Multifractality in random-bond Potts models

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Outline

Self-averaging

Physical observables ϕ should be averaged over thermal fluctuations AND disorder realizations. Using Bertrand's notations

$$\begin{aligned}
\overline{\langle\phi\rangle} &= \int \mathcal{D}[K_{ij}]\mathcal{P}[K_{ij}]\langle\phi\rangle_{[K_{ij}]} \\
&= \int \mathcal{D}[K_{ij}]\mathcal{P}[K_{ij}] \int \mathcal{D}[\sigma_i]\phi[\sigma_i]\frac{e^{-\beta H[K_{ij},\sigma_i]}}{\mathcal{Z}[K_{ij}]}
\end{aligned}$$

Self-averaging (2)

$\langle\phi\rangle_{K_{ij}}$ is a random variable (fluctuates from sample to sample) with probability distribution :

$$\wp(\phi) = \int \mathcal{D}[K_{ij}]\mathcal{P}[K_{ij}]\delta\left(\phi - \langle\phi\rangle_{[K_{ij}]}\right)$$

and average

$$\overline{\langle\phi\rangle} = \int \phi\, \wp(\phi) d\phi$$

Naive assumption: when the system size tends to ∞, all disorder realizations become equivalent, i.e.

$$\wp(\phi) \sim \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\phi-\overline{\langle\phi\rangle})^2/2\sigma^2} \longrightarrow \delta(\phi - \overline{\langle\phi\rangle})$$

Self-averaging (3)

Sample-to-sample fluctuations should vanish

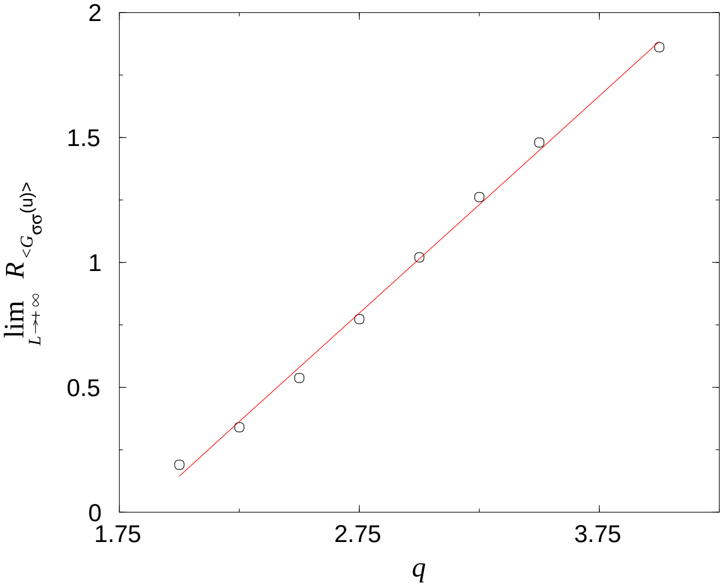
$$R_\phi = \frac{\overline{\langle \phi \rangle^2} - \overline{\langle \phi \rangle}^2}{\overline{\phi}^2} \longrightarrow 0$$

according to the central limit theorem.

Wiseman and Domany (1995) observed that it is not true: R_m and R_χ tend to finite values in the thermodynamic limit !

m and χ are examples of non self-averaging quantities.

Self-averaging (4)



Self-averaging (5)

Aharony, Harris and Wiseman (1998) showed that

- Stable random fixed point: (non self-averaging)

$$R_\phi(L) \sim R_\phi(\infty) + \mathcal{A}\, L^{(\alpha/\nu)^{\mathrm{rand.}}}$$

  $R_\phi(\infty)$ is a universal value.

- Unstable random fixed point, i.e. $\alpha < 0$: (weak self-averaging)

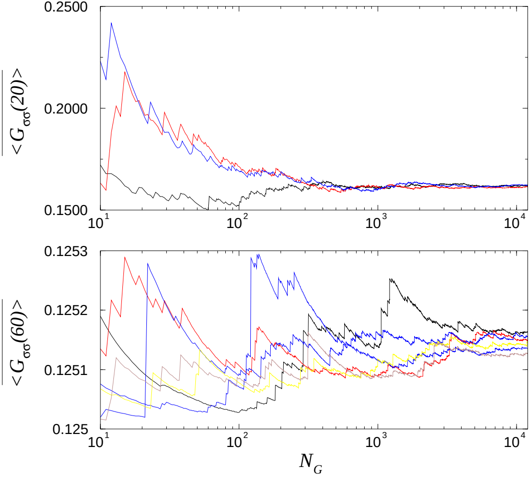
$$R_\phi(L) \sim L^{\alpha/\nu}$$

- Out of the critical point (self-averaging)

$$R_\phi(L) \sim L^{-d}$$

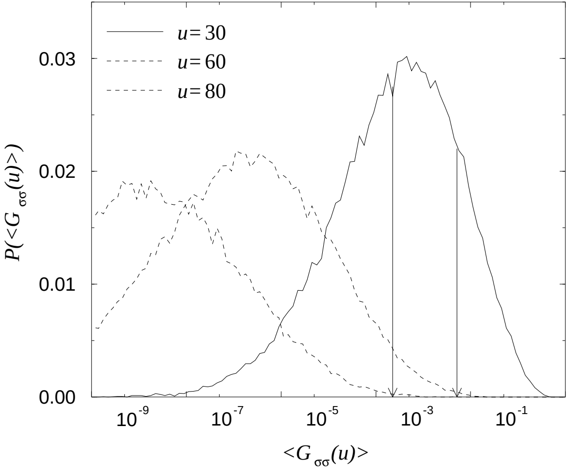
Rare events

The average is dominated by rare events. Average and typical values are different.



Rare events (2)

The distribution $\wp(\phi)$ displays a long tail:



Impossibility of numerical simulations?

The Ising chain

$$-\beta H = \sum_i K_i \sigma_i \sigma_{i+1}, \quad (\sigma_i \in \{-1; +1\})$$

Spin-spin correlation function for a given disorder realization

$$\langle G_{\sigma\sigma}(i,j)\rangle = \langle \sigma_i \sigma_j \rangle = \prod_{k=i}^{j-1} \tanh K_k$$

The Ising chain (2)

If the K_k are uncorrelated independent random variables, the central limit theorem applies to the sum

$$\ln\langle G_{\sigma\sigma}(i,j)\rangle = \sum_{k=i}^{j-1} \ln\tanh K_k$$

leading to ($|i-j| \gg 1$)

$$\wp\left(\ln\langle G_{\sigma\sigma}(i,j)\rangle = \ln C\right) \sim \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\ln C - \overline{\ln C})^2/2\sigma^2}$$

The Ising chain (3)

Typical

$$G_{\sigma\sigma}^{\text{typical}}(i,j) = e^{|j-i|\overline{\ln \tanh K}} = e^{\overline{\ln\langle G_{\sigma\sigma}(i,j)\rangle}}$$

and average values

$$\overline{\langle G_{\sigma\sigma}(i,j)\rangle} = \int e^{x}\wp(x)dx = e^{|j-i|\ln\overline{\tanh K}} \gg G_{\sigma\sigma}^{\text{typical}}(i,j)$$

are different (Derrida, 1981). Rare events: large number of strong couplings.

The Ising chain (3)

One can define two different correlation lengths

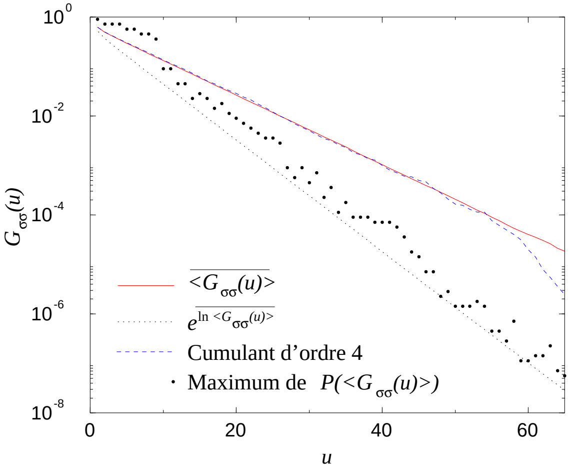
$$\ln G_{\sigma\sigma}^{\text{typical}}(r) \sim -\frac{r}{\xi^{\text{typical}}}, \qquad \ln \overline{\langle G_{\sigma\sigma}(r)\rangle} \sim -\frac{r}{\xi^{\text{avg}}}$$

with different critical behavior

$$\xi^{\text{typical}} \sim |T - T_c|^{-\nu^{\text{typical}}}, \qquad \xi^{\text{avg}} \sim |T - T_c|^{-\nu^{\text{avg}}}$$

The stability of the random fixed point imposes $\nu^{\text{avg}} \geq 2/d$ but ν^{typical} can violate this constraint.

Multifractality



Multifractality (2)

For a self-averaging quantity, for instance a Gaussian distributed random variable, the cumulants satisfy the recursion relation (Wick Theorem)

$$\begin{aligned}\overline{(\phi-\overline{\phi})^n} &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} d\phi\,(\phi-\overline{\phi})^n e^{-(\phi-\overline{\phi})^2/2\sigma^2} \\ &= (n-1)\sigma^2\overline{(\phi-\overline{\phi})^{n-2}}\end{aligned}$$

The scaling dimension $x_\phi(n)$ of $\overline{(\phi-\overline{\phi})^n}^{1/n}$ is thus

$$x_\phi(n) = x_\phi(2) = x_\phi$$

Perturbative results

Replica trick

$$F = -\overline{\ln \mathcal{Z}} = -\lim_{p\to 0} \frac{1}{p}\left[\overline{\mathcal{Z}^p} - 1\right]$$

For disorder $m(\vec{r})$ coupled to the energy density $\epsilon(\vec{r})$

$$\begin{aligned}
\overline{\mathcal{Z}^p} &= \overline{\int \mathcal{D}[\phi_1(\vec{r})]\ldots\int \mathcal{D}[\phi_p(\vec{r})]\ e^{-\sum_{\alpha=1}^{p}\left(H_0[\phi_\alpha]+\sum_{\vec{r}}\ m(\vec{r})\epsilon_\alpha(\vec{r})\right)}} \\
&\simeq \int \mathcal{D}[\phi_1(\vec{r})]\ldots\int \mathcal{D}[\phi_p(\vec{r})]\ e^{-\sum_\alpha H_0[\phi_\alpha]} \\
&\qquad \times e^{-\sum_{\vec{r}}\left[\overline{m}\sum_\alpha \epsilon_\alpha(\vec{r})+\frac{1}{2}(\overline{m^2}-\overline{m}^2)\sum_{\alpha,\beta}\epsilon_\alpha(\vec{r})\epsilon_\beta(\vec{r})+\ldots\right]}
\end{aligned}$$

Perturbative results (2)

Renormalization group predictions for the decay exponents of the moments

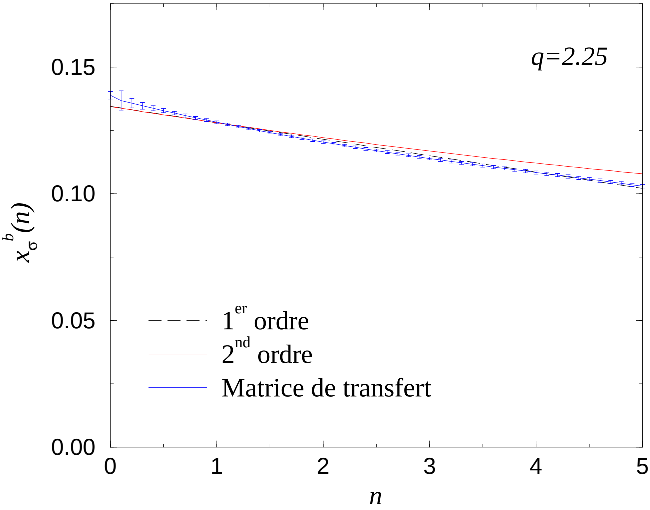
$$\overline{\langle G_{\sigma\sigma}(u)\rangle^n}^{1/n} \sim u^{-2x_\sigma^b(n)}$$

of the spin-spin correlation function of conformal minimal models (Ludwig, Dotsenko, Lewis)

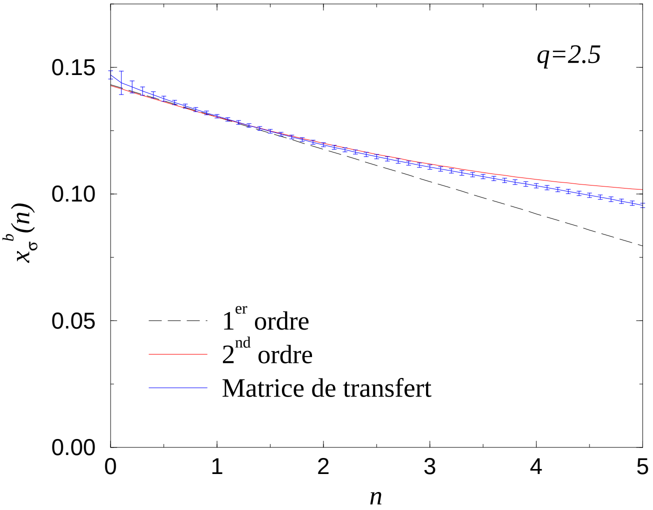
$$x_\sigma^b(n) = x_\sigma^{b,\mathrm{Pure}} - \frac{n-1}{16}\left\{y_H + \left[\frac{11}{12} - 4\ln 2 + \frac{n-2}{24}\left(33 - \frac{29\pi}{\sqrt{3}}\right)\right]\frac{y_H^2}{2}\right\} + \mathcal{O}(y_H^3)$$

where $y_H = \alpha^{\mathrm{Pure}}/\nu^{\mathrm{Pure}}$ for disorder coupled to the energy density.

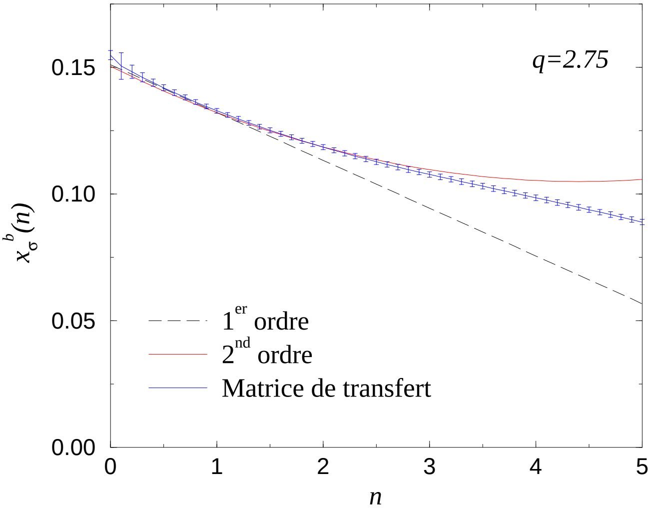
Perturbative results (3)



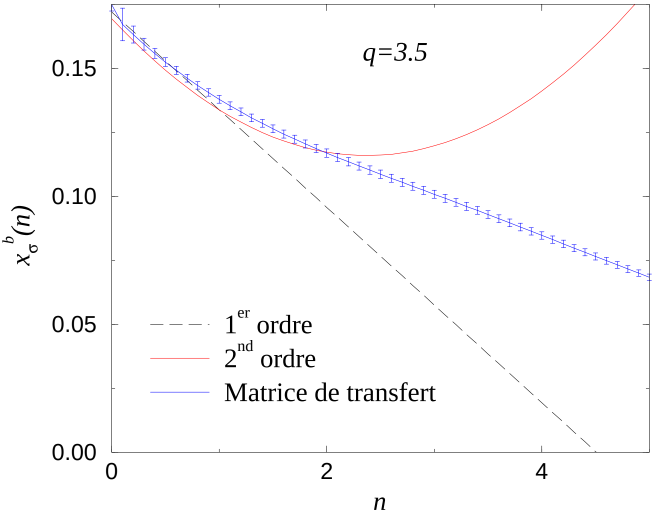
Perturbative results (4)



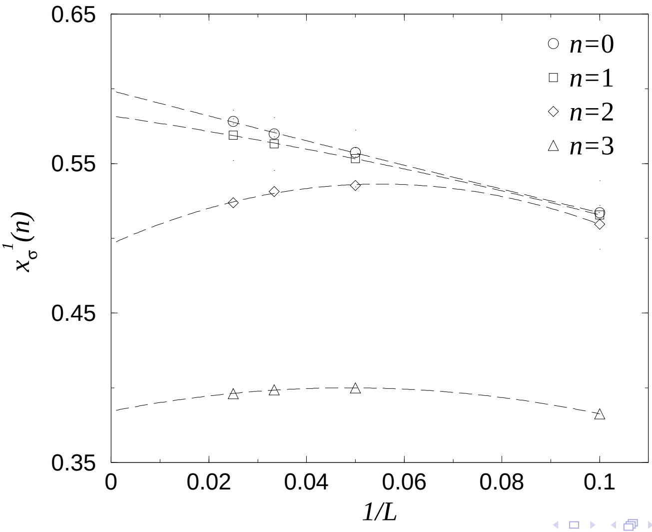
Perturbative results (5)



Perturbative results (6)



Multifractality at boundaries



Replica Symmetry Breaking

Even for uncorrelated disorder, i.e.

$$\overline{m(\vec{r})m(\vec{r}\,')} = \overline{m^2}\delta(\vec{r}-\vec{r}\,')$$

it appears a coupling between different replicas. All interactions are symmetric under permutation between replicas. But the associated random fixed point may be unstable (random field XY model, ϕ^4) and the system flows under renormalization toward a new fixed point where this symmetry is broken !

$$\sum_{\alpha,\beta=1}^{p} g_{\alpha\beta}\epsilon_\alpha(\vec{r})\epsilon_\beta(\vec{r})$$

Replica Symmetry Breaking (2)

For minimal models, renormalization group calculations leads to

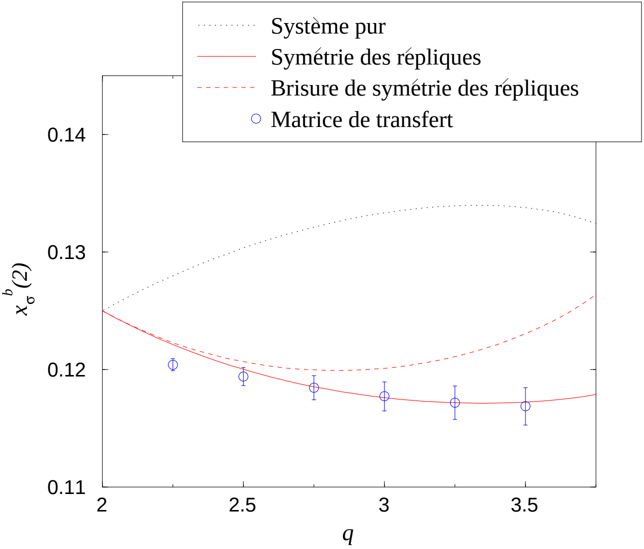
- Replica symmetry

$$x_\sigma^b(2) = x_\sigma^{b,\mathrm{Pur}} - \frac{1}{16}y_H + \frac{1}{32}\left(4\ln 2 - \frac{11}{12}\right)y_H^2 + \mathcal{O}(y_H^3)$$

- Broken replica symmetry

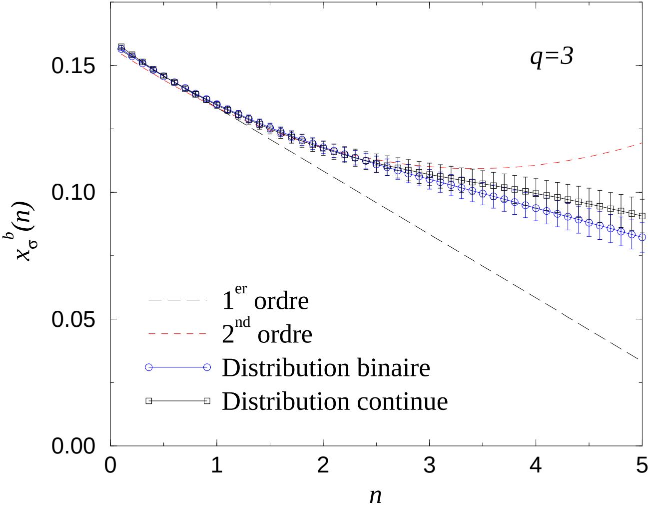
$$x_\sigma^b(2) = x_\sigma^{b,\mathrm{Pur}} - \frac{1}{16}y_H + \frac{1}{32}\left(4\ln 2 - \frac{5}{12}\right)y_H^2 + \mathcal{O}(y_H^3)$$

Replica Symmetry Breaking (3)



Universality

Universality of the multifractal spectrum



Prob. distrib. and multifractal spectrum

The multifractal spectrum is entirely determined by the proba. distribution.

$$\overline{\langle G_{\sigma\sigma}(u)\rangle^n} = \int_0^1 dG \; \wp_u(G) G^n \sim \mathcal{A}_n u^{-2X(n)}$$

where $X(n) = n x_\sigma^b(n)$. Let $y = -\ln G$ and $\tilde{\wp}_u(y)dy = \wp_u(G)dG$

$$\int_0^{+\infty} dy \; e^{-ny} \tilde{\wp}_u(y) \sim \mathcal{A}_n u^{-2X(n)}$$

By Mellin-Fourier transform

$$\tilde{\wp}_u(y) \sim \frac{1}{2i\pi} \int_{\delta - i\infty}^{\delta + i\infty} dn \; e^{ny - 2X(n) \ln u + \ln \mathcal{A}_n}$$

Prob. distrib. and multifractal spectrum (2)

Let $\alpha = y/2 \ln u$. In the saddle point approximation

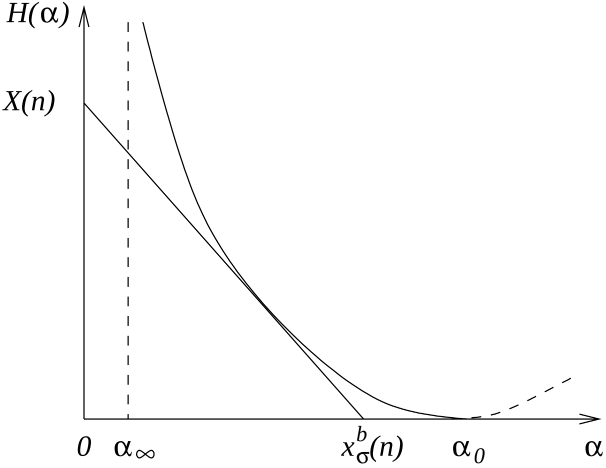
$$\tilde{\wp}_u(y) \sim \frac{1}{2i\pi} \int_{\delta - i\infty}^{\delta + i\infty} dn \; e^{-2 \ln u [X(n) - n\alpha]} \sim e^{-2 \ln u \;\; H(\alpha)}$$

where $H(\alpha)$ is the Legendre transform of the exponent $X(n)$

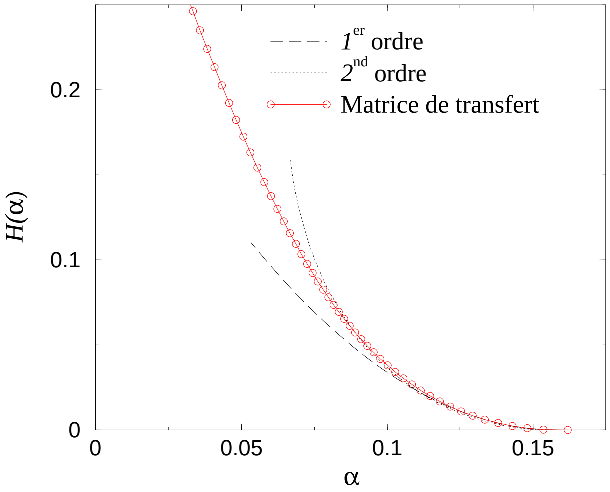
$$H(\alpha) = X(n^*) - \alpha n^*, \qquad \alpha = \left(\frac{\partial X(n)}{\partial n} \right)_{n^*}$$

Prob. distrib. and multifractal spectrum (3)

$$H(\alpha) = X(n) - \alpha n$$



Prob. distrib. and multifractal spectrum (4)



Conclusions

- Self-averaging because of rare events
- Multifractality
- Multifractal spectrum determined by the probability distribution