Discrete random geometries: genealogy and computational techniques

Martin Weigel

Institut für Physik, KOMET 331

Johannes-Gutenberg-Universität

Mainz, Germany

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Junior Research Group

“Complex Systems with Frustration”

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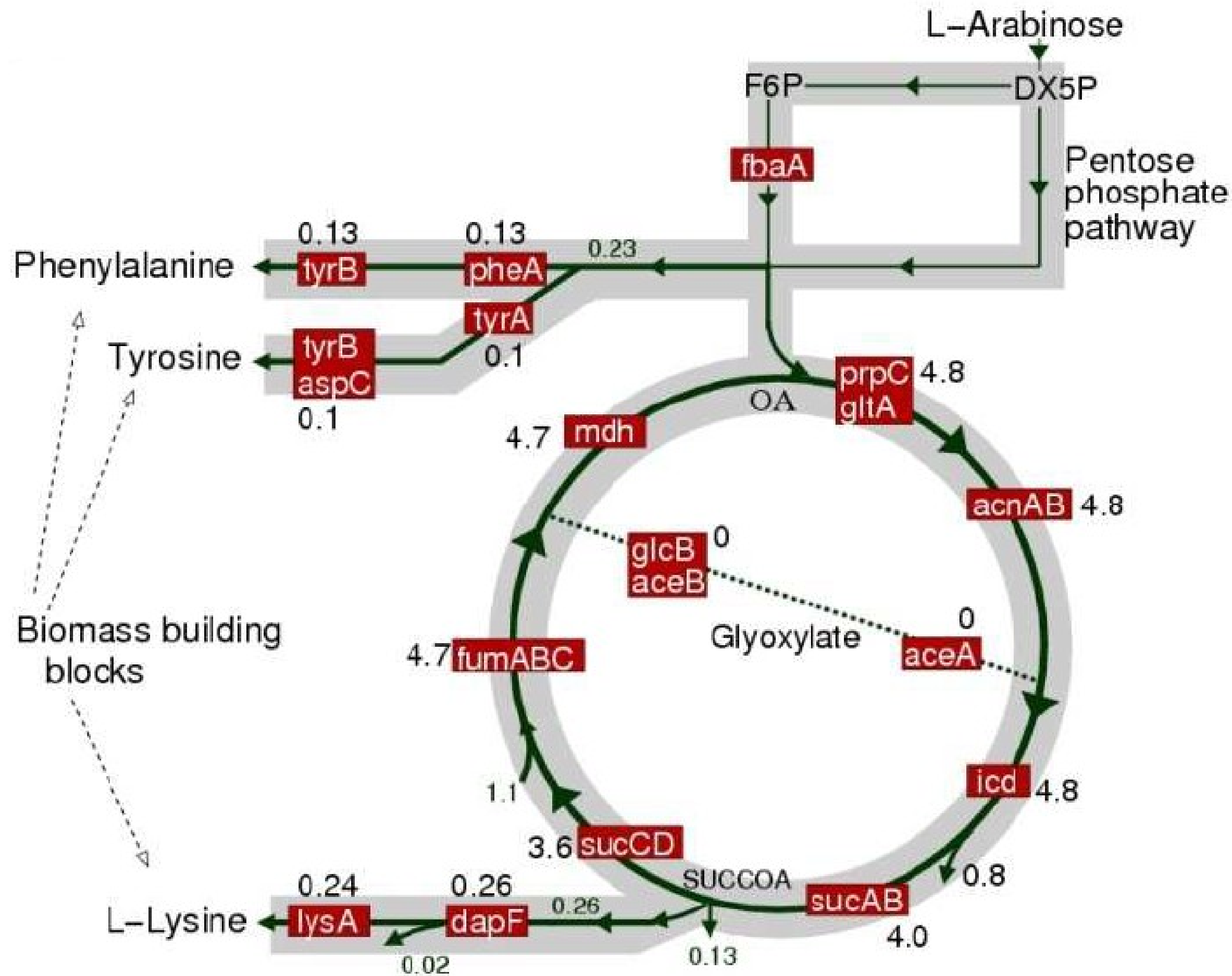

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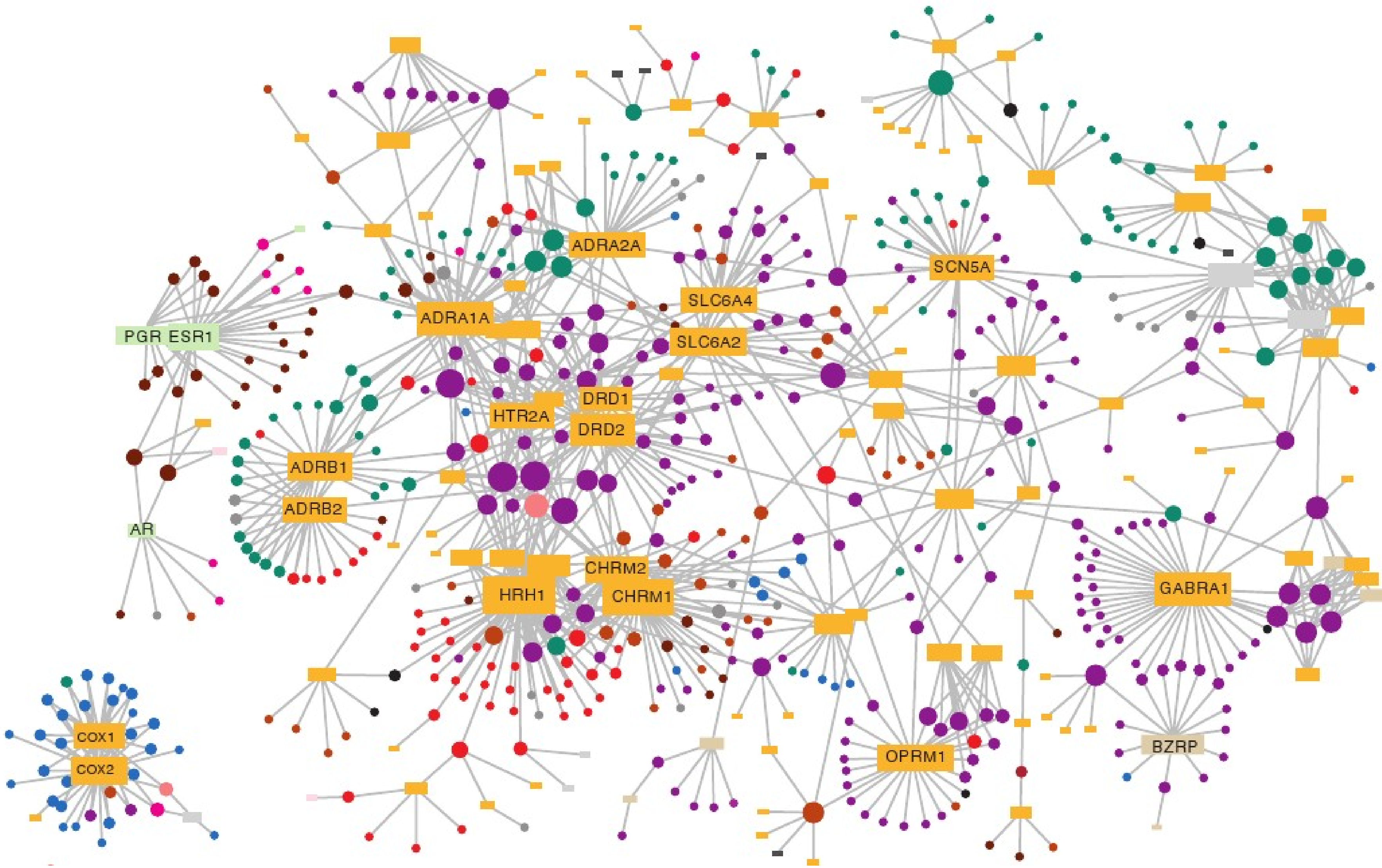

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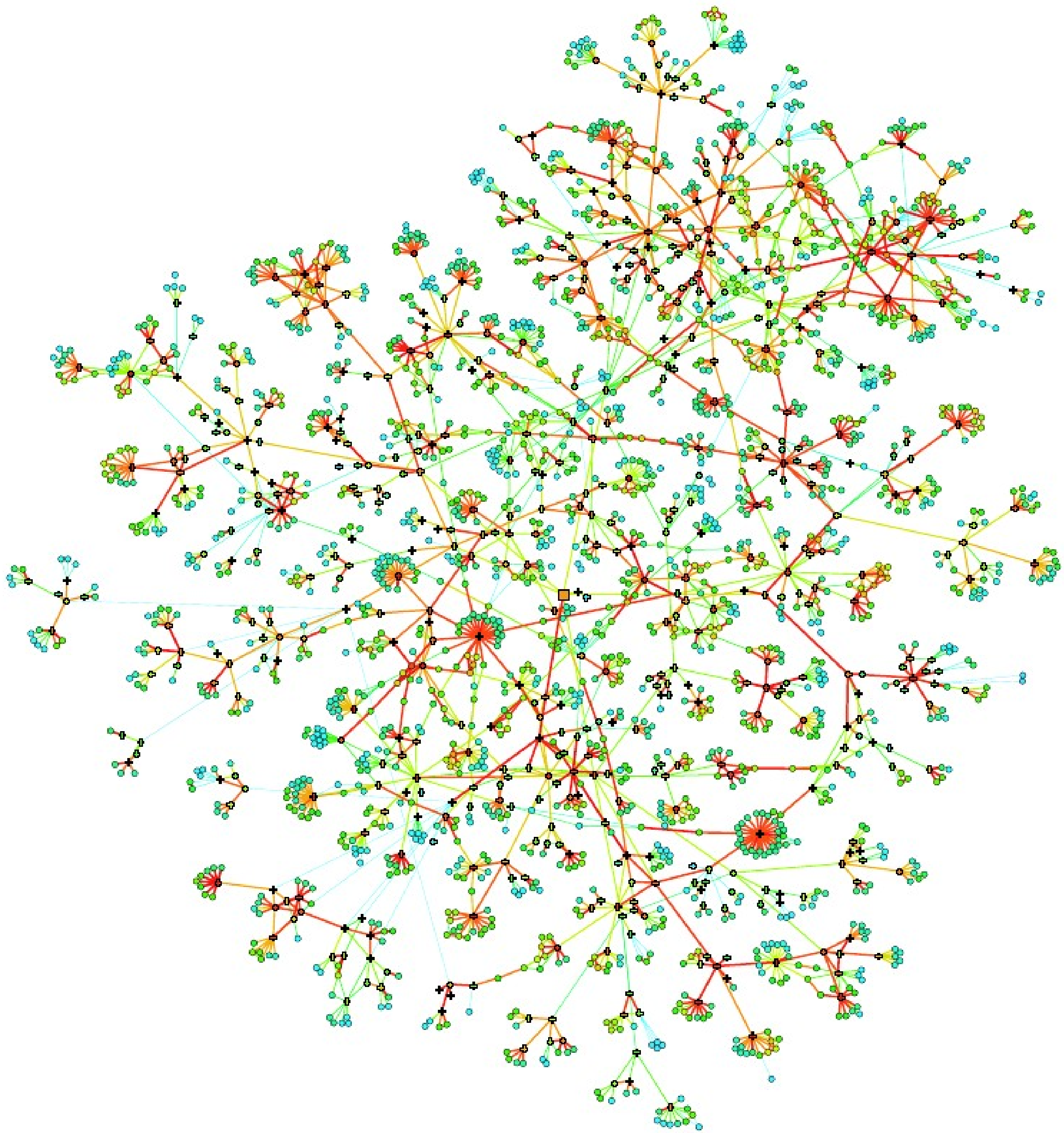

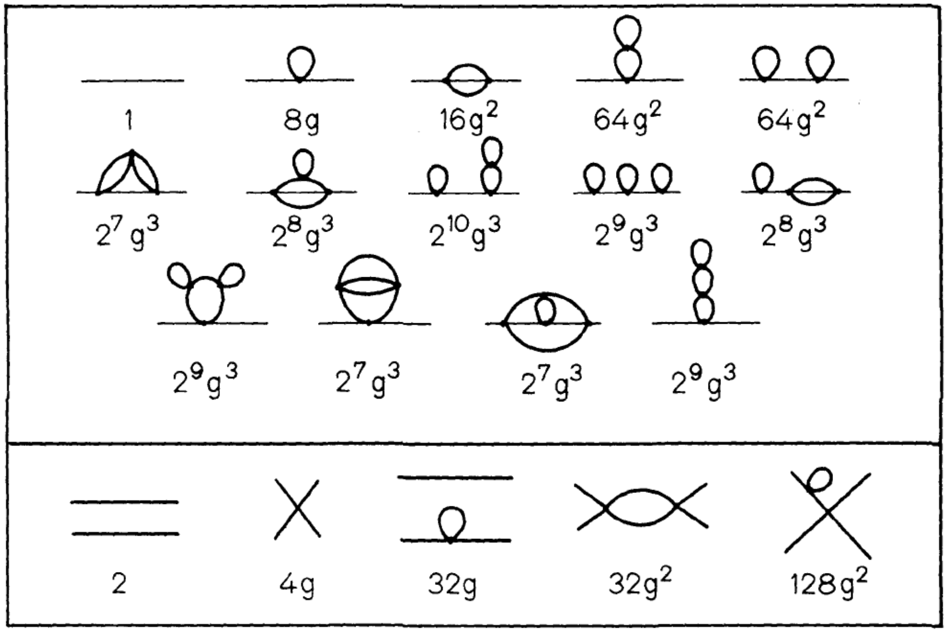
1
8g
16g²
64g²
64g²
2⁷g³
2⁸g³
2¹⁰g³
2⁹g³
2⁸g³
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2⁷g³
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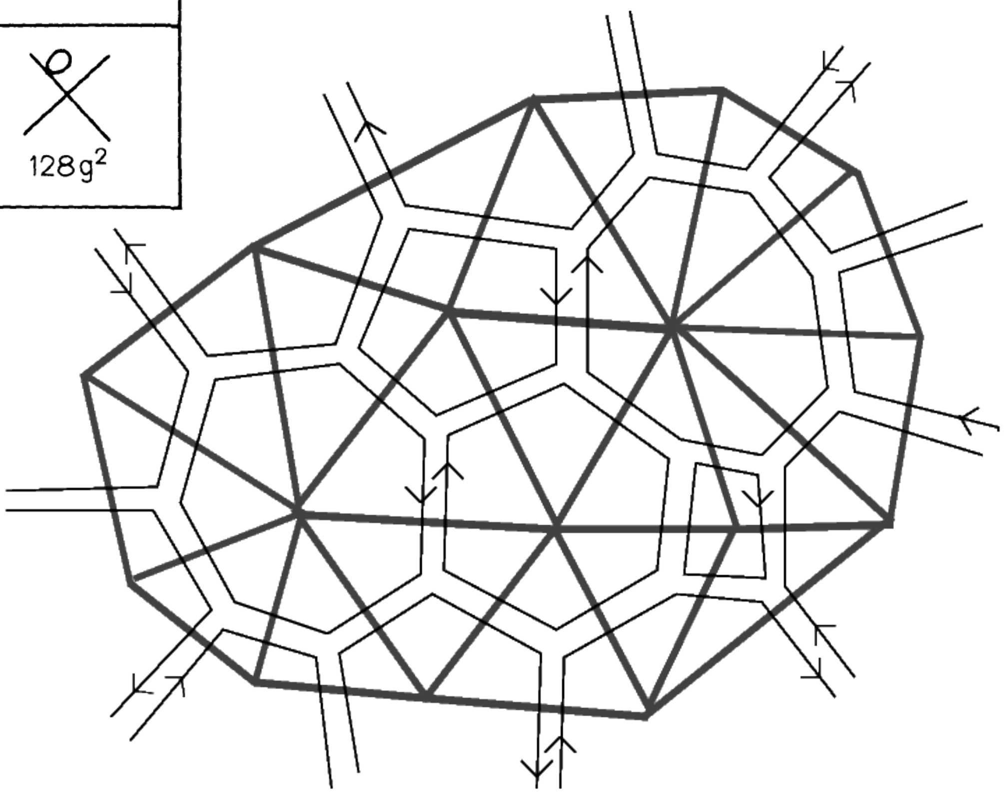
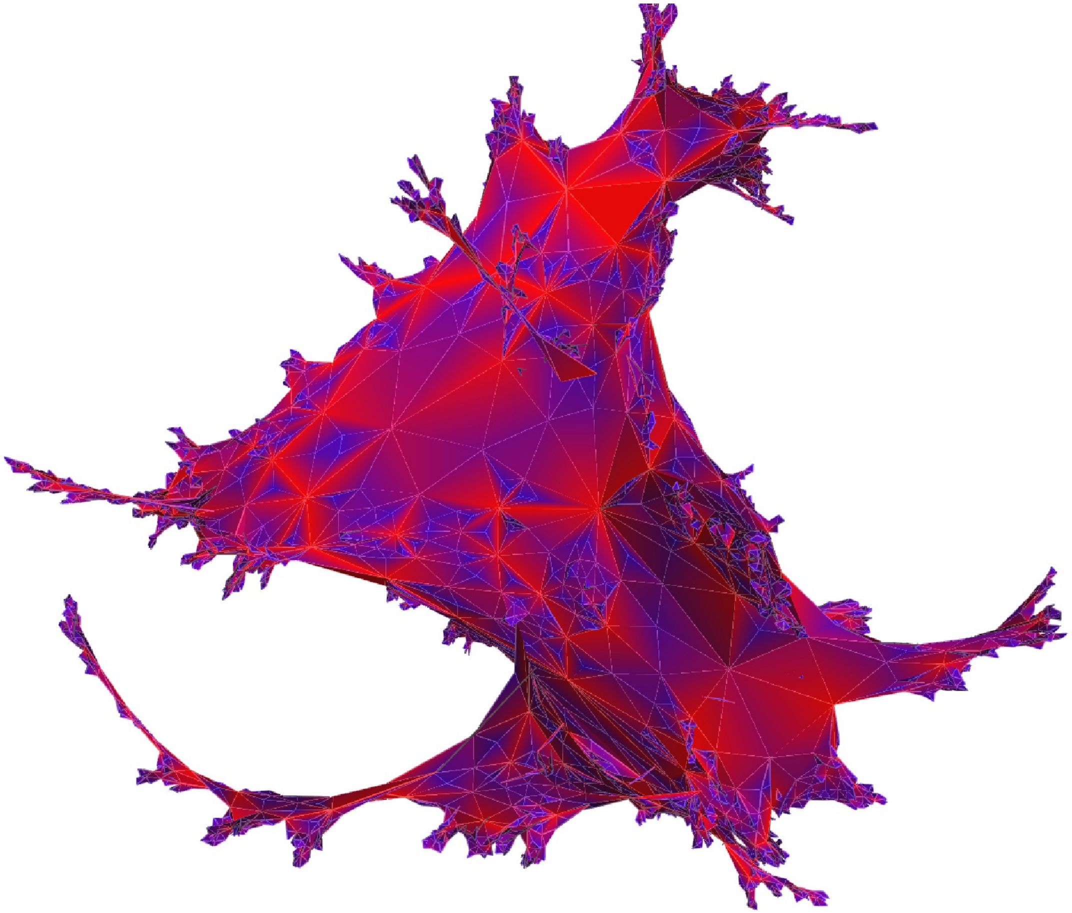
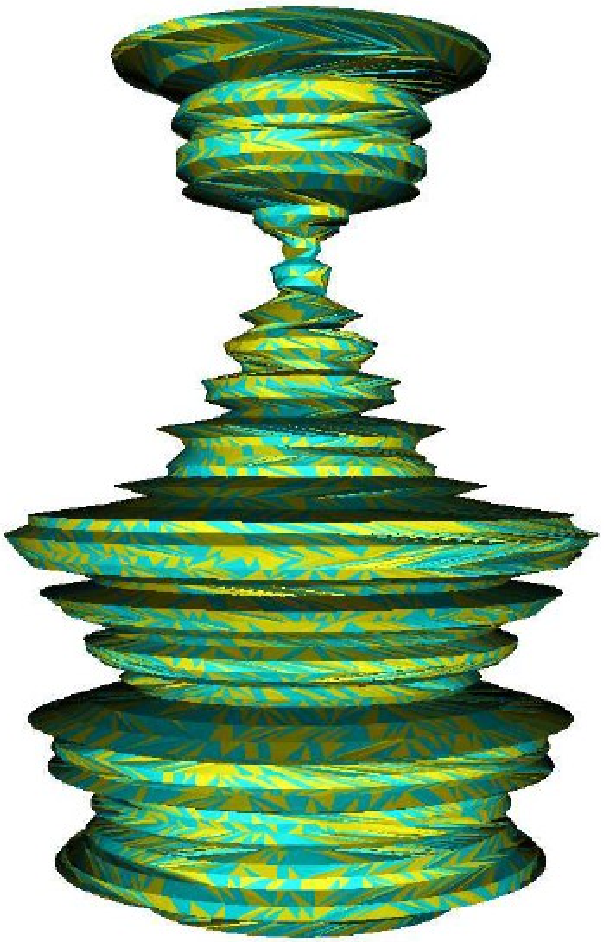
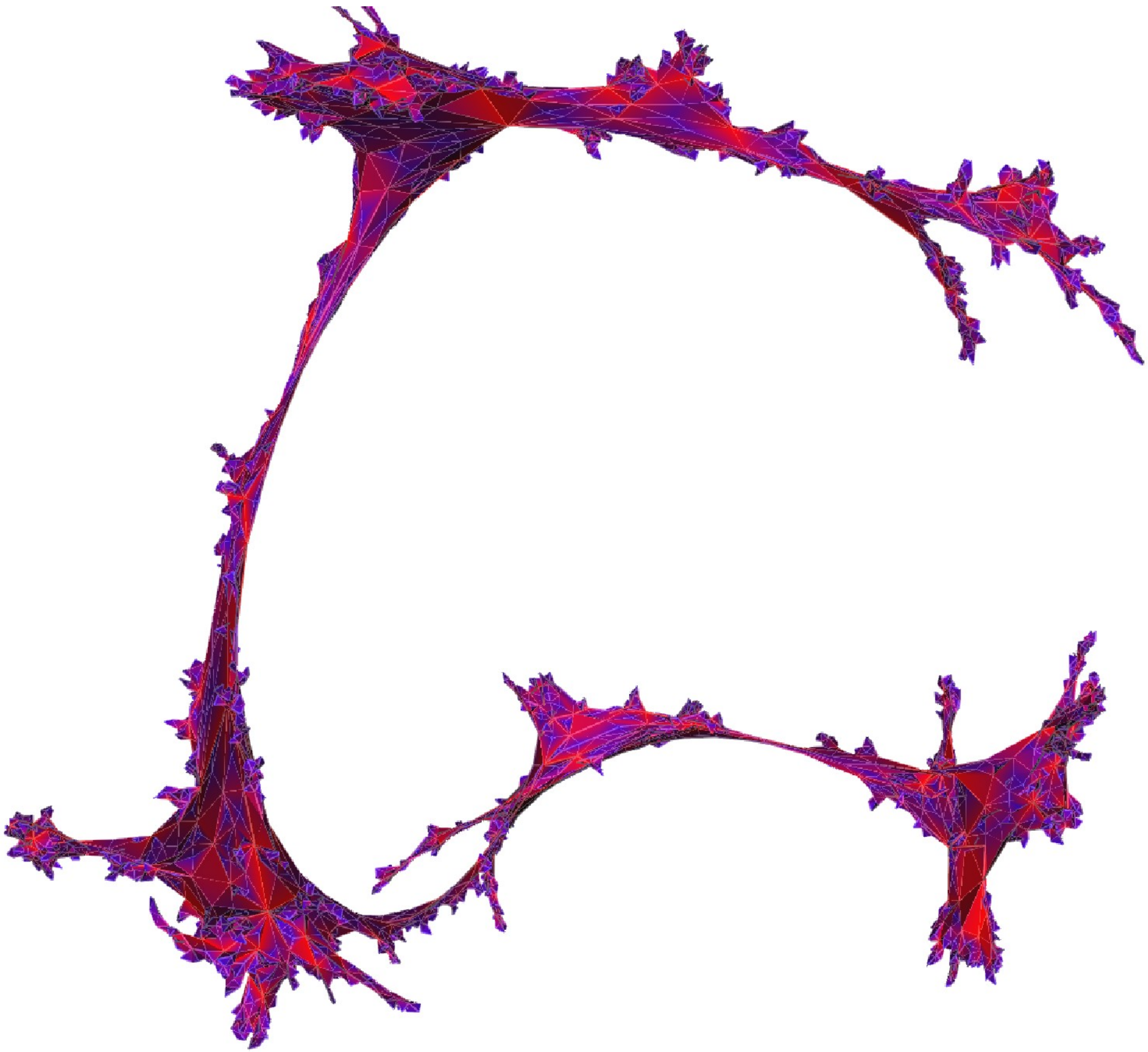
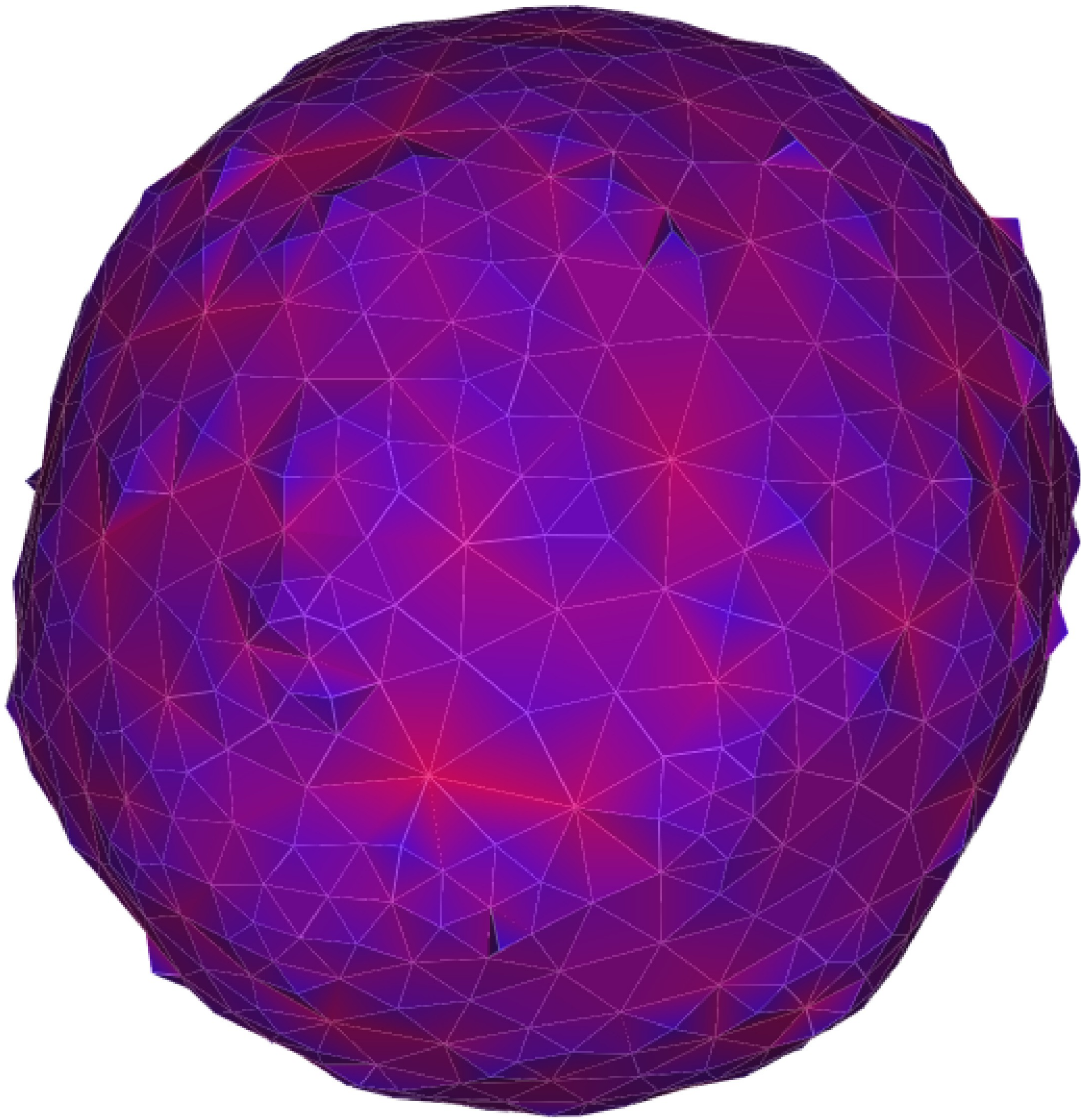
4g
32g
32g²
128g²

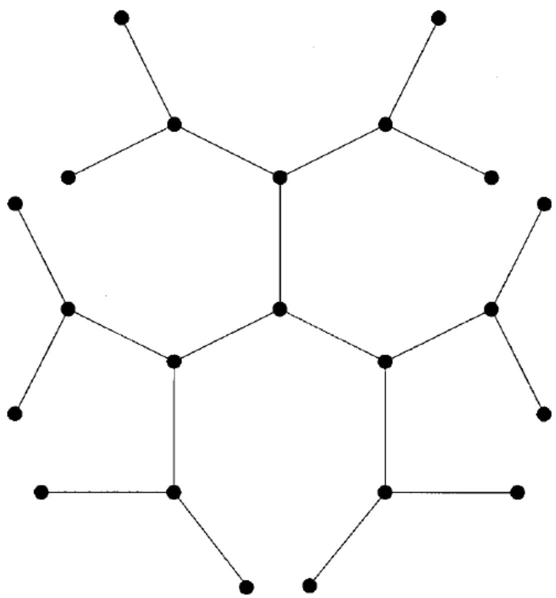


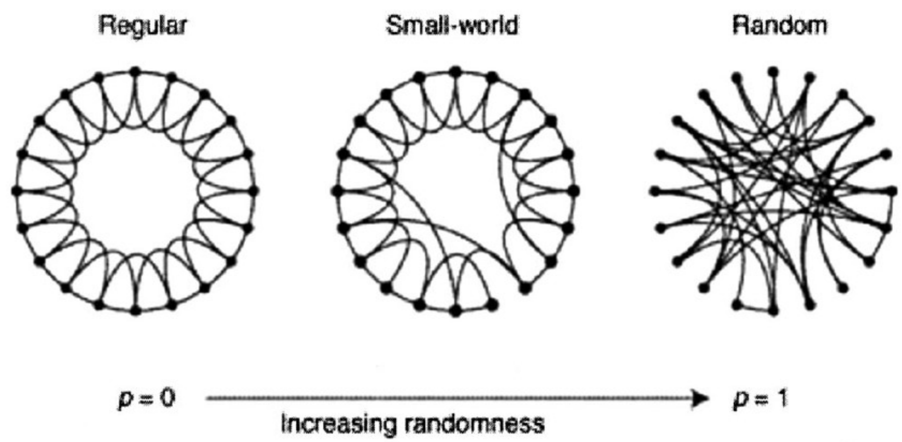


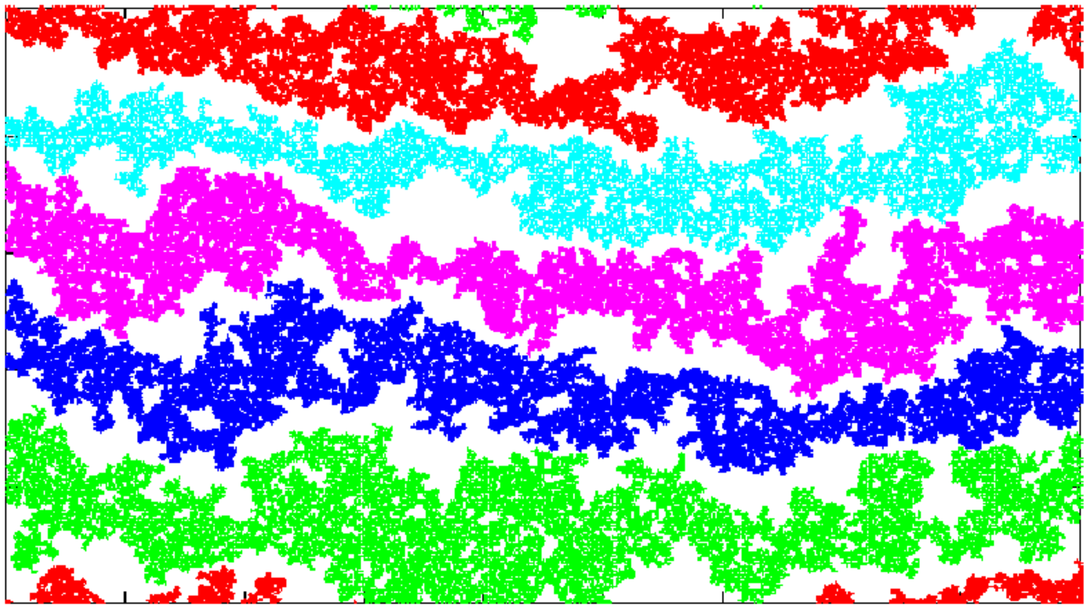









Regular
Small-world
Random
p = 0
Increasing randomness
p = 1



Outline

(1) Complex networks

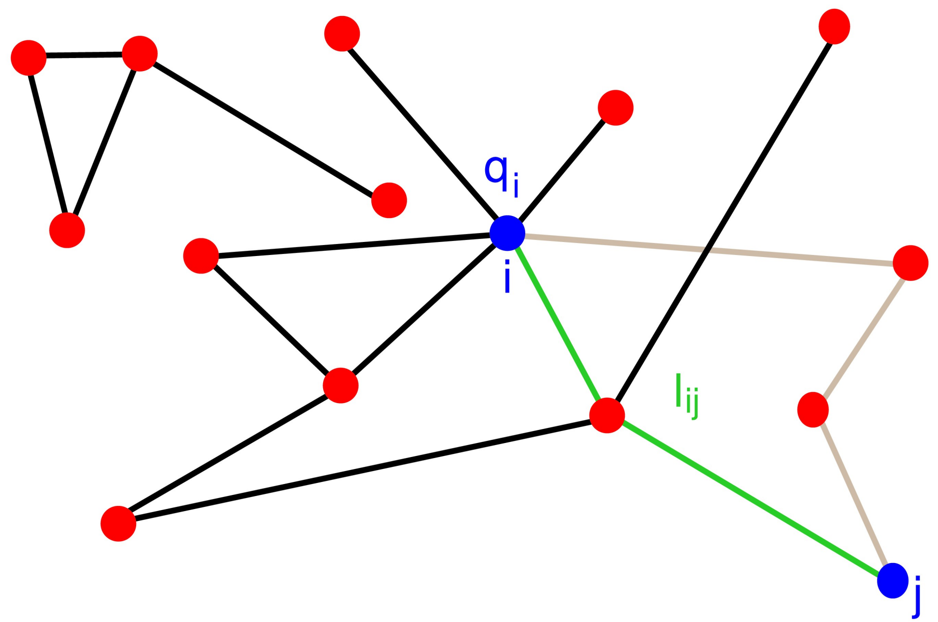
(2) Thin random graphs

(3) Fat random graphs

(4) Poissonian Voronoï-Delaunay triangulations

(5) Coupling matter to random graphs

General graph characteristics



Co-ordination number distribution:

$$P(q)$$

Clustering coefficient:

$$C(q_i) = t_i/[q_i(q_i - 1)/2]$$

Intervertex distance: ℓ_{ij}

$$\bar{\ell}(N) \sim \ell^{1/d_h}$$

Small worlds: $d_h = \infty, \ \ \bar{\ell}(N) \sim \ln N$

Betweenness centrality, giant connected component, ...

Erdős-Rényi random graphs

Poissonian coordination numbers:

$$P(q) = e^{-\langle q \rangle} \langle q \rangle^q / q!$$

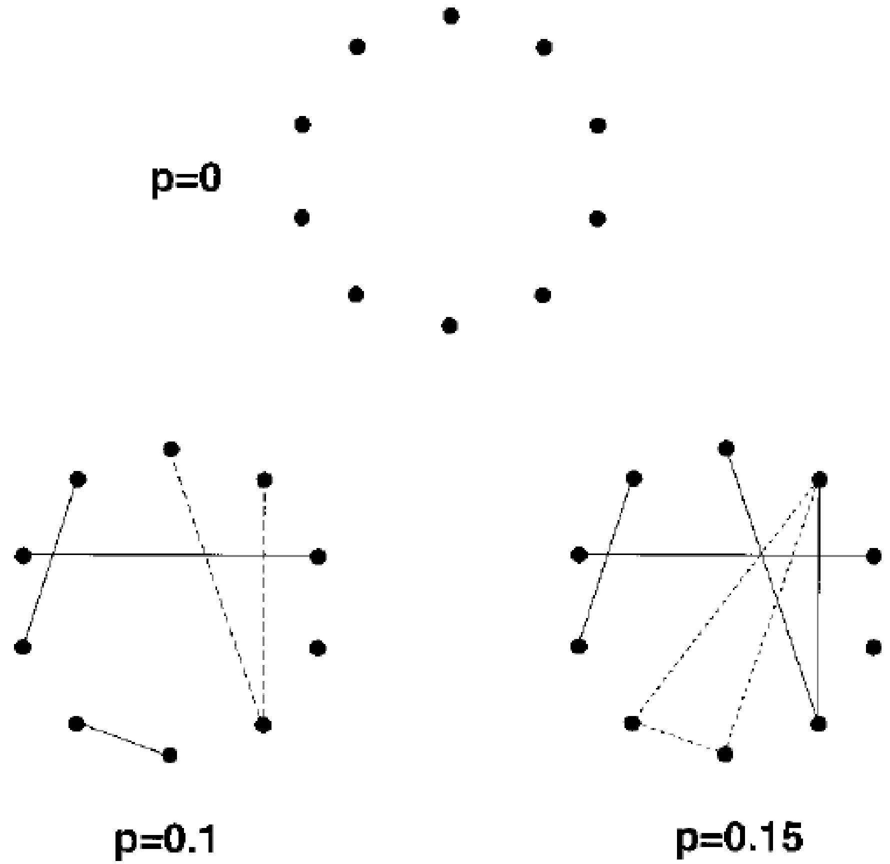
i.e., maximally random graphs under the single constraint

$$\langle q \rangle = q_0$$

Formation of giant component:

$$p_c = \frac{\langle q \rangle}{\langle q^2 \rangle - \langle q \rangle}$$

Small world: $$\bar{\ell}(N) \sim \frac{\ln N}{\ln \langle q \rangle}$$



Uncorrelated networks with arbitrary degrees

e.g., generated by equilibrium re-wiring of
random graphs

Critical version has power-law coordination number distribution:

$$P(q) \sim q^{-\gamma}$$

which is often seen in real networks

Correlated networks:

- Random under the constraint of fixed $P(q, q')$
  -> still tree-like
- Evolving networks: preferential attachment (Barabasi-Albert) leading to self-organized criticality, power-law degrees
- Short-cuts in a regular lattice (Watts-Strogatz)

General properties of this class

- Equilibrium models: maximally random under the constraint of P(q) only or of P(q) and P(q,q')
- Growing networks may auto-tune to criticality
- Usually small-world, i.e., $d_h = \infty$
- Locally tree-like: no short loops
- Not enough structure to define a surface

→ Talk by B. Waclaw today

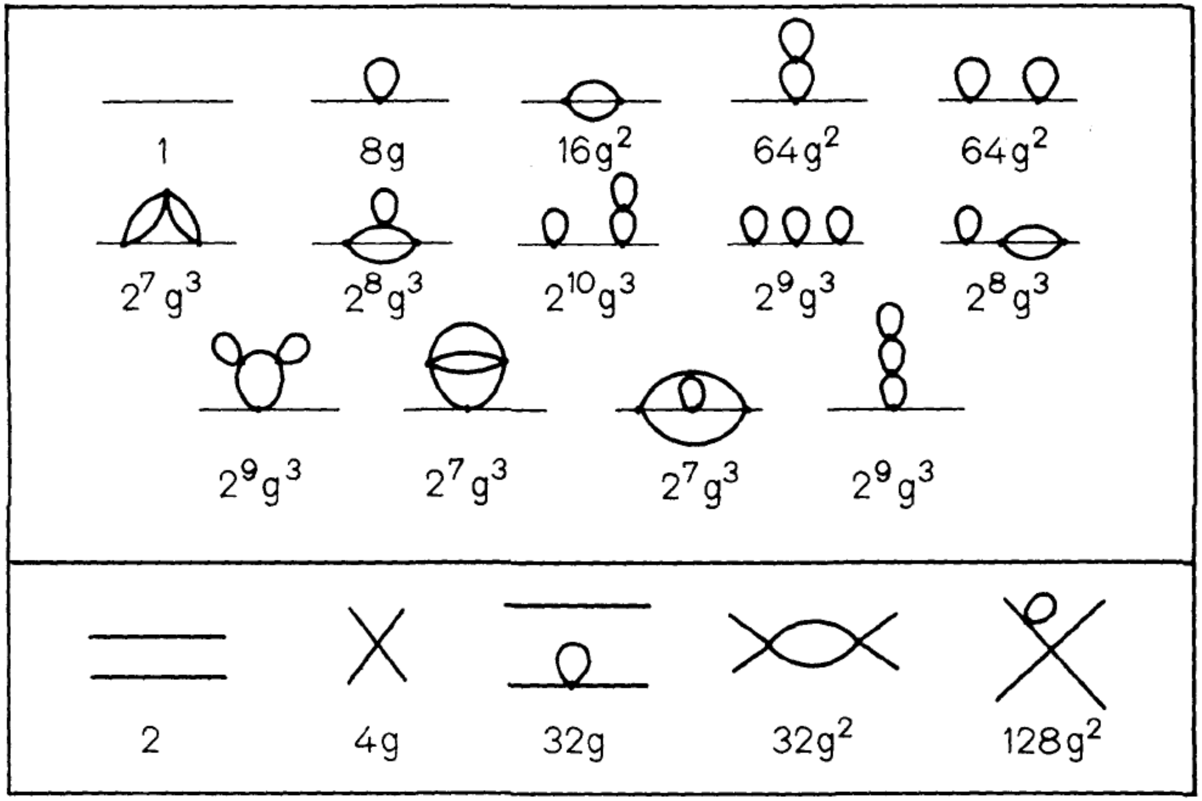
R. Albert and A. Barabási, *Statistical mechanics of complex networks*, Rev. Mod. Phys. 74 (2002) 47.

S. N. Dorogovtsev, A. V. Goltsev and J. F. F. Mendes, *Critical phenomena in complex networks*, to appear in Rev. Mod. Phys.

Definition

Consider the integral

$$\frac{1}{2\pi i}\oint \frac{\mathrm{d}g}{g^{2N+1}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi}{2\pi}\exp[-\frac{1}{2}\phi^2+\frac{1}{6}g\phi^3]$$



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Definition

Consider the integral

$$\mathcal{N}_n = \frac{1}{2\pi i}\oint \frac{\mathrm{d}g}{g^{2N+1}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi}{2\pi}\exp[-\frac{1}{2}\phi^2+\frac{1}{6}g\phi^3]$$

In the saddle-point limit $n\to\infty$, after a rescaling $\phi\to\phi/g$ and Gaussian integration, one finds

$$\mathcal{N}_n \approx (n/e)^n\hat{S}^{-n}(-2\pi n\,\det\hat{S}'')^{-1/2}$$

Here, $S=\phi^2/2-\phi^3/6$, $\hat{S}=S(\hat{\phi})$ and $\hat{\phi}$ is the dominant saddle point, i.e., a solution of

$$\frac{\partial S}{\partial\phi}=0$$

For $\hat{\phi}=2,\ \hat{S}=\frac{2}{3},\ -\hat{S}''=1$ one finds

$$\mathcal{N}_n = \left(\frac{1}{6}\right)^{2n}\frac{(6n-1)!!}{(2n)!}$$

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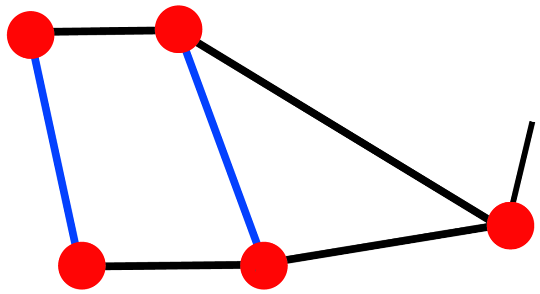
Properties

- Very similar to Erdős-Rényi random graphs, but
- Fixed coordination number, nevertheless still
- Disorder in distribution of loop lengths
- Maximally random network under the constraint of constant degree
- Same formalism holds for any k-regular class of graphs
- Again no/few short loops (smaller than $\ln N$)
- Drop-in replacement for calculations on the Bethe lattice, since no boundary effects

Computational treatment:

Sampling with Monte Carlo simulation through re-wiring of links



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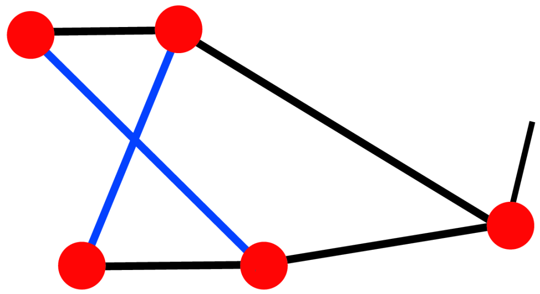

(1) Complex networks

(2) Thin random graphs

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Fat Graphs

Definition

Now, consider the matrix integral

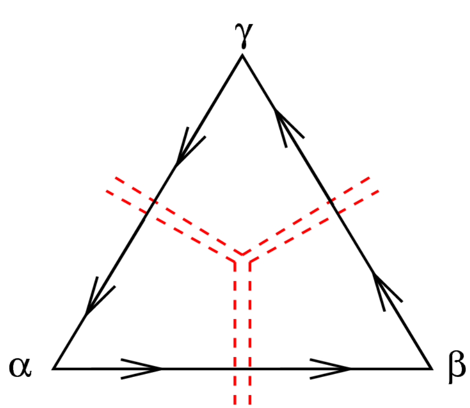
$$W(g,\mathcal{N}) = \int \mathrm{d}\phi\, e^{-\frac{1}{2}\mathrm{Tr}\phi^2 + \frac{g}{3\sqrt{\mathcal{N}}}\mathrm{Tr}\phi^3} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{g}{3\sqrt{\mathcal{N}}}\right)^k \langle \mathrm{Tr}\phi^{3k}\rangle$$

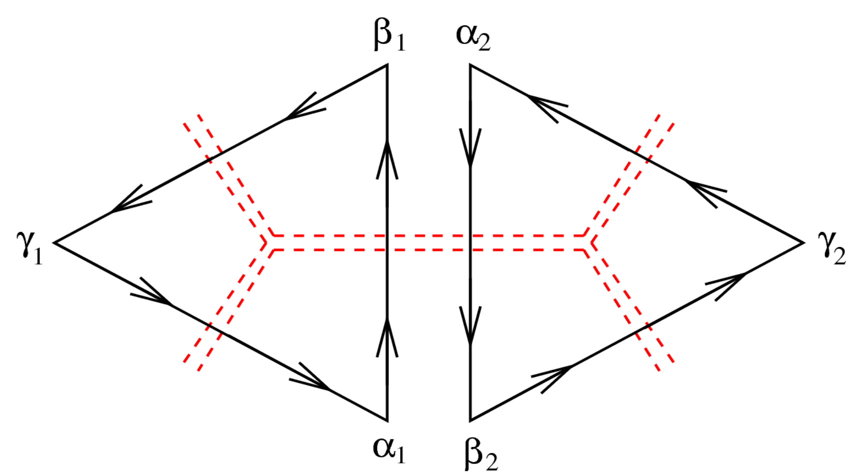
Where now ϕ is an $\mathcal{N}\times\mathcal{N}$ Hermitian matrix and

$$\mathrm{d}\phi \equiv \prod_{\alpha\le\beta} \mathrm{d}\,\mathrm{Re}\,\phi_{\alpha\beta} \prod_{\alpha<\beta} \mathrm{d}\,\mathrm{Im}\,\phi_{\alpha\beta}$$

The propagator is

$$\langle \phi_{\alpha\beta}\phi_{\alpha'\beta'}\rangle = \int \mathrm{d}\phi\, e^{-\frac{1}{2}\sum_{\alpha\beta}|\phi_{\alpha\beta}|^2} \phi_{\alpha_\beta}\phi_{\alpha'\beta'} = \delta_{\alpha\beta'}\delta_{\beta\alpha'}$$





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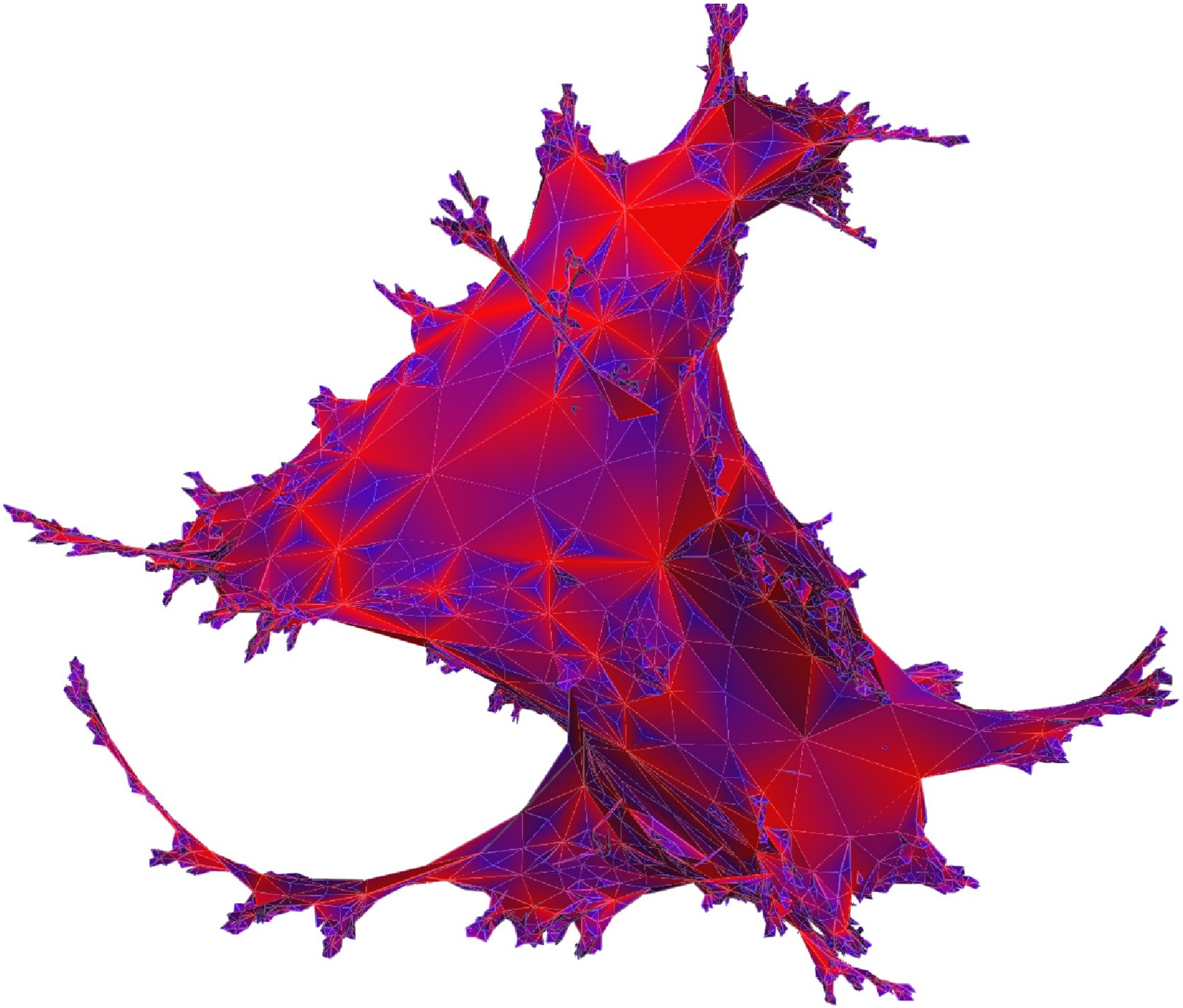
Weight of each graph is

$$g^{\mathcal{N}_2(T)}\mathcal{N}^{N_0(T)-N_2(T)/2}\frac{1}{C(T)} = g^{N_2(T)}\mathcal{N}^{\chi(T)}\frac{1}{C(T)}$$

i.e., in the limit $\mathcal{N} \to \infty$ one counts planar triangulations of size N_2

Fat graphs



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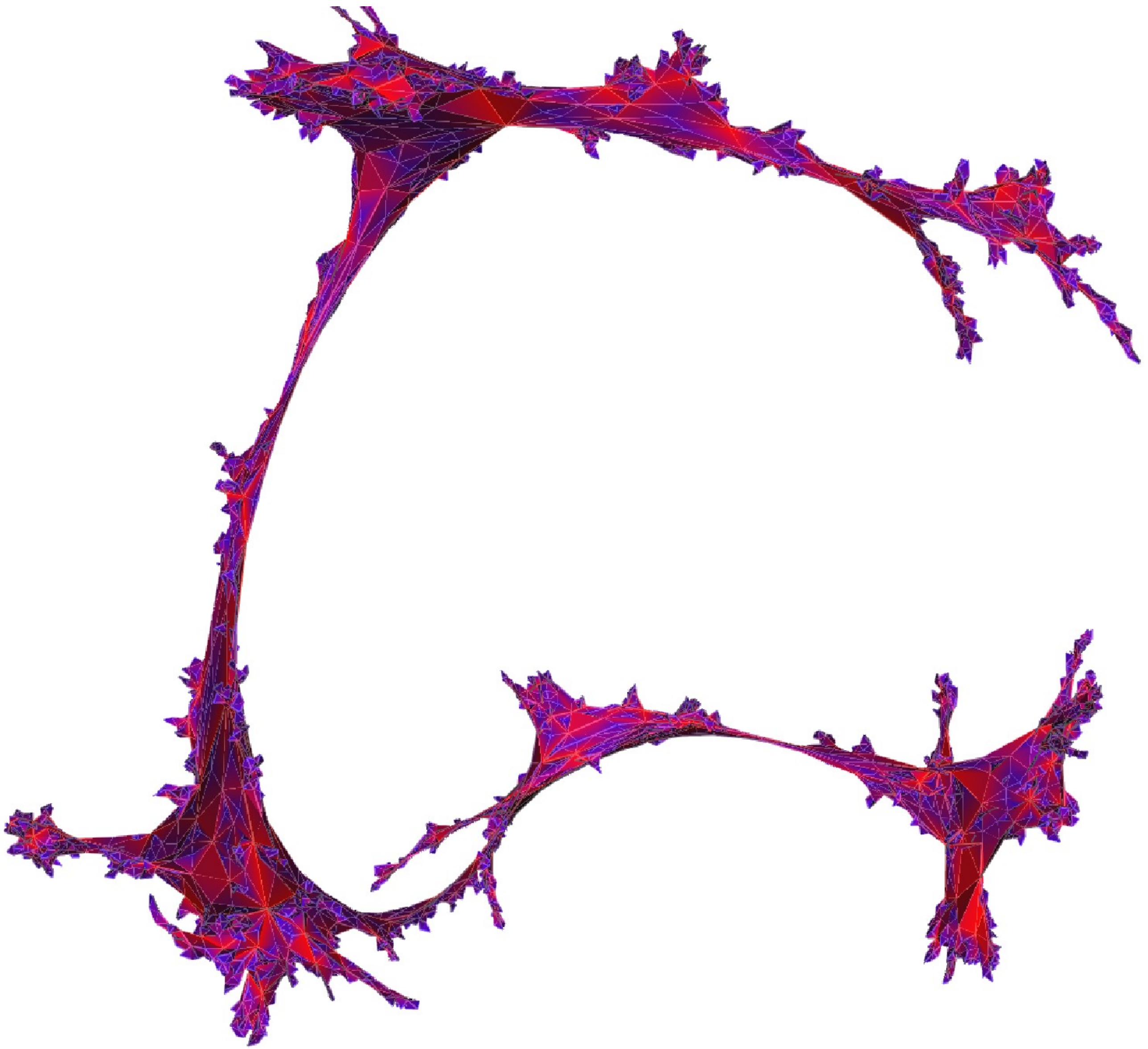
Fat graphs



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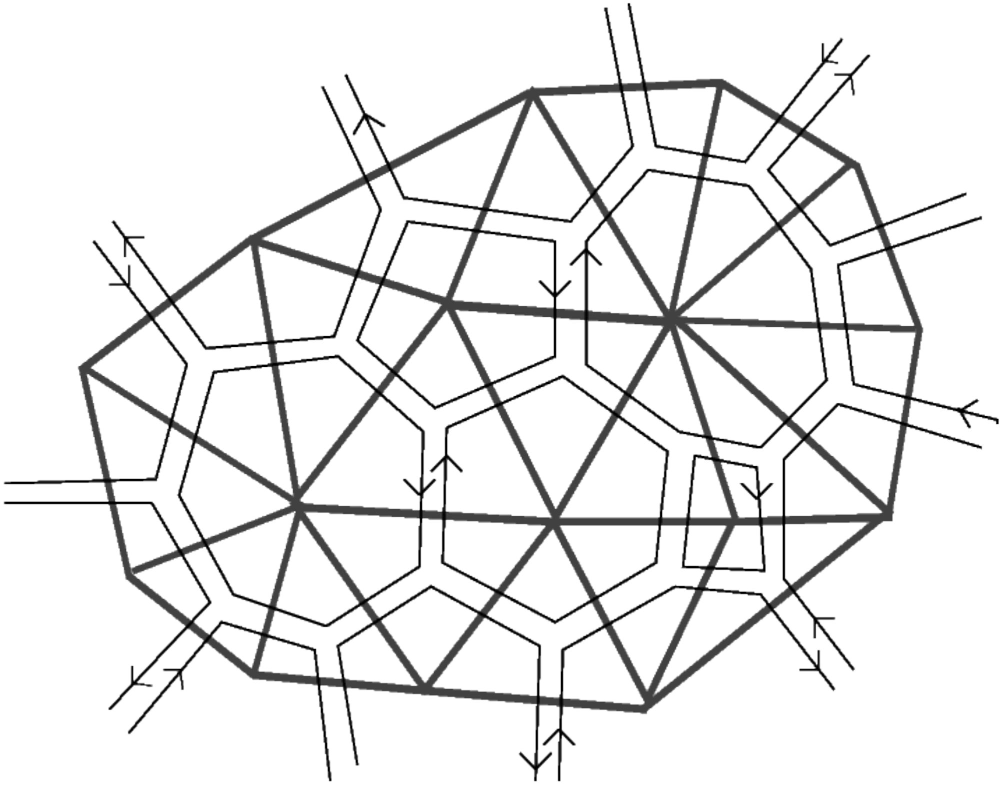
Some properties

- Fattening of propagators to ribbons generates orientability: surface instead of simple graph
- Duality: ϕ^3 graphs and triangulations, similarly ϕ^4 graphs and quadrangulations



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Some properties

- Fattening of propagators to ribbons generates orientability: surface instead of simple graph
- Duality: ϕ^3 graphs and triangulations, similarly ϕ^4 graphs and quadrangulations
- Coordination number distribution

$$P_\infty(q) = 16\left(\frac{3}{16}\right)^q \frac{(q-2)(2q-2)!}{q!(q-1)!}$$

which decays exponentially for large q
- Naturally many short loops
- Average distance

$$\bar{\ell}(N) \sim \ell^{1/d_h}$$

with $d_h = 4$, i.e., not small-world, but still "smaller" than regular lattice
- Large number of exact results due to matrix model and combinatorial techniques
- Coordination numbers are correlated algebraically,

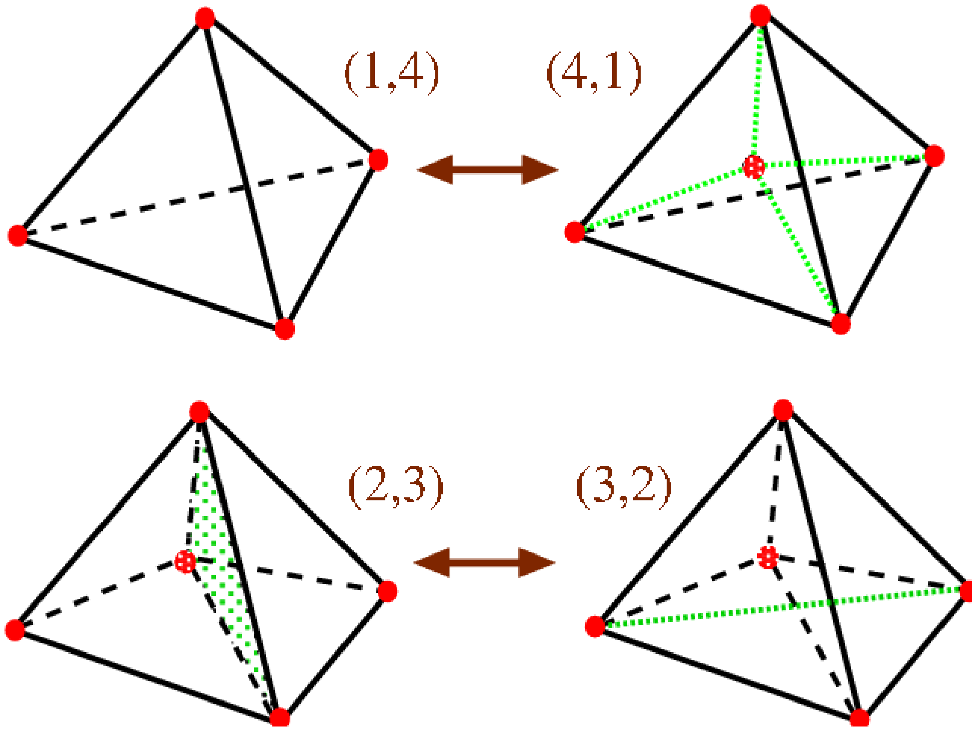
$$\langle \delta q_i \delta q_j \rangle \sim \mathrm{dist}(i,j)^{-a}$$

with $a \approx 2$

Numerical simulations

For a a general simplicial complex, define the (k,l) moves

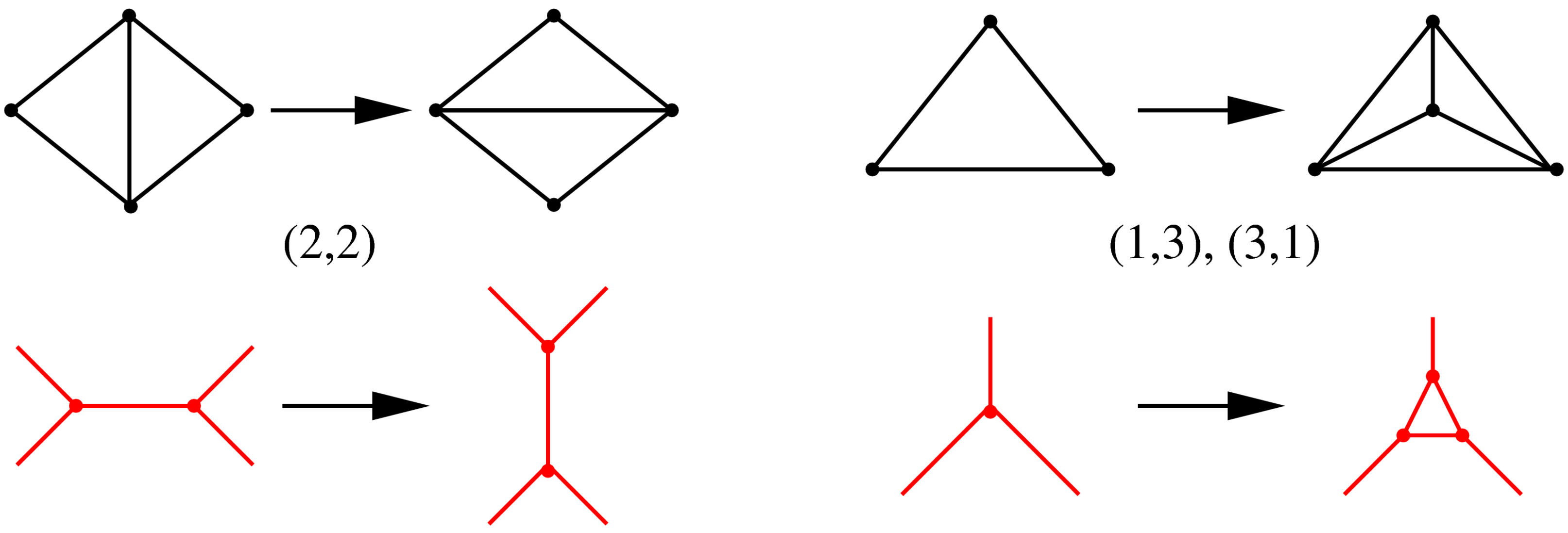
$$a_1 \dots a_l \overline{b_1 \dots b_k} \to \overline{a_1 \dots a_l} b_1 \dots b_k,$$



These can be shown to be ergodic in the space of homeomorphic simplicial manifolds (for d < 4).

Numerical simulations

In two dimensions:



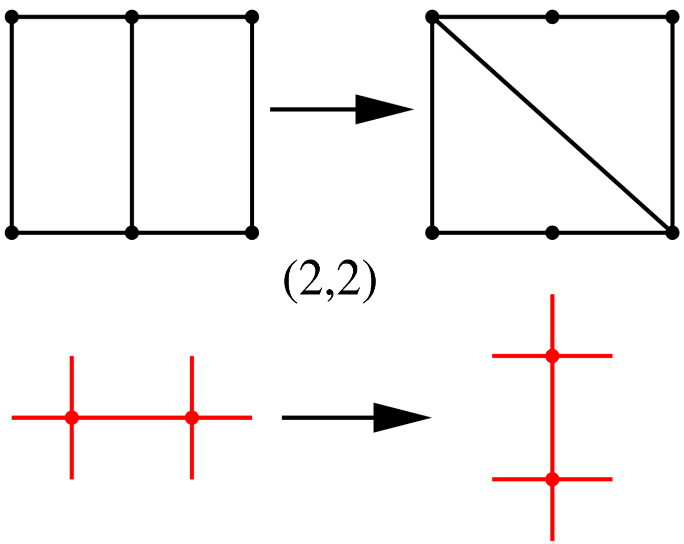
Canonical move

Grand-canonical move

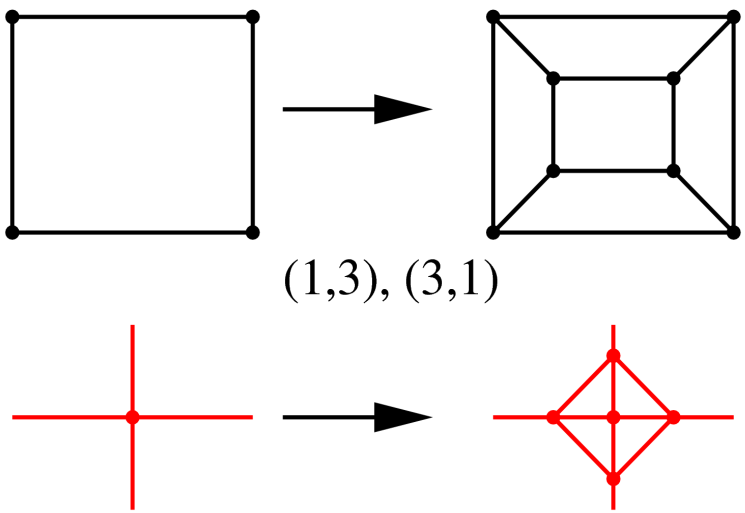
Canonical move alone is ergodic for simulations in the canonical ensemble.

Numerical simulations

What about ϕ^4 graphs and quadrangulations?



Canonical move



Grand-canonical move

Moves not ergodic in general!

Numerical simulations

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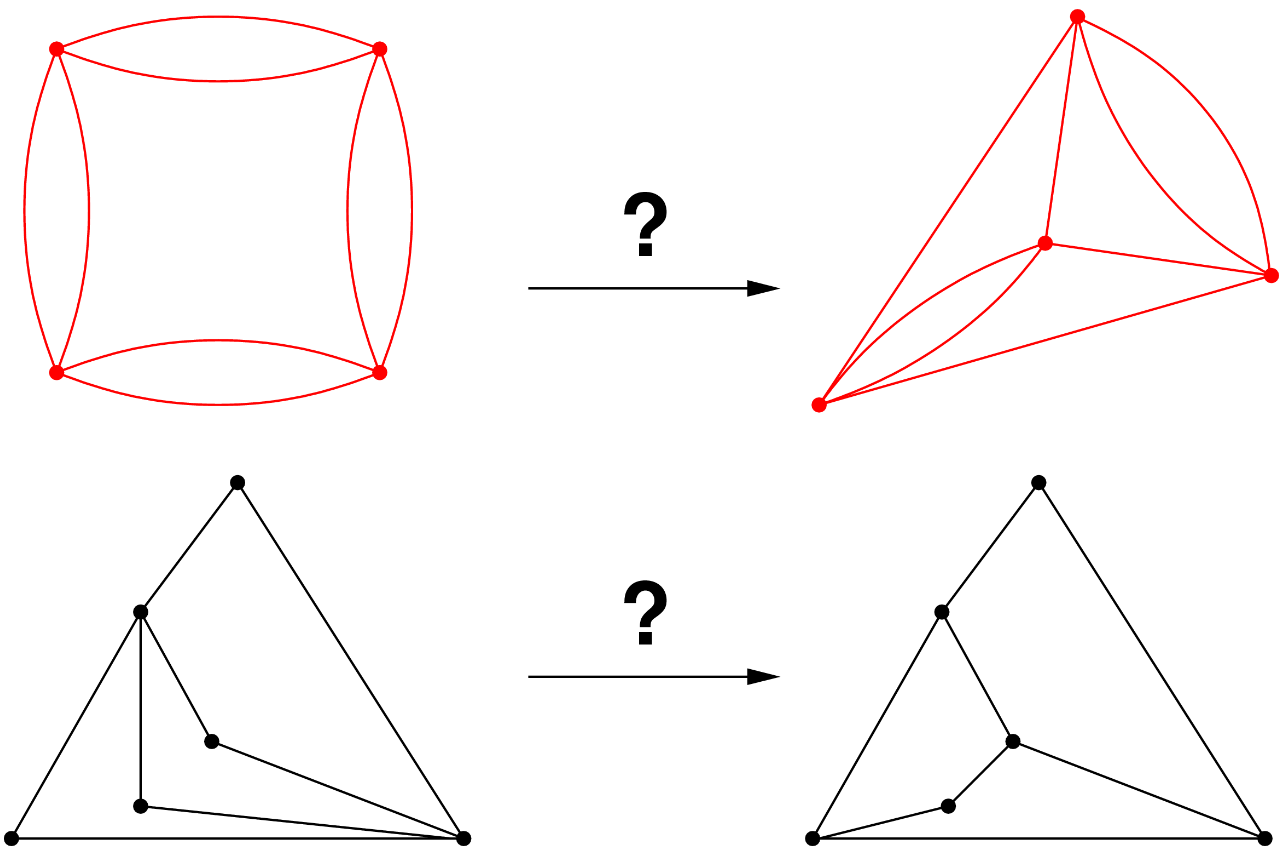
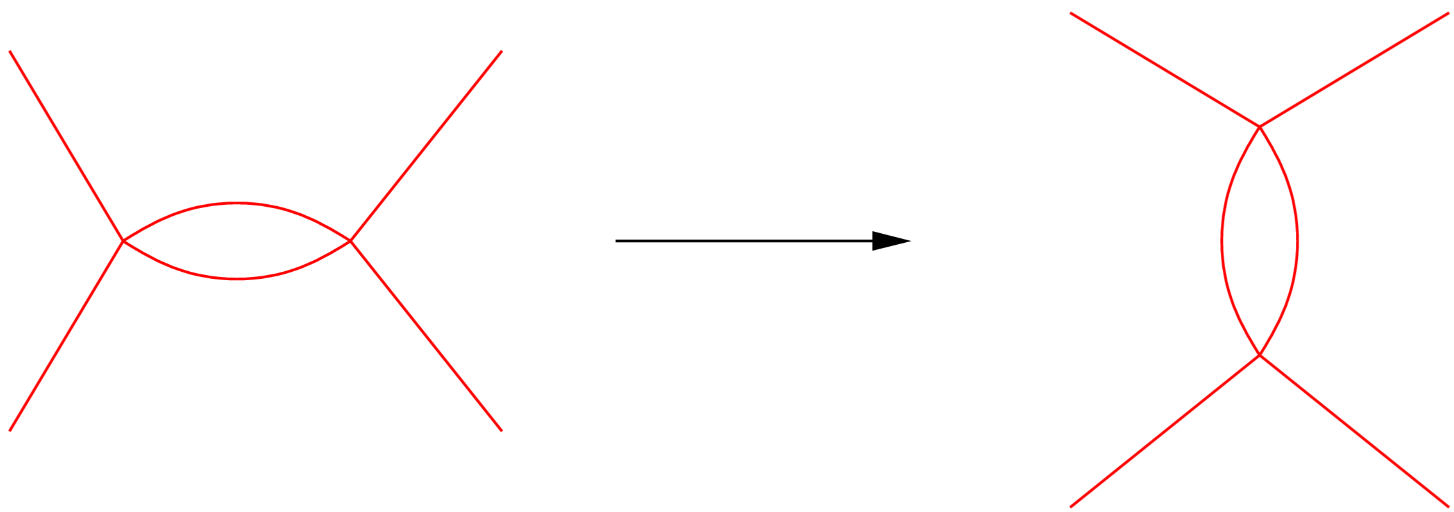


Numerical simulations

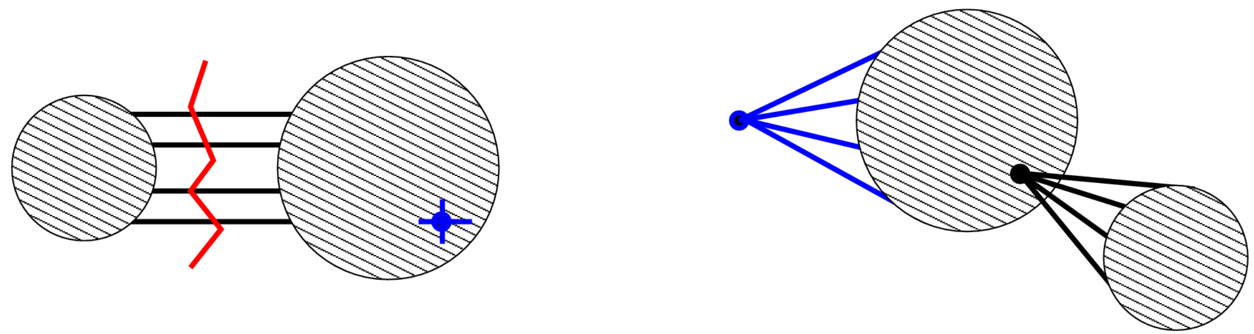
What about ϕ^4 graphs and quadrangulations?

One needs two-link flip to restore ergodicity



Numerical simulations

Non-local dynamics: "*minimal-neck baby universe surgery*"



J. Ambjørn, B. Durhuus, and T. Jonsson, *Quantum Geometry — A Statistical Field Theory Approach* (Cambridge University Press, Cambridge, 1997).

J. Ambjørn, M. Carfora, and A. Marzuoli, *The Geometry of Dynamical Triangulations* (Springer, Berlin, 1997).

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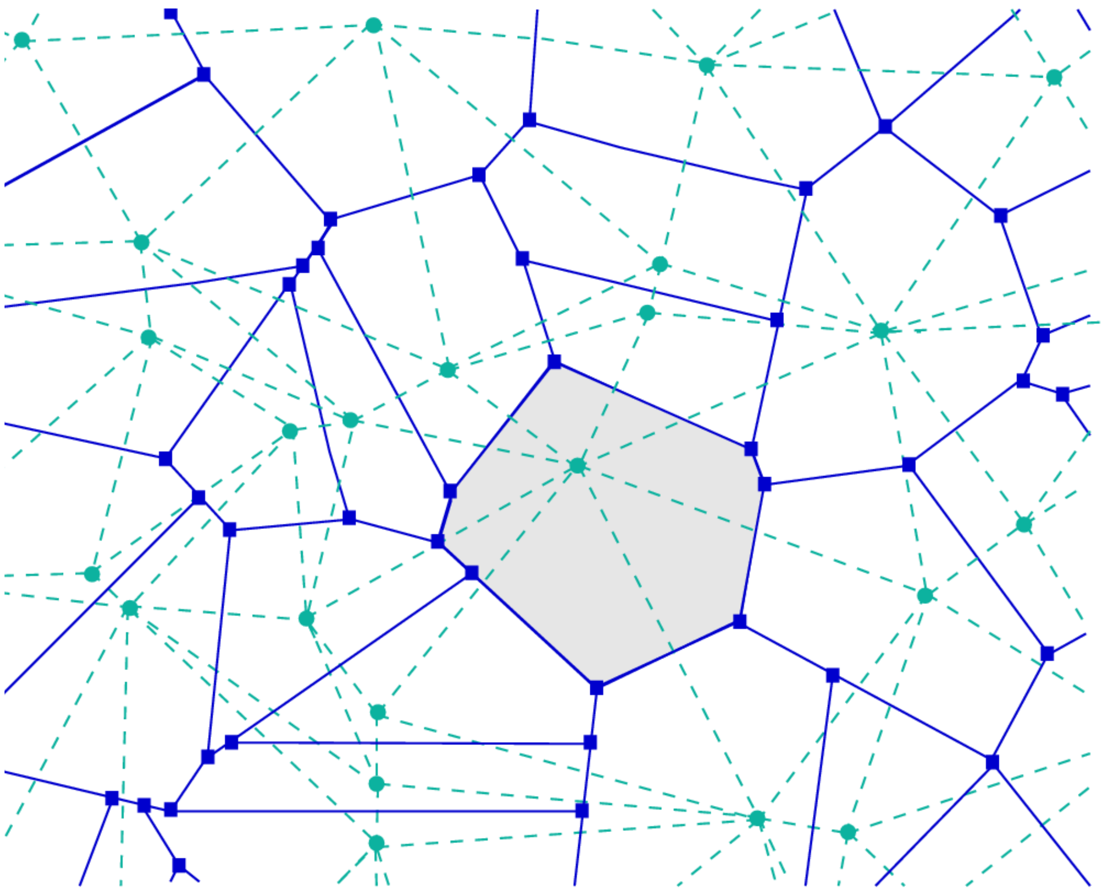
Construction

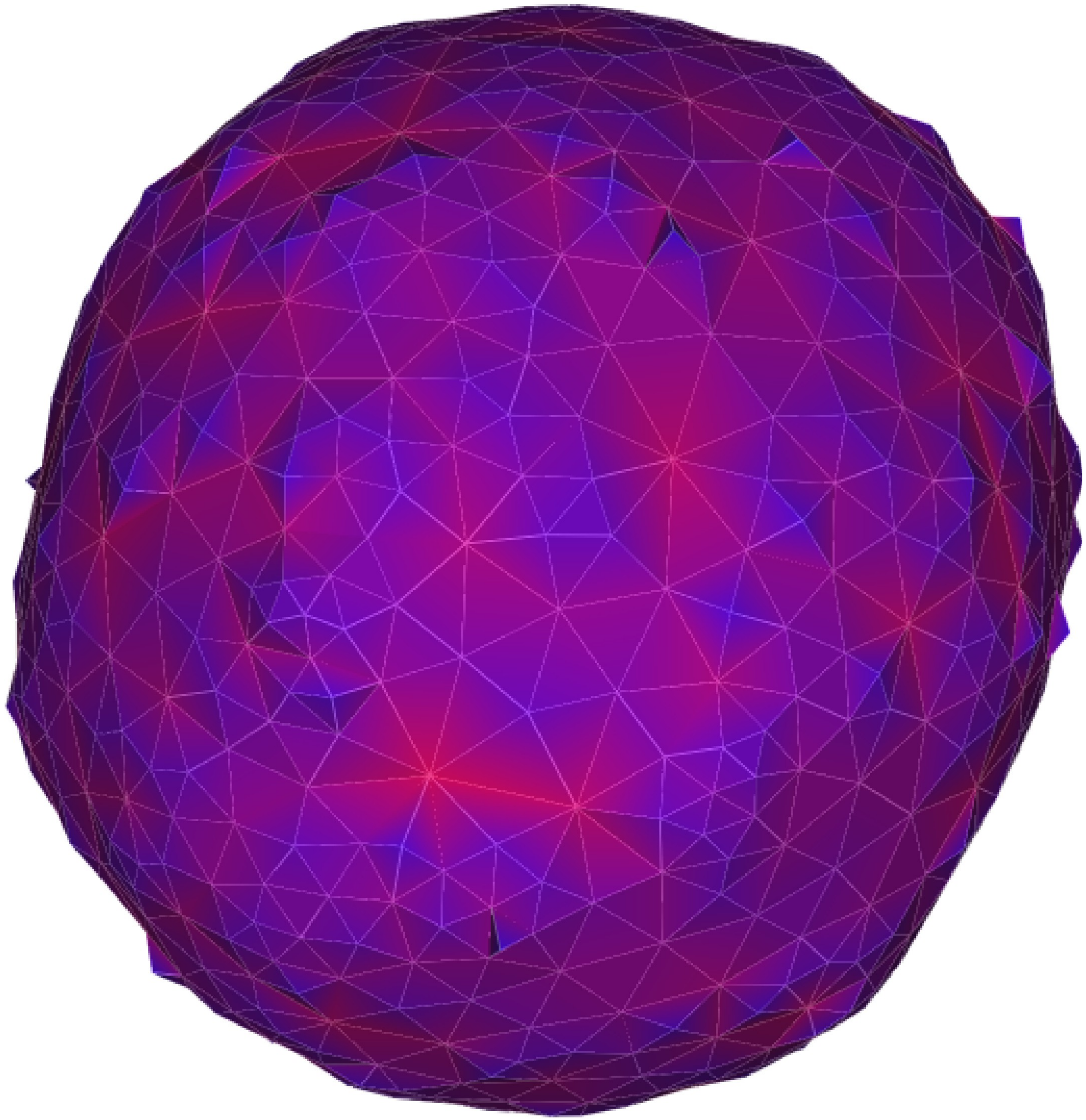
Generalize crystallographic Wigner-Seitz construction to random arrangement of points



Delaunay triangulation

Voronoï cell

Construction



Some properties

- Statistics are found to be surprisingly complex, only few exact results
- Recent studies provided asymptotic expansion of coordination number distribution for the planar case

$$\ln P(q) = -2q \ln q + q \ln(2\pi^2 e^2) - \frac{1}{2} \ln(2^6 \pi^5 C^{-2} q) + \mathcal{O}(q^{-1/2})$$

  i.e., asymptotically exponential
- Exponentially decaying correlations of the coordination numbers
- Many short loops, not small world, $d_h = 2$
- Aboav-Weaire law

$$m(q) = (6 - a) + b/q$$

  "many-sided cells tend to have few-sided neighbors and vice versa"
  from the asymptotic expansion, one instead finds

$$m(q) = 4 + 3\sqrt{\pi/q} + \cdots$$

- Also a number of results on the spatial *geometry* (cell diameter, surface etc.)

Generation

- Naïve method using plane intersections: $\mathcal{O}(n^3)$
- Adaptive modification of regular triangulation: $\mathcal{O}(n^2)$
- Add generators one by one: $\mathcal{O}(n^2)$
- Divide-and-conquer technique: $\mathcal{O}(n \ln n)$
- Plane sweep method: $\mathcal{O}(n \ln n)$
- ...

A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu, *Spatial Tessallations – Concepts and Applications of Voronoi Diagrams* (Wiley, Chichester, 2000).

H.J. Hilhorst and P. Calka, *Random line tessellations of the plane: statistical properties of many-sided cells*, Preprint arXiv:0802.1869.

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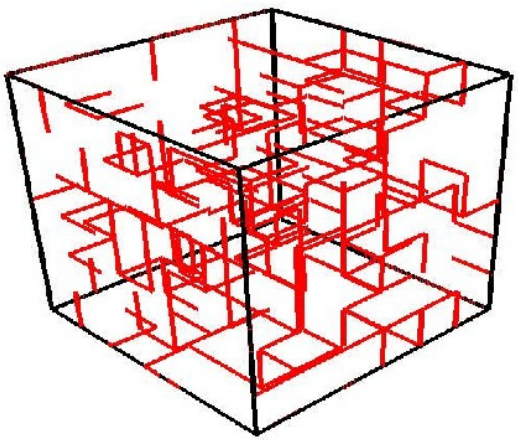
Coupling matter to random graphs

Effects of coupling spin models to random graphs instead of regular lattices:

- For sufficient connectivity, ordered phase should persist (at least for ferromagnets)
- Order of transition and universality class might change:
  - Regular lattice: Harris criterion

$$\sigma_R(J) \sim R^{-d/2} \;\Rightarrow\; \sigma_\xi(J) \sim \xi^{-d/2} \sim t^{\nu d/2}$$



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relevant if $\nu d/2 < 1$ or $\alpha > 0$

  - Finite-dimensional random graph: consider average coordination number in patch of size R,

$$J(R) \equiv \frac{1}{B(R)} \sum_{i \in P} q_i$$

then,

$$\sigma_R(J) \equiv \langle |J(R) - J_0| \rangle / J_0 \sim \langle B(R) \rangle^{-(1-\omega)} \sim R^{-d_h(1-\omega)}$$

disorder is relevant if $d_h \nu (1-\omega) > 1$

Wandering exponent ω is related to correlation function exponent a as

$$\omega = 1 - a/2d_h$$

Coupling matter to random graphs

Disorder

Annealed | **Quenched**

Uncorrelated

Harris criterion:

$\alpha > 0$

– random bond model

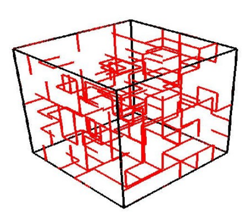
Correlated

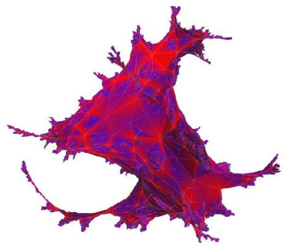
Luck criterion:

$(1-\alpha)/(2-\alpha) < \beta$

– QG graphs

– Voronoi/Delaunay





Coupling matter to random graphs

Quenched case

Complex networks

- In general very different from finite-dimensional graphs since diameter grows logarithmically (small worlds), but correlation length usually diverges algebraically
- Mean-field if all moments of $P(q)$ finite
- Deviation from mean-field, including non-universal exponents and infinite-order transitions for divergent moments

Thin graphs

- Mean-field since asymptotically equivalent to Bethe lattice
- E.g., for Ising model

$$Z_n(\beta) \times N_n = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda^{2n+1}} \int \frac{d\phi_+ d\phi_-}{2\pi\sqrt{\det K}} \exp(-S),$$

with

$$S = \frac{1}{2} \sum_{a,b} \phi_a K_{ab}^{-1} \phi_b - \frac{\lambda}{3}(\phi_+^3 + \phi_-^3).$$

Quenched case

Fat graphs

- Wandering exponent found to be $\omega = 3/4$, i.e., disorder relevant for $\alpha > -2$
- Confirmed by exact result for percolation and numerical results for q = 2, 3, 4 Potts models
- First-order transition for q > 4 softened to second order

Voronoï-Delaunay triangulations

- No exact results
- Simulations yield unchanged universality class for Ising in 2D ($\alpha = 0$) and 3D (α small)
- 3-State Potts model with $\alpha = 1/3$ also yields unchanged exponents, in apparent contradiction to the relevance criterion

Annealed fat graphs

Ferromagnet

Dressing of conformal weights according to KPZ/DDK formula

$$\tilde{\Delta} = \frac{\sqrt{1-c+24\Delta} - \sqrt{1-c}}{\sqrt{25-c} - \sqrt{1-c}}$$

well understood

Antiferromagnet

New phenomena resulting from back-reaction of matter onto graph structure (adaptive graph):

- Transition in the universality class of the FM case to Néel state for bipartite graphs
- Complete wipe-out of transition in other cases
- Infinite-order transition to magnetic Néel state combined with bipartite graph state in third cases

Conclusions

- Compared to regular lattices, random structures as modelled by random graphs exhibit a host of novel phenomena (to be) explored by statistical physicists
- Many more examples than those covered here, e.g., trees, combs, brushes, percolation clusters, polymers and flux lines, ...
- Many of the graph ensembles considered are attractive subjects since they are amenable to analytical treatment *as well as* numerical simulations
- Fat graphs and related models interesting intermediates between regular lattices and (infinite-dimensional, small world) random graphs
- Current focus of research:
  - Matter-graph back-reaction, adaptive graphs, annealed averages
  - Dynamics on graphs and networks
  - Evolving networks, non-equilibrium effects