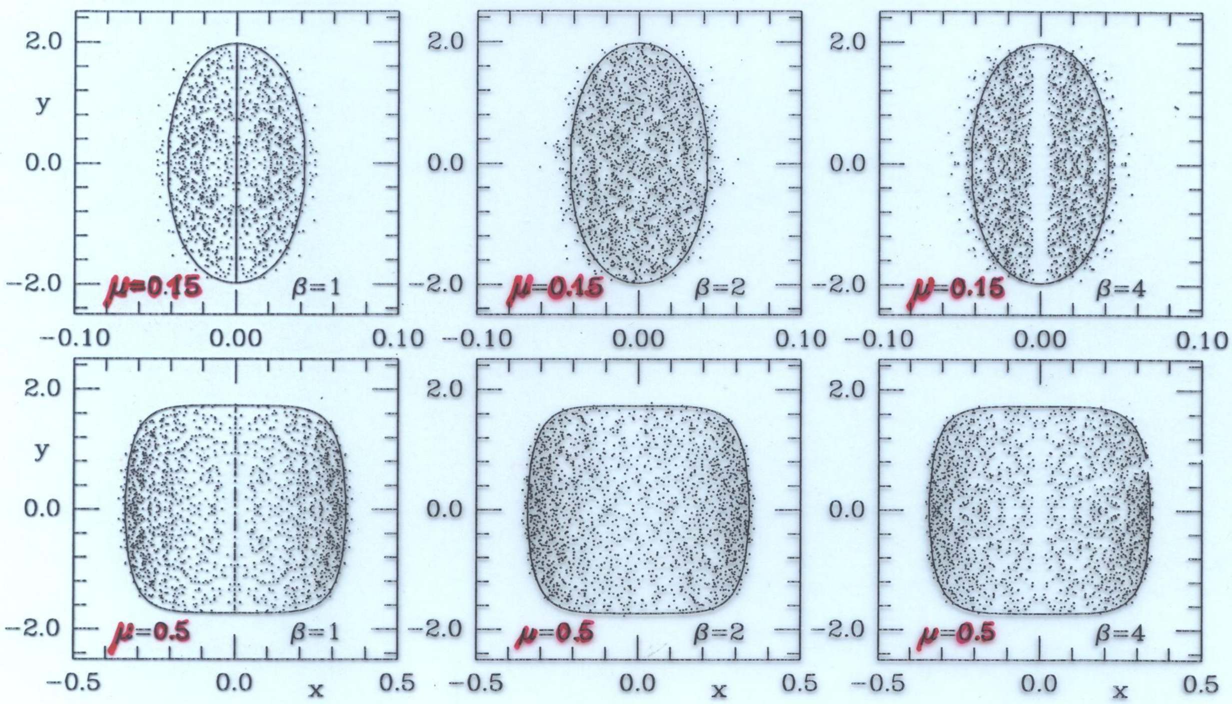
quenched SU(2)

Berbenni-Bitsch et al. 98

Symmetry classes of complex Matrix Models

$$\mathcal{Z}_{MM} \equiv \int d\Phi \ \ \det \begin{pmatrix} 0 & i\Phi + \mu \\ i\Phi^\dagger + \mu & 0 \end{pmatrix} \exp[-N \ \mathrm{Tr} \ \Phi^\dagger \Phi]$$

with Φ complex (real, symplectic)



Halasz, Osborn, Verbaarschot 97

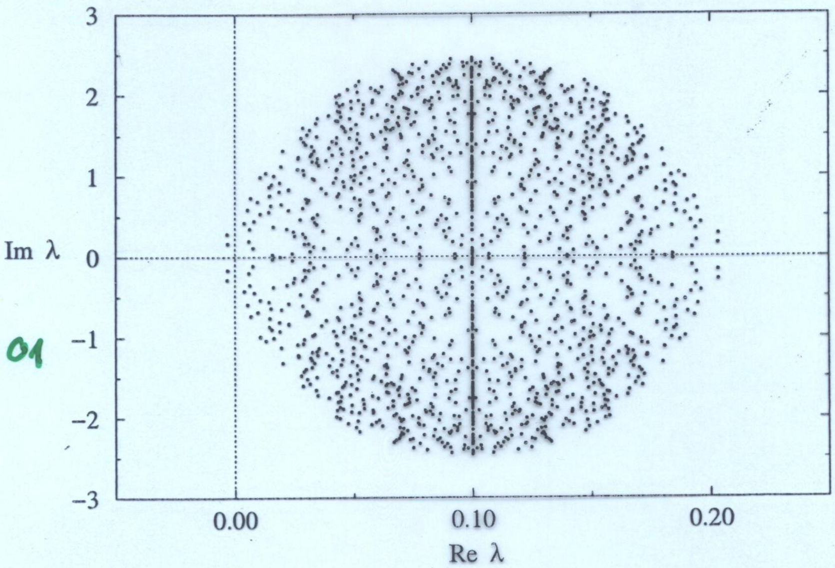
SU(2) *fund.* *SU*($N_c \geq 3$) *fund.* *SU*(N_c) *adj.*

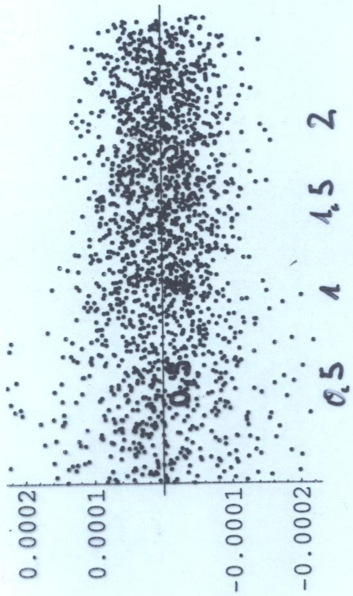
- Aim: **analytic calculation of zoomed spectra** at zero & comparison to lattice data

2) Symmetry classes of Dirac spectra with $\mu \neq 0$

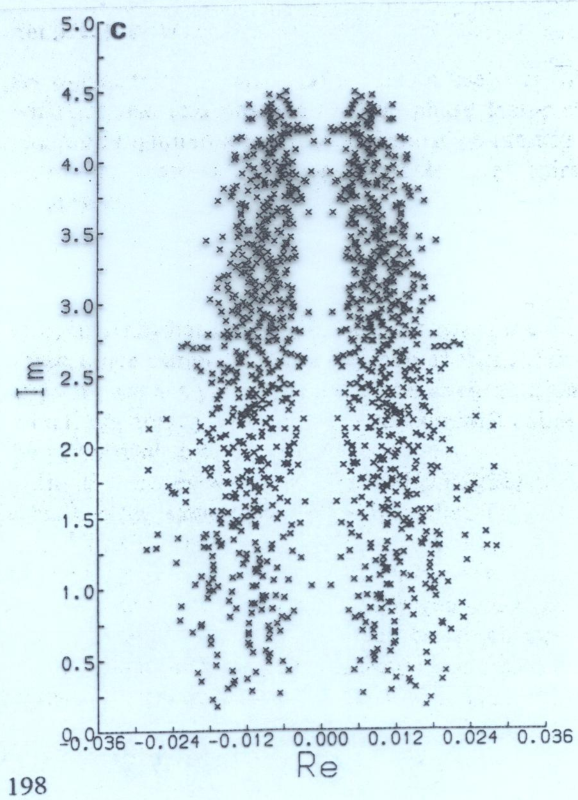
Lattice data:

SU(2)
adj.
staggered
$\mu = 0.35$
Montvay et al 01





SU(3) fund
$\mu = 0.006$



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SU(2) fund
staggered
$\mu = 0.1$
Baillie et al. 87

Limit QCD $\longrightarrow \chi$PT $\longrightarrow$ MM

$\mathcal{Z}_{\mathbf{QCD}}$	$\longrightarrow$	$\mathcal{Z}_{\chi PT}$	$\longrightarrow$	$\mathcal{Z}_{\varepsilon}$
q,A	$\langle \bar{q} q \rangle \neq 0$	pions $\pi(x)$	$\lvert V \rvert < \infty$	$SU(N_f)$ integral
		"Goldstone"	0-mode	$\Leftrightarrow \mathcal{Z}_{MM}$

approx. $$\boxed{1/\Lambda \ll V^{1/4} \ll 1/m_\pi}$$

unphysical **finite Volume IR limit $\Rightarrow$ analytic solution**

χPT to leading order

$$\mathcal{Z}_{\chi PT} \equiv \int [d\Sigma(x)] \; \exp[- \int \mathrm{Tr}\mathcal{L}_{eff}(\Sigma, \partial\Sigma)]$$

$$\mathcal{L}_{eff}(\Sigma, \partial\Sigma) = \frac{F_\pi^2}{4}\partial\Sigma(x)\partial\Sigma(x)^\dagger - \frac{1}{2}\langle\bar{q}q\rangle M \left(\Sigma(x) + \Sigma(x)^\dagger\right) + \ldots$$

parametrize

$$\Sigma = U \in \; SU(N_f) \quad \text{for } \beta_D = 2$$

$$\Sigma = U \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U^T \text{ for } \beta_D = 1$$

$$\Sigma = U \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U^T \text{ for } \beta_D = 4$$

χPT + μ: global $\longrightarrow$ local symmetry

- $\exists$ χ**PT** $+\ \mu$ for quenched **QCD** & $\beta_D = 1, 4$

- **local symmetry on fermion action** (SU(2))

$$\mathcal{L}_q = \bar{q}\not{D}q \ - \ \mu q^\dagger B q, \quad B = \begin{pmatrix} +1 & 0 \\ & -1 \end{pmatrix} \text{ baryon charge}$$

$\beta_D = 1, 4$ (similar for $B_{L,R}$ at $\beta_D = 2$)

$q \to Uq$ and $B_\nu \to UB_\nu U^\dagger - \frac{1}{\mu} U \partial_\nu U^\dagger$

$\Rightarrow$ **local symmetry on Goldstone action**

$$\partial_\nu \Sigma \to \mathcal{D}_\nu \Sigma = \partial_\nu \Sigma - \mu(B_\nu \Sigma + \Sigma B_\nu) \qquad B_\nu = (B, \mathbf{0})$$

- zeromode term

$$\mathcal{S}_{eff}(\Sigma_0) = -\frac{F_\pi^2}{2}\mu^2 V \Sigma_0 B^T \Sigma_0^\dagger B - \frac{1}{2}\langle \bar{q}q \rangle VM \left(\Sigma_0 + \Sigma_0^\dagger\right)$$

- fixes all coupling constants & **scaling**:

$\mu^2 V$ and VM

Matrix Model(s) for QCD + $\mu \neq 0$

$$\mathcal{Z}_{MM} \equiv \int d\Phi \ \Pi_f^{N_f} \det \begin{pmatrix} m_f & i\Phi + \mu\Psi \\ i\Phi^\dagger + \mu\Psi^\dagger & m_f \end{pmatrix} \exp[-N \ \mathrm{Tr}\ \Phi^\dagger\Phi + \Psi^\dagger\Psi]$$

(i) **no eigenvalue representation**, difficult

(ii) **complex $\not{D} + \mu\gamma_0$ eigenvalue model** ($\beta_D = 2, 4$)

$$\mathcal{Z}_{MM} = \int \Pi_j^N \ d^2 z_j |z_j|^{2\nu+1} \Pi_f^{N_f} (z_j^2 + m_f^2) \exp\left[-N \frac{|z_j|^2 - \frac{\tau}{2}\Re e(z_j^2)}{1-\tau^2}\right] |\Delta(z^2)|^2$$

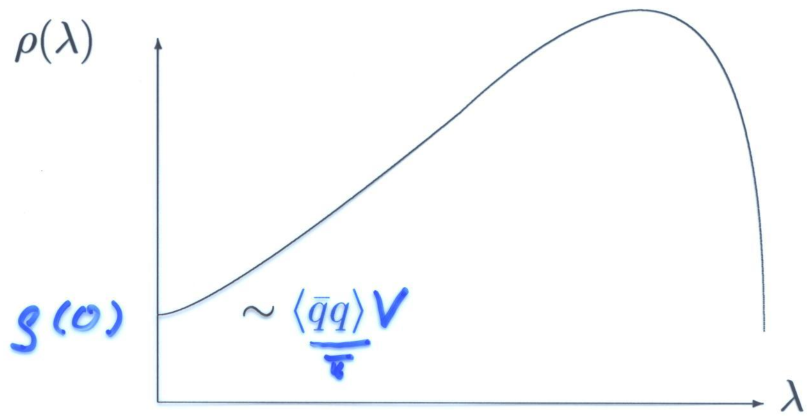
$\boxed{\mu^2 \sim (1-\tau^2)}$ **non-Hermiticity parameter**

- **no flavor–topology duality**
- action has a sign problem, **but $\mathcal{Z}$ remains real**

(iii) **two-matrix model** with blocks $i\Phi + \mu\Psi$: $\exists$ eigenvalue rep.

* all 3 models **agree for small** μ: $1 \gg \mu^2/N|z_{min}|^2$

A typical $\not{D}$ spectrum on the lattice $(\mu = 0)$

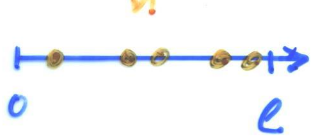


scaling: correlations at 0 important $\Rightarrow$ **zoom**

microscopic limit $\boxed{\lambda V \langle \bar{q}q \rangle \; = \; X}$ const. for $V \to \infty$, $\lambda \to 0$

$\neq$ free scaling $\lambda \sim 1/L$

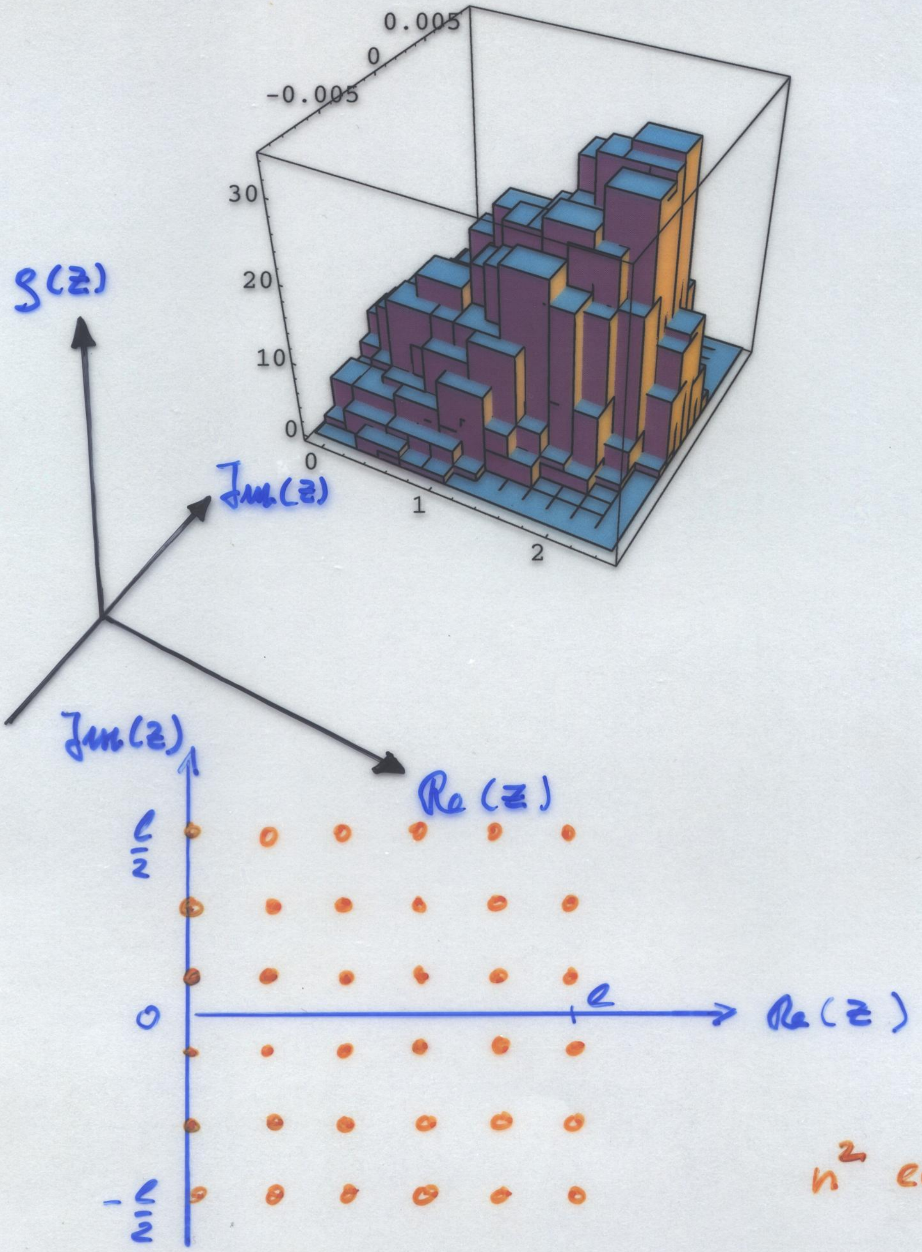
$$\rho(\lambda) \quad = \text{const.} \quad = (\sharp \lambda_k)/l \quad \text{on interval } [0, l]$$



$$\Rightarrow \textbf{spacing} \quad = l/(\sharp \lambda_k) \quad = \pi / V \langle \bar{q}q \rangle \quad \textbf{between } \lambda_k$$

- remains true at weak non-Hermiticity $\mu^2 V \sim 1$

A typical complex $\not{D}$ spectrum on the lattice: $\mu \neq 0$



$\mu \neq 0$

n^2 eigenwerte

$$\rho(z) = \text{const} = \frac{n^2}{\ell^2} \sim V$$

spacing $\sim \frac{\ell}{n} \sim \frac{1}{\sqrt{V}}$

microscopic limit $\quad z\sqrt{V}$

Universality on C ?

- $\mathcal{Z}^{N_f=K+L}{}_{MM} \sim \langle \Pi_k^K \det(\not{D} - m_k)\, \Pi_l^L \det(\not{D}^\dagger - m_l^*)\rangle_w \sim \det(L_N, \mathcal{K}_N)$

$\Rightarrow$ **all fermion. part. funct. universal** $\forall\, w(z)$ $(N \leq \infty)$

$\mathcal{Z}_w = \mathcal{Z}_{(ii)} = \mathcal{Z}_{(iii)}$, also computed $= \mathcal{Z}_{(i)}$ $\surd$ (without OP)

- **universal correlation functions ?**

  * example weak limit: $\xi = Nz$

$(ii) \quad \rho(\xi) \sim |z|\ \mathrm{e}^{-\frac{(\Im\mathrm{m}\xi)^2}{\alpha^2}} \int_0^1 dt \mathrm{e}^{-\alpha^2 t} |J_\nu(\xi\sqrt{t})|^2$

$(iii) \quad \rho(\xi) \sim |z|^2 K_\nu(\frac{|\xi|^2}{2\alpha^2})\ \mathrm{e}^{\frac{\Re\mathrm{e}(\xi^2)}{4\alpha^2}} \int_0^1 dt \mathrm{e}^{-\alpha^2 t} |J_\nu(\xi\sqrt{t})|^2$

Toda lattice: $\sim |z|^2 \mathcal{Z}^{N_f=-2}\ \mathcal{Z}^{N_f=+2}$

$\rightarrow$ **bosonic part. funct. differ !**

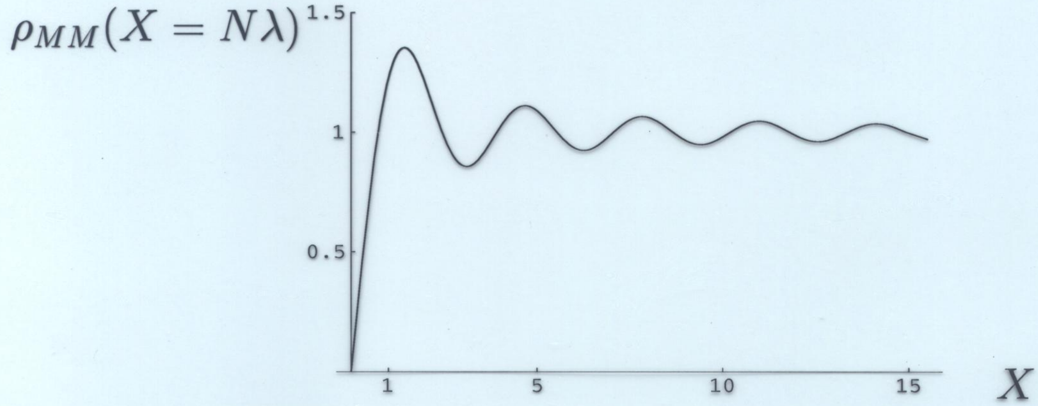
conjecture: $\mathcal{Z}^{N_f=-2}{}_{MM} \sim w(z)$, universal for same limiting $w(z)$

* similar at strong non-Hermiticity $\zeta = z\sqrt{N}$:

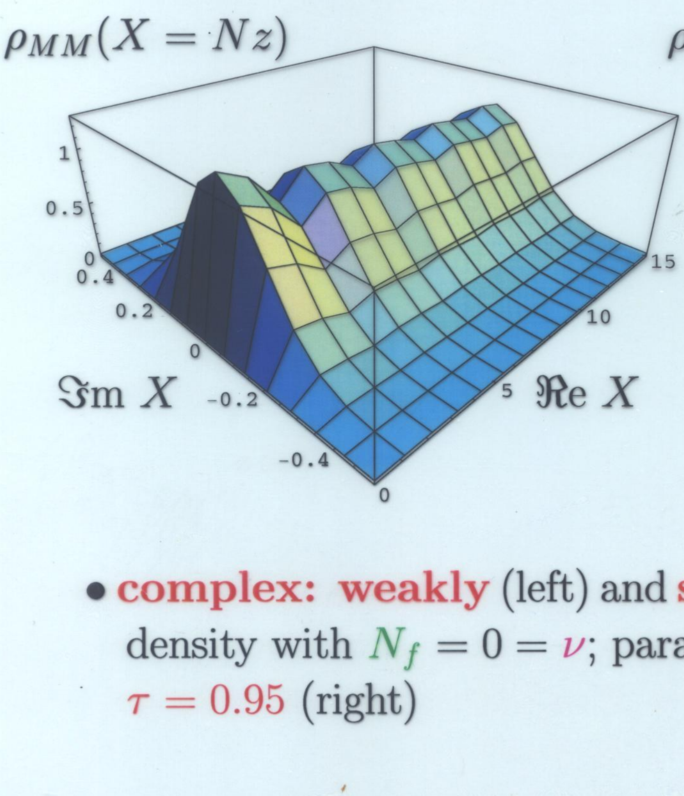
$(ii) \quad \rho(\zeta) \sim |\zeta| \exp[-\frac{|\zeta|^2}{\mu^2}] I_\nu(\frac{|\zeta|^2}{\mu^2})$

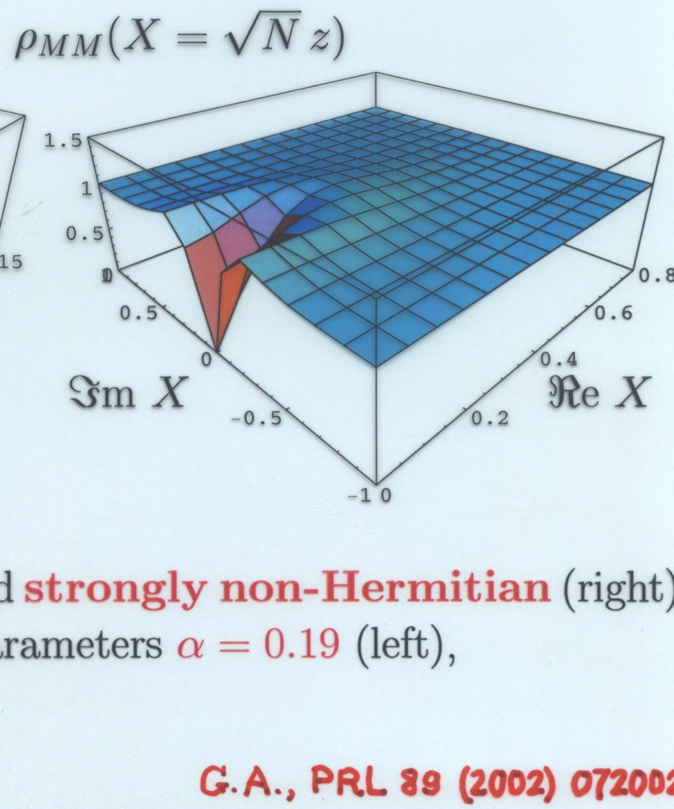
Quenched QCD

Real versus complex Matrix Model correlations



- **real density** with $N_f = 0 = \nu$





- **complex: weakly** (left) and **strongly non-Hermitian** (right) density with $N_f = 0 = \nu$; parameters $\alpha = 0.19$ (left), $\tau = 0.95$ (right)

G.A., PRL 89 (2002) 072002

MM for $SU(N_c)$ adj. / $SU(2)$ fund. staggered $+ \mu$

$$\mathcal{Z}_{MM} = \int \Pi_j^N \, d^2 z_j |z_j|^{4\nu+3} \, \Pi_f^{N_f} |z_j^2 + m_f^2| \, w(z_j) \mathbf{J}(\{\mathbf{z_i}\}, \{\mathbf{z_i^*}\})$$

μ

- eigenvalue or symplectic two-matrix model:

  Jacobian **repels eigenvalues from Re and Im**-axis

  $\mathbf{J}(\{\mathbf{z_i}\}, \{\mathbf{z_i^*}\}) = \Pi_{k>l}^N |z_k^2 - z_l^2|^2 |z_k^2 - z_l^{*\,2}|^2 \, \Pi_{m=1}^N |z_m^2 - z_m^{*\,2}|^2$

- eigenvalues in complex conjugate pairs: $\det(\not{D} + \gamma_0 \mu + M) \in \mathbf{R}$

  $\Rightarrow$ massless flavor – topology duality **holds**: $N_f + 2\nu$

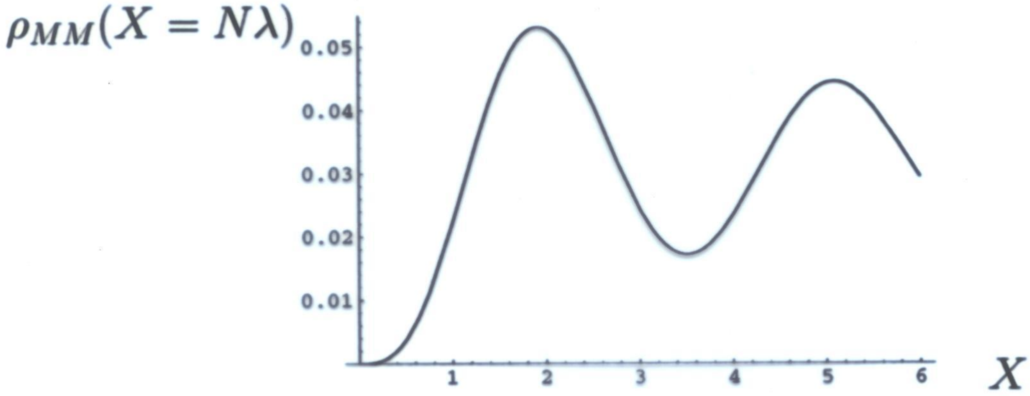
- solution: **skew** orthogonal Laguerre polynomials $\in \mathbf{C}$:

  $\langle f, g \rangle \;\equiv\; \int d^2 z (z^2 - z^{*\,2}) w(z) [f(z) g(z^*) - f(z^*) g(z)]$ scalar prod.

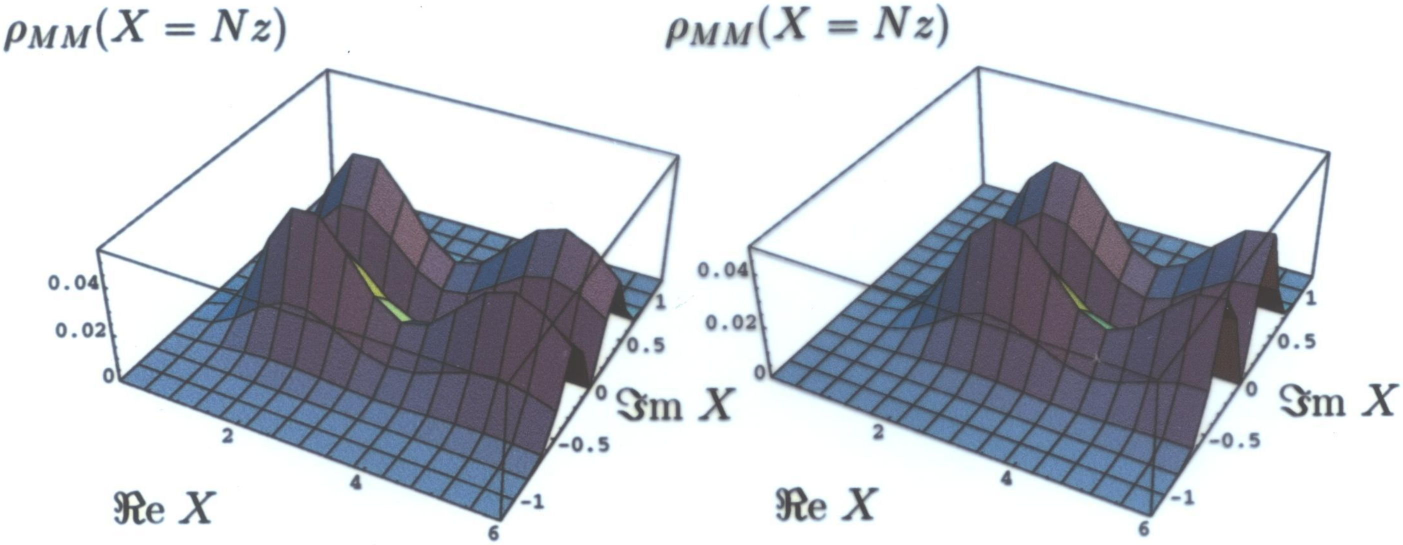
  $\langle q_{2k+1}, q_{2l} \rangle \sim \delta_{kl}$ **and** $\langle q_{2k+1}, q_{2l+1} \rangle = \langle q_{2l}, q_{2k} \rangle = 0$

  $q_{2k+1}(z) \sim L_{2k+1}^{\nu}\left(\frac{N z^2}{2\tau}\right)$ **and** $q_{2k}(z) \sim \Sigma_{j=0}^{k} \tau^{2j} c_j L_{2j}^{\nu}\left(\frac{N z^2}{2\tau}\right)$

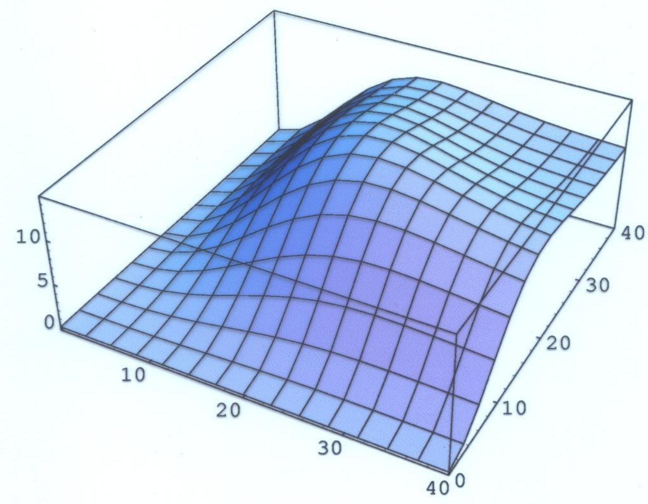
Real versus complex correlations for $\beta_D = 4$



- **real density** with $N_f = \nu = 0$



- **complex: weakly non-Hermitian**
  density with $N_f = \nu = 0$ (left) and $N_f = 1$, $\nu = 0$ (right), $\alpha = 0.4$



- strong non-Hermiticity

Conclusions

- $\exists$ a variety of **new Matrix Model predictions** for **complex Dirac spectra:**

- **MM for quenched QCD** from lattice & χPT $\surd$
  comparison to **unquenched ?**

- $\beta_D = 4$ **MM for** $SU(N_c)$ **adj.** (or $SU(2)$ fund. staggered) $\surd$
  $\longrightarrow$ Lattice

- $\beta_D = 1$ **for** $SU(2)$ **fund.** (or $SU(N_c)$ adj. staggered)
  challenging for MM – not for Lattice