First-passage area distribution and optimal fluctuations of fractional Brownian motion

Alexander K. Hartmann[1], Baruch Meerson[2]

1: Institut für Physik, Universität Oldenburg, Germany
2: Racah Institute of Physics, Hebrew University, Jerusalem, Israel

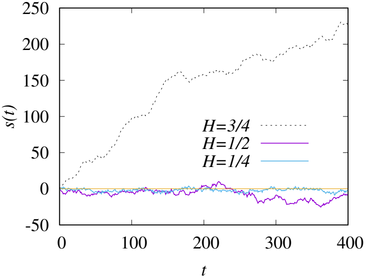
Leipzig, 19. December 2023

Outline

- Fractional Brownian Motion
- Area under first passage
- Sampling Algorithm
- Results
- Summary

Fractional Brownian Motion

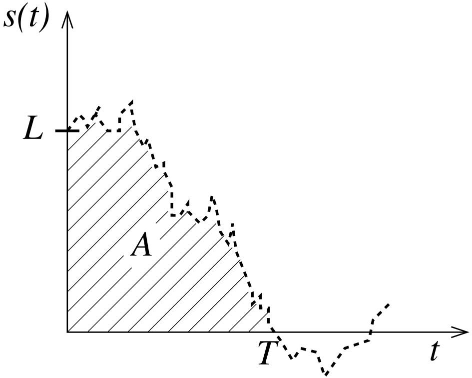
- Gaussian process $s(t)$ with correlation $\langle s(t_1)s(t_2)\rangle \sim t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H}$
- H: Hurst exponent: $H = 1/2$: Brownian motion, $H > 1/2$: correlation, $H < 1/2$: anticorrelation



- Efficient generation: Fourier transformation of desired correlation, multipl. with Gaussian random numbers $\vec{x}$, back transformation: increments $\Delta_i \to$ walk $s(t) = \sum_{i<t} \Delta_i$

Area under walk

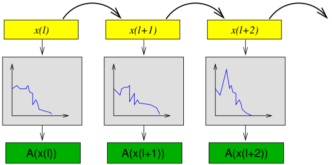
- Walks $s(t)$ start at distance L
- First passage to $s = 0$, A: area under curve



- Wanted: distribution $P(A)$ even for very small probabilities

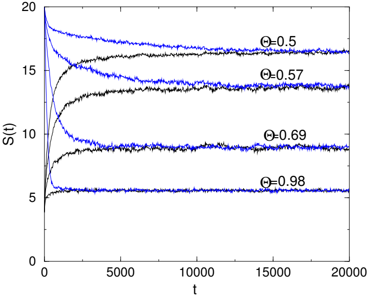
Large-deviation Approach

- Basic idea: Markov chain of Random vector instances
  $\vec{x}^{(0)} \to \vec{x}^{(1)} \to \vec{x}^{(2)} \to \dots$
  $\vec{x}^{(l)}$ calc. area $\to A(\vec{x}^{(l)})$
  step: change fraction of $\vec{x}^{(l)}$
  accept with $\min\{1, e^{+\Delta A/\Theta}\}$



[AKH, PRE 2002; PRE 2014]

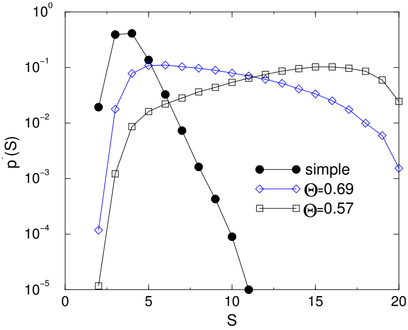
- Simulation at different Θ (using $(MC)^3$/PT)
  Example for general "Score" S (e,g., $S = A$)
  equilibration:
  start with ground state/
  with random state



- possible: Wang-Landau

Distribution of Scores

- Raw result $\longrightarrow$ (simple $\leftrightarrow \Theta = \infty$) at low Θ: high scores prefered
- MC moves: $\vec{x} \rightarrow \vec{x}'$ change one "element" probability $= f_a$



$$\Pr(\text{acceptance}) = \min\{1, \frac{\exp(S(\vec{x}')/\Theta)}{\exp(S(\vec{x})/\Theta)}\} = \min\{1, e^{\Delta S/\Theta}\}$$

- $\Rightarrow$ equilibrium distribution $Q_\Theta(\vec{x}) = P(\vec{x})e^{S(\vec{x})/\Theta}/Z(\Theta)$
  with $P(\vec{x}) = \prod_i f_{x_i}, \;\; Z(\Theta) = \sum_{\vec{x}} P(\vec{x})e^{S(\vec{x})/\Theta}$

$$\Rightarrow \;\; p_\Theta(S) = \sum_{\vec{x},S(\vec{x})=S} Q_\Theta(\vec{x}) = \frac{\exp(S/\Theta)}{Z(\Theta)} \sum_{\vec{x},S(\vec{x})=S} P(\vec{x})$$

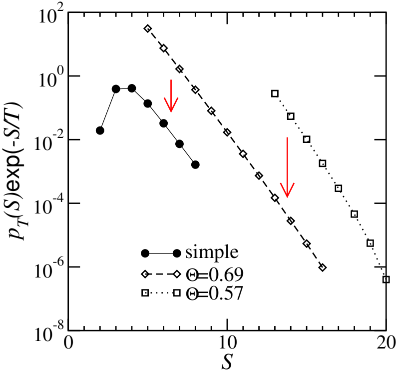
$$\Rightarrow \quad p(S) = p_\Theta(S)Z(\Theta)e^{-S/\Theta}$$

[AKH, PRE 2001]

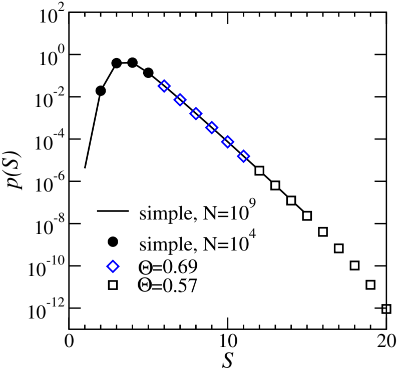
Match Distributions

$\left[\ p(S) = p_\Theta(S) Z(\Theta)\exp(-S/\Theta)\ \right]$

rescaling with $\exp(-S/\Theta)$



$Z(\Theta)$ by “matching”



agrees with large statistics simple sampling

agrees with (for some examples) known exact results

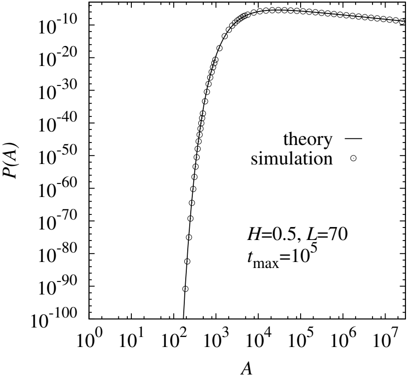
Results

Exact result known for standard RW case ($H = 1/2$)

$$P(A) = \frac{3^{-2/3} L}{\Gamma\left(\frac{1}{3}\right)\left(DA^4\right)^{1/3}}\, e^{-\frac{L^3}{9DA}}$$

[M.J. Kearney & S.N. Majumdar (2005)]

[S.N. Majumdar & B. Meerson (2020)]



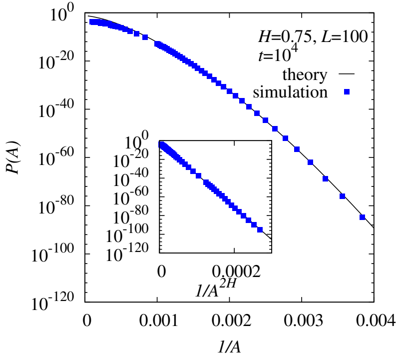
[AKH & B. Meerson, arXiv: 2310.14003]

$\rightarrow$ perfect agreement

general H: scaling form $$P(A) = \frac{D^{\frac{1}{2H}}}{L^{1+\frac{1}{H}}} \Phi_H \left(\frac{D^{\frac{1}{2H}} A}{L^{1+\frac{1}{H}}} \right)$$

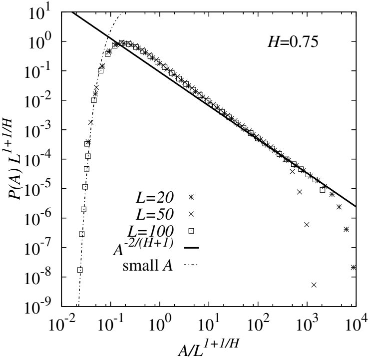
$\Phi_H(z \to 0) \sim \exp\left[-\sigma_H z^{-2H}\right]$

[B. Meerson & G. Oshanin, PRE (2022)]



$\Phi_H(z \to \infty) \sim z^{-\frac{2}{H+1}}$

[AKH & B. Meerson, arXiv: 2310.14003]

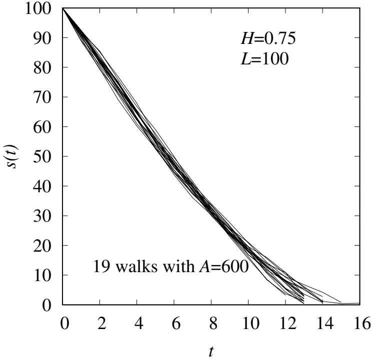


Corresponding results and good agreement also for $H = 1/4$

Paths

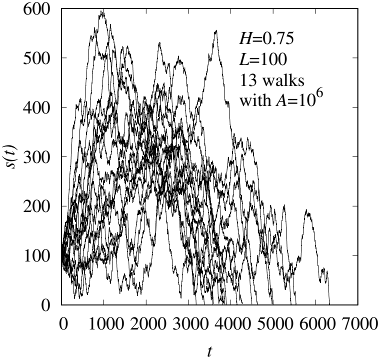
How do the rare paths look like?

left tail of $P(A)$



only “optimal” paths

right tail of $P(A)$



many different paths

Summary

- Fractional Brownian motion
- First-passage area
- Markov chain approach:
  evolve vector of random numbers
- Good agreement theory/simulation for $P(A)$
- Left tail: optimal paths
  right tail: many paths