Quantum Fluctuations in Simplicial Gravity

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Outline

1. Path integral formulation of Quantum Gravity
2. Causal Dynamical Triangulations
3. Background spacetime
4. The Minisuperspace Model
5. Quantum fluctuations
6. The Sturm-Liouville Operator
7. Conclusions

Path integral formulation of Quantum Gravity

- The partition function of quantum gravity is defined as a formal integral over all geometries weighted by the Einstein-Hilbert action.
- To make sense of the gravitational path integral one uses the standard method of regularization - discretization.
- The path integral is written as a nonperturbative sum over all causal triangulations $\mathcal{T}$.
- Wick rotation is well defined due to proper time foliation.
- Using Monte Carlo techniques we can calculate expectation values of observables.

$$Z = \int \frac{\mathcal{D}_{\mathcal{M}}[g]}{Diff_{\mathcal{M}}} e^{iS^{EH}[g]} \quad \rightarrow \quad Z = \sum_{\mathcal{T}} \frac{1}{s(\mathcal{T})} e^{iS^{Regge}[g]}$$

Causal Dynamical Triangulations (CDT) is a background independent approach to quantum gravity.

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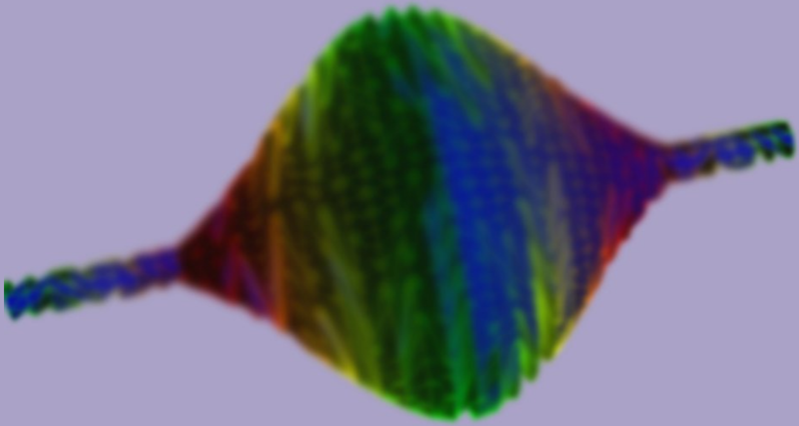
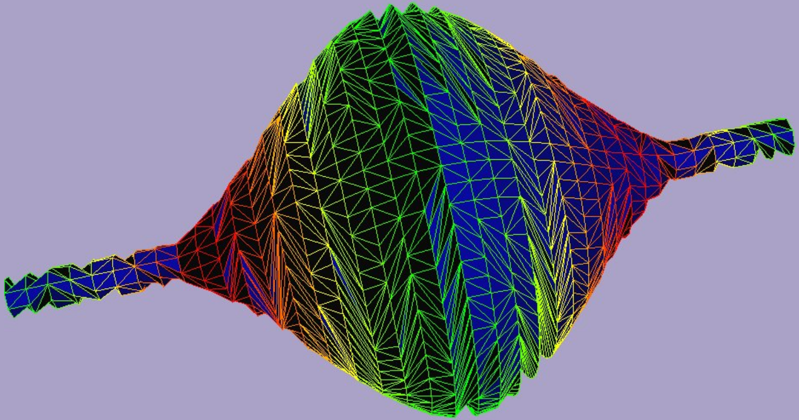
Dynamical Triangulations

A manifold with topology $S^3 \times S^1$ ...

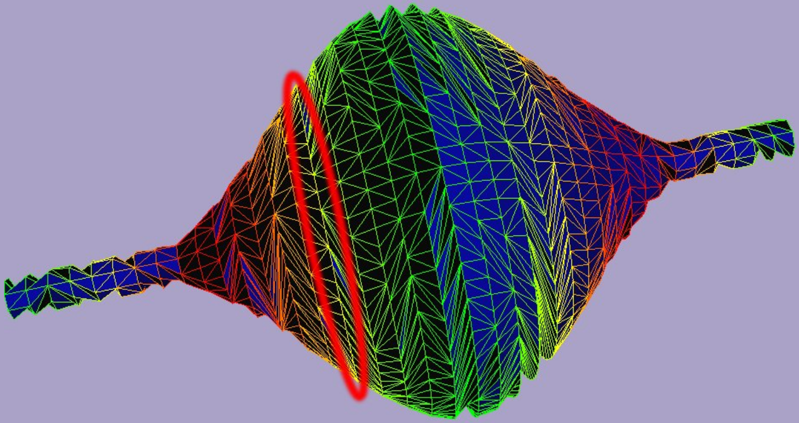


Dynamical Triangulations

. . . is discretized by gluing 4-simplices - triangulation

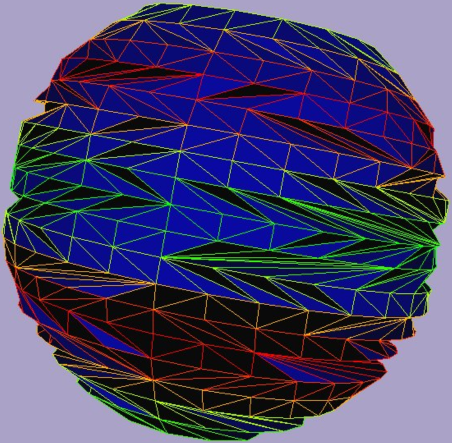


The spatial slices (of constant time) . . .



Causal Dynamical Triangulations

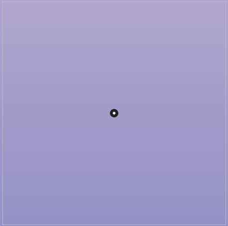
. . . have S^3 topology



Fundamental building blocks of CDT

- d-dimensional simplicial manifold can be obtained by gluing pairs of d-simplices along their $(d-1)$-faces.
- Lengths of the time and space links are constant. Simplices have a fixed geometry.
- The metric is flat inside each d-simplex.
- The angle deficit (curvature) is localized at $(d-2)$-dimensional sub-simplices.

0D simplex - point



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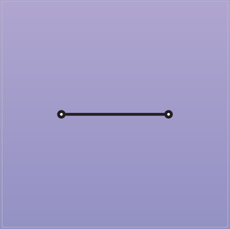
1D simplex - link



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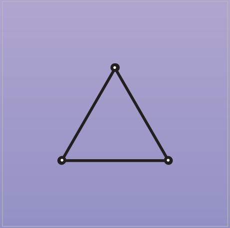
2D simplex - triangle



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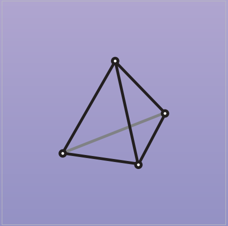
3D simplex - tetrahedron



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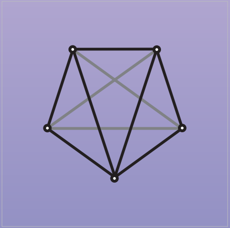
4D simplex - 4-simplex



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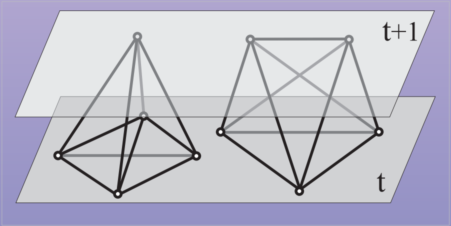
4D simplex, two types in CDT



We construct a starting space-time manifold with given topology ($S^3 \times S^1$) and perform a random walk over configuration space.

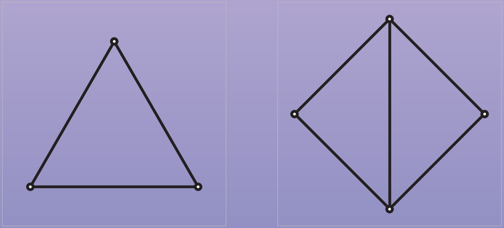
Ergodicity In the dynamical triangulation approach all possible configurations are generated by the set of Alexander moves.

Fixed topology The moves don't change the topology.

Causality Only moves that preserve the foliation are allowed.

4D CDT We have 4 types of moves.

Examples in 2D



Monte Carlo simulations - Alexander moves

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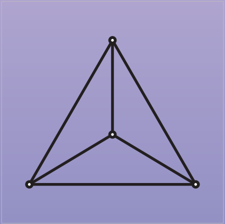
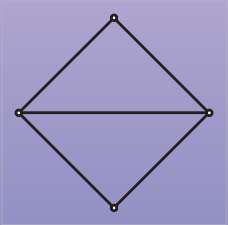




Causal Dynamical Triangulation - properties

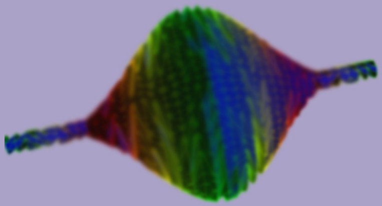
- Manifolds are approximated by simplicial manifolds.
- Sum over triangulations (gluings).
- By construction we have foliation in proper time. We do not allow the spatial slices to change topology.
- Wick rotation is well defined.
- Such formulation involves only geometric invariants like length and angles.
- We don't introduce coordinates.
- Manifestly diffeomorphism-invariant.

We generate a large number of such configurations with the probability

$$P[configuration] \propto e^{-S}$$

We use Einstein-Hilbert action ...

$$S = -\frac{1}{G} \int \mathrm{d}t \int \mathrm{d}\Omega \sqrt{g}(R - 6\lambda)$$



- G gravitational constant
- λ cosmological constant
- g determinant of a spacetime metric
- R scalar curvature

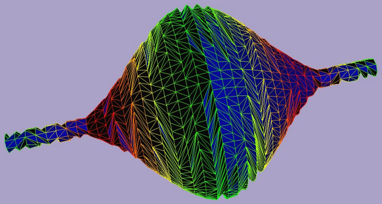
Path integral - The action

We generate a large number of such configurations with the probability

$$P[configuration] \propto e^{-S}$$

...or the Regge action in the discrete version

$$S = -K_0 N_0 + K_4 N_4 + \Delta(N_{14} - 6N_0)$$



- N_0 number of vertices
- N_4 number of simplices
- N_{14} number of simplices of type $\{1, 4\}$
- K_0 K_4 Δ bare coupling constants

Background

- For a certain range of bare coupling constants, a typical configuration has a "bloblike" shape and behaves as a well defined four-dimensional manifold.
- This isn't trivial. In the Euclidean version, without imposed causality, one either got
  - a "crumpled phase" with infinite Hausdorff dimension
    or
  - a "branched polymer phase" dominated by spacetimes where the 4-simplices form treelike structures with Hausdorff dimension two,

  even though they are built from 4D simplices. Unfortunately, the phase transition between them is of the 1^{st} order.

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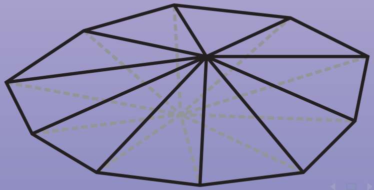

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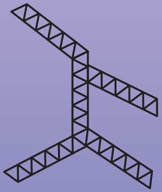
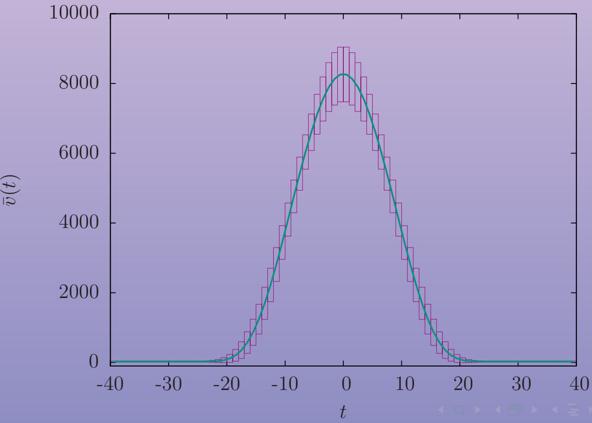


Background

- For a certain range of bare coupling constants, a typical configuration has a "bloblike" shape and behaves as a well defined four-dimensional manifold.
- The averaged spatial volume $\bar{v}(t)$ is proportional to $\cos^3(t/B)$.



Background - classical solution

- Such behaviour of the average volume occurs when we assume spatial homogeneity and isotropy.
- We "freeze" all degrees of freedom except the volume (scale factor).
- We introduce the following metric on $S^3 \times S^1$ spacetime

$$\mathrm{d}s^2 = \mathrm{d}t^2 + v^{2/3}(t)\mathrm{d}\Omega_3^2$$

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$$S = \frac{1}{G} \int \frac{\dot{v}^2}{v} + v^{\frac{1}{3}} - \lambda v \mathrm{d}t$$

Minisuperspace

The classical trajectory $\bar{v}(t) \propto \cos^3(t/B)$ is perfectly recovered in this model.

Question?

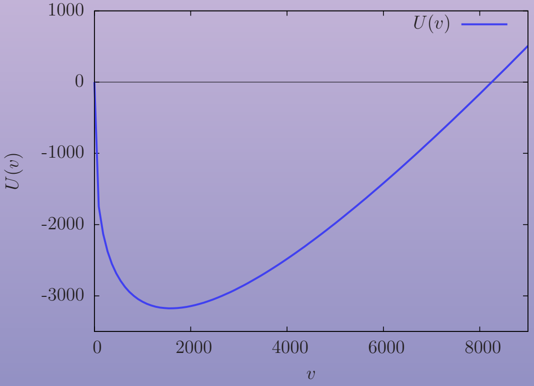
How well does the minisuperspace model describe the quantum fluctuations computed from Monte Carlo simulations?

The minisuperspace action

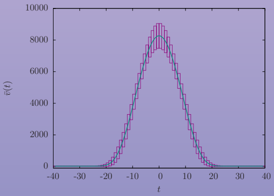
$$S = \frac{1}{G} \int \frac{\dot{v}^2}{v} + v^{\frac{1}{3}} - \lambda v \mathrm{d}t$$

The potential

- Restricting our considerations to the volume $v(t)$ we reduce the problem to one-dimensional quantum mechanics.
- The minisuperspace action describes a motion of a particle in a potential:



$$U(v) = -v^{\frac{1}{3}} + \lambda v$$



The regularized potential - bounce

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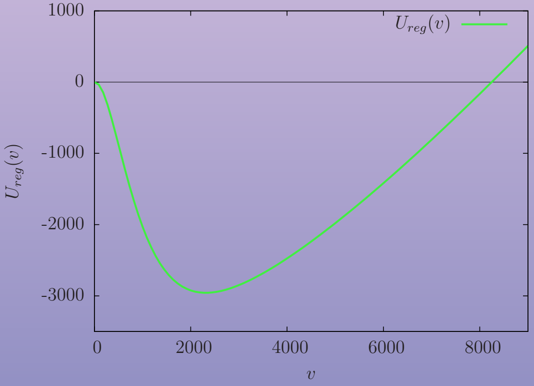

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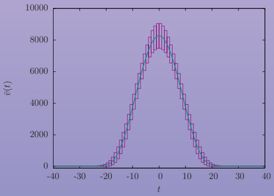

Quantum fluctuations - semiclassical approximation

- The spatial volume fluctuations are described by a Hermitian Sturm-Liouville operator $D(t)$.

$$S[v = \bar{v} + x] \approx S[\bar{v}] + \frac{1}{2} \int x(t) D(t) x(t) \mathrm{d}t$$

$$D(t) = -\partial_t \frac{1}{\bar{v}(t)} \partial_t + \frac{\partial^2 U}{\partial v^2}\Big|_{v=\bar{v}}$$

- For a discrete time it is a matrix $M_{tt'}$

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$$x_t M_{tt'} x_{t'} = c_1 \sum_t \frac{(x_{t+1} - x_t)^2}{\bar{v}_t} + \frac{\partial^2 U}{\partial v^2}\Big|_{v=\bar{v}} x_t^2$$

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Propagator vs action

- There exists a direct relationship between the propagator - fluctuation correlation matrix - and the matrix M

$$C_{tt'} = \langle x_t x_{t'} \rangle = \frac{1}{Z} \int x_t x_{t'} e^{-\frac{1}{2} \sum_{t,t'} x_t M_{tt'} x_{t'}} \prod_s \mathrm{d}x_s = {M^{-1}}_{tt'}$$

- The propagator C can be measured. $x_t = v_t - \bar{v}_t$.
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The propagator C

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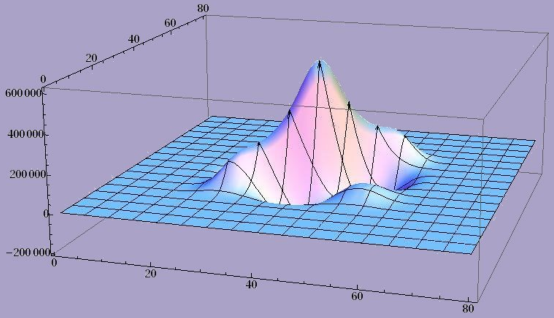

The Sturm-Liouville operator M

$$M = C^{-1}$$

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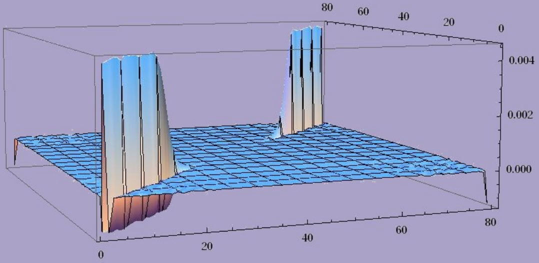
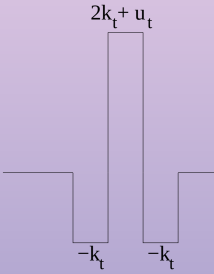


M matrix decomposition

- Analysing the matrix M row by row we can find the kinetic k_t and potential u_t coefficients.
- This allows us to compare the theoretical predictions with numerical data.
- We expect that $\bar{v}_t \approx \frac{c_1}{k_t}$ and $u_t \approx -U''(\bar{v}_t)$.



$x_t M_{tt'} x_{t'}$

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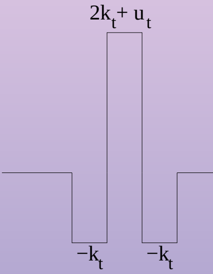

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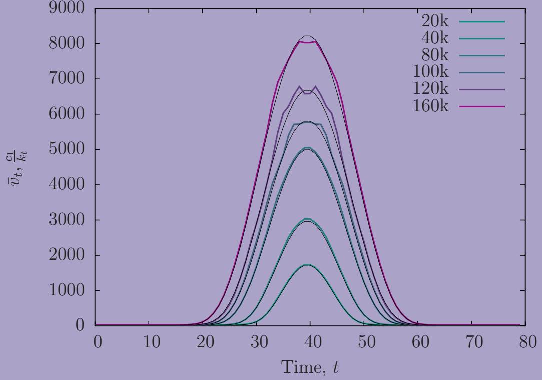


The kinetic part

- Coefficients k_t fully agree with the predictions

$$\bar{v}_t = \frac{c_1}{k_t}$$

- The constant c_1 doesn't depend on the total volume.
- We can relate the cut-off with the gravitational constant G, which is responsible for the fluctuation amplitude.

$$G = \mathrm{const}\frac{a^2}{c_1}$$

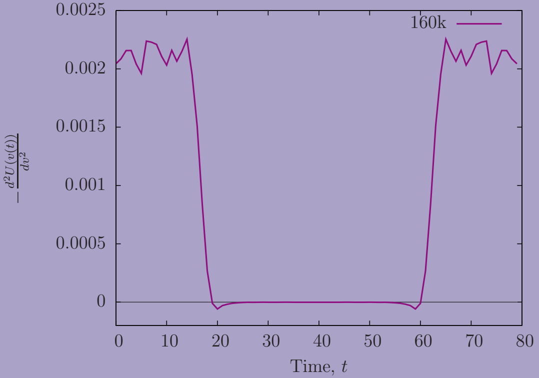
- The Universe built of 362000 simplices has a radius of about 20 Planck lengths.

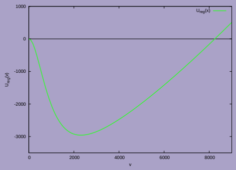
The potential part

- The coefficients u_t also agree with the regularized potential

$$u_t \approx -U''(\bar{v}_t)$$

Potential well - $U''(v(t))$ plot

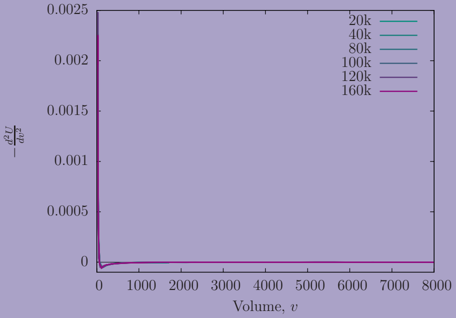


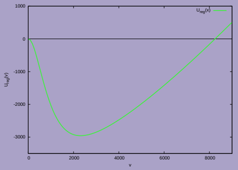


The potential part

- If the model is correct, $U''(v)$ should be universal and not depend on total volume. The "cosmological constant" λ controls the volume, but it gives no contribution to $U''(v)$.
- The numerical results are in full agreement with the regularised potential.

$U''(v)$ plot

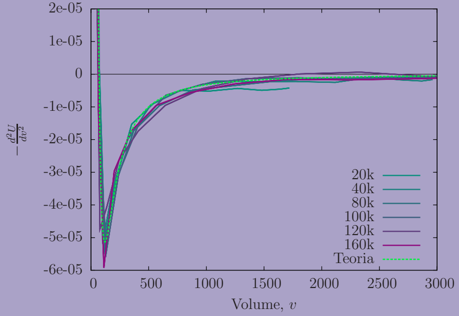


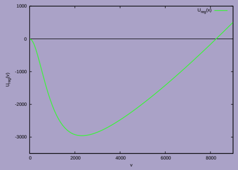


The potential part

- If the model is correct, $U''(v)$ should be universal and not depend on total volume. The "cosmological constant" λ controls the volume, but it gives no contribution to $U''(v)$.
- The numerical results are in full agreement with the regularised potential.

$U''(v)$ plot

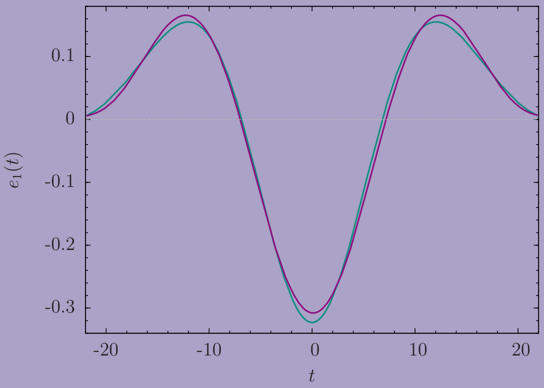




Eigenvectors

- Eigenvectors of the numerical matrix C and theoretical matrix M are very similar.

The eigenvectors of C i M



Conclusions

1. We observe a four-dimensional universe with well defined time and space extension.
2. The background geometry exactly corresponds to the classical solution of the minisuperspace model (classical Einstein theory).
3. Quantum fluctuations are properly described by this simple model.
4. The gravitational constant G controls the fluctuation amplitude. We may estimate that the Universe built of 362000 simplices has a radius of about 20 Planck lengths.
5. The minisuperspace model correctly predicts the eigenvectors of the propagator.

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