Ground states of the random-field Potts model

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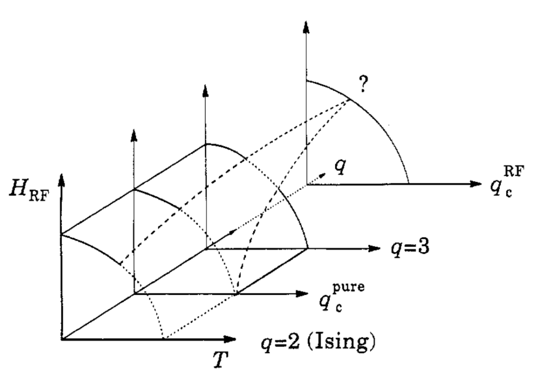
Possible generalisations:

- continuous spins (XY, Heisenberg etc.): some field-theoretic and numerical results
- random anisotropies
- more than two states: random-field Potts model (RFPM)

Random-field Potts model

Very little work to date:



Blankschtein, Shapir, Aharony, 1984

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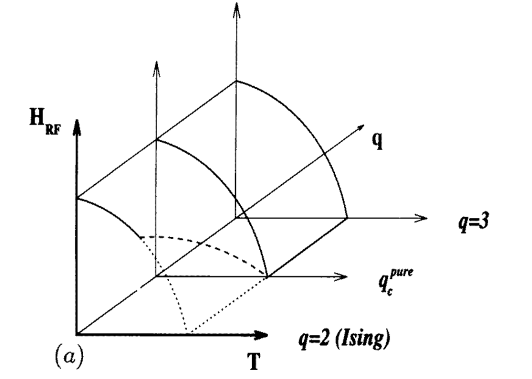

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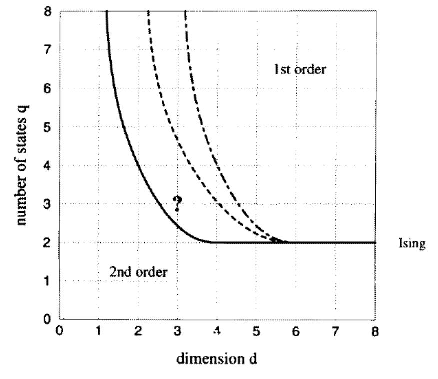

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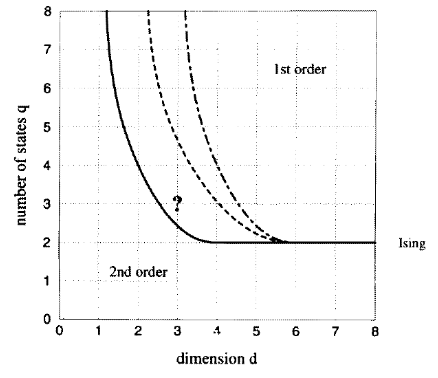

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Most recent study by Eichhorn and Binder (1995/96): possible 2nd order transition for 3D $q = 3$ model.

Maximum flows and graph cuts

Split up Ising model Hamiltonian,

$$-\mathcal{H} = \sum_{\langle ij \rangle} J_{ij} s_i s_j = W^+ + W^- - W^\pm = K - 2W^\pm, \tag{1}$$

where $K = \sum_{\langle ij \rangle} J_{ij}$, and

$$W^+ = \sum_{\substack{\langle ij \rangle \\ s_i = s_j = +1}} J_{ij}, \quad W^- = \sum_{\substack{\langle ij \rangle \\ s_i = s_j = -1}} J_{ij}, \quad W^\pm = \sum_{\substack{\langle ij \rangle \\ s_i \neq s_j}} J_{ij} \tag{2}$$

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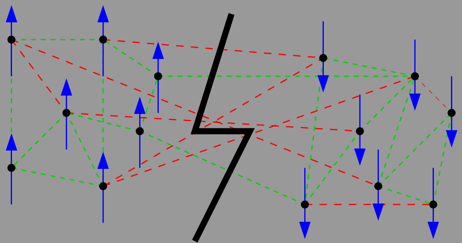
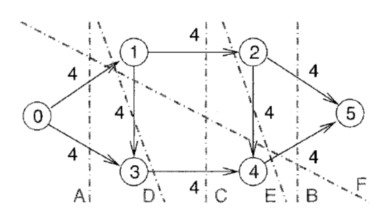


Then, a ground state is given by a configuration with minimal cut $W^\pm$, which divides the spins between the "up" and "down" states.

Maximum flows and graph cuts (2)



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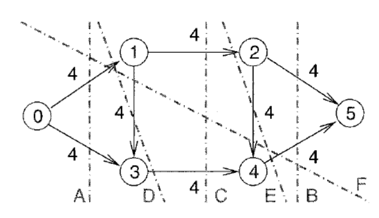

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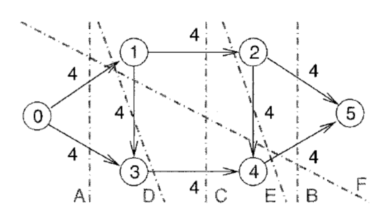

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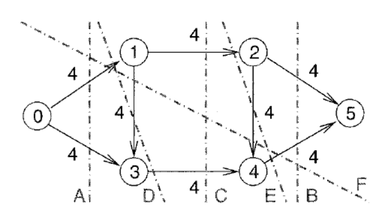

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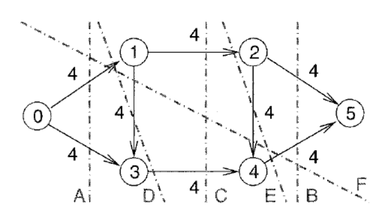

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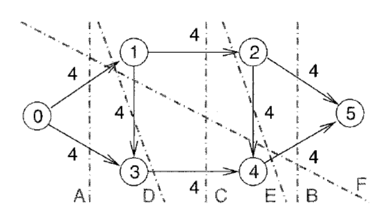

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- a cut separates the two classes of sites, the energy of the configuration corresponds to the weight of the cut
- due to the max-flow–min-cut theorem, the ground-state (min-cut) configuration occurs for maximum flow through the network
- there are efficient (polynomial time) algorithms to solve maximum flow exactly (Ford-Fulkerson, Edmonds-Karp, push relabel, ...)

Graph cuts and the Potts model

We consider the Hamiltonian

$$\mathcal{H} = -J\sum_{\langle ij\rangle}\delta_{s_i,s_j} - \sum_i\sum_{\alpha=0}^{q-1} h_i^\alpha \delta_{s_i,\alpha},$$

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We need to revert to approximation methods.

Approximate graph cuts

Boykov, Veksler and Zabih (2001) propose a method for problems in computer vision:

$$E(\{s_i\}) = \sum_{i,j} V_{ij}(s_i, s_j) + \sum_i D_i(s_i).$$

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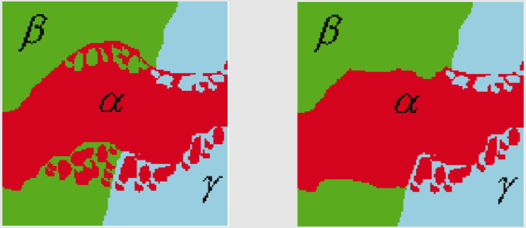
- α-β-**swap move**
  picks two labels $\alpha \neq \beta \in \{0, 1, \ldots, q-1\}$ and freeze all labels apart from α and β



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Works well in computer vision (paper has 8000 citations). How about the RFPM?

Benchmark: parallel tempering

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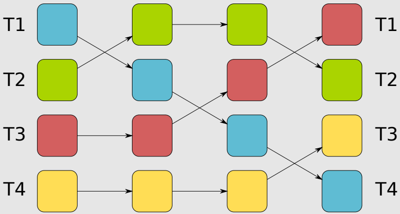
Parallel tempering:

- in principle converges to equilibrium
- optimize temperature protocol for optimum tunneling, based on

$$T_m = m^\eta T_{\mathrm{norm}} + T_{\mathrm{min}},$$

where

$$T_{\mathrm{norm}} = \frac{T_{\mathrm{max}} - T_{\mathrm{min}}}{(N_T - 1)^\eta}.$$



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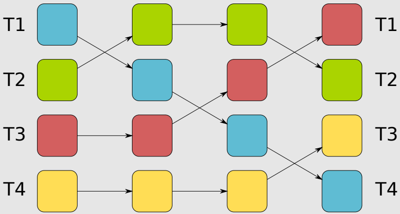
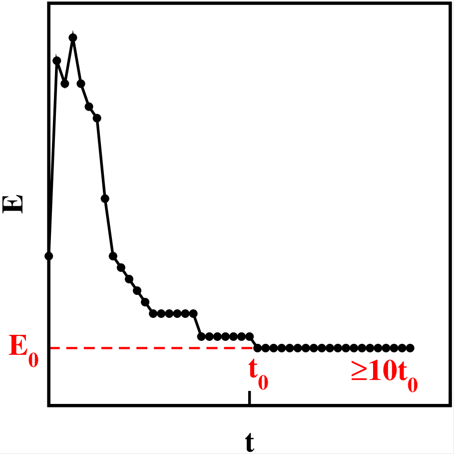


Ensure ground states are found almost always using some form of self-consistent bootstrapping procedure.

Benchmark: parallel tempering (2)

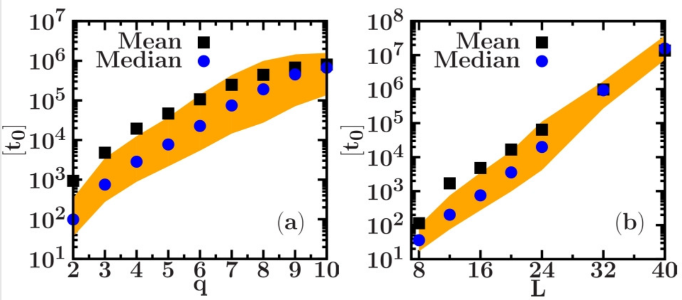
Ground-state procedure in parallel tempering:



Ensure that simulation time T is at least 10 times the onset time.

Benchmark: parallel tempering (3)

Onset times for systems of size $L \times L$ and numbers of Potts states q.



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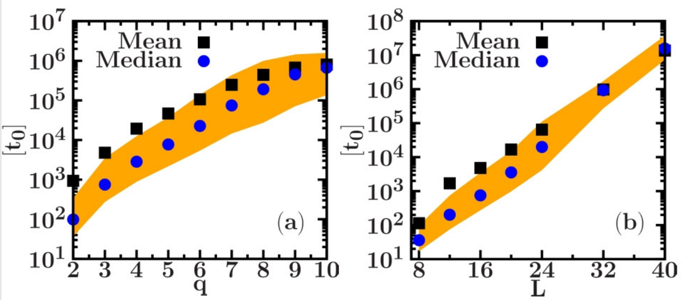
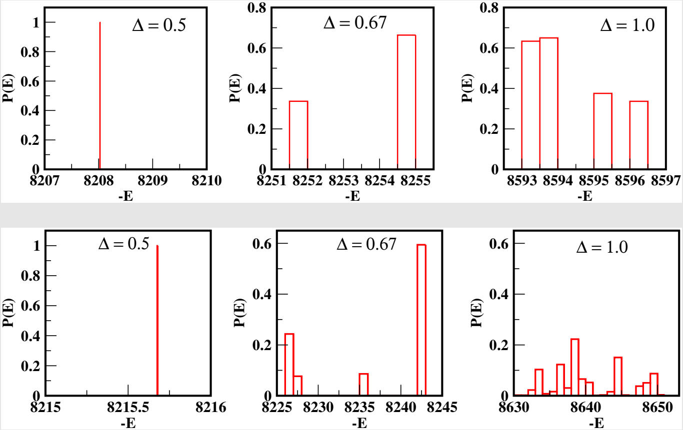


Exponential increase of hardness with L, maybe slightly slower with q.

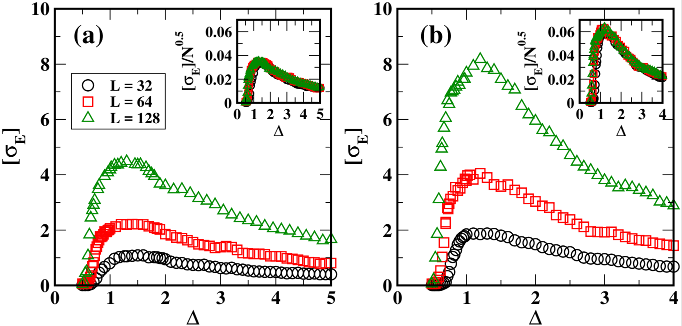
Graph cuts: histograms

Distribution of energies found:



Graph cuts: histograms (2)

Width of distributions:



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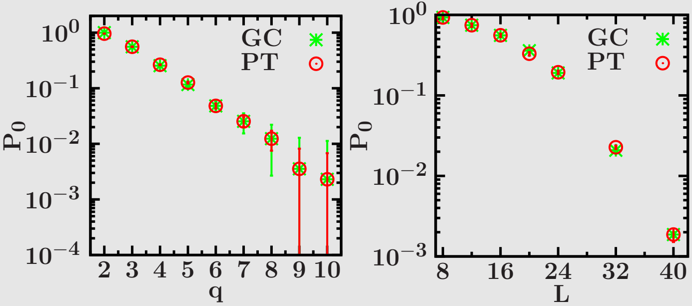

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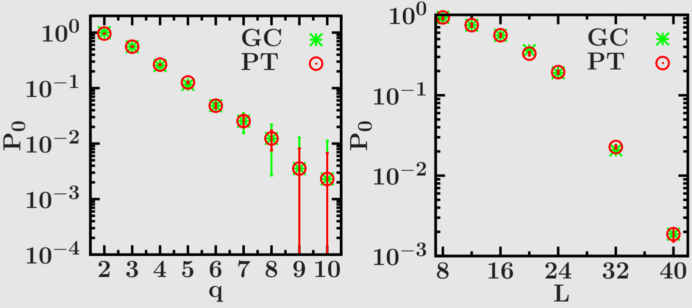

Repeated runs can be used for both methods to increase success probability,

$$P_s(\{h_i^\alpha\}) = 1 - [1 - P_0(\{h_i^\alpha\})]^m.$$

Comparison (2)

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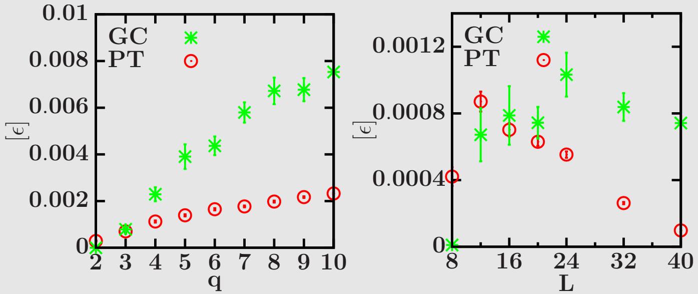

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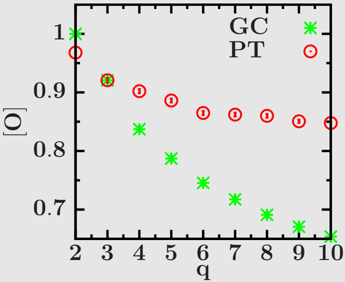
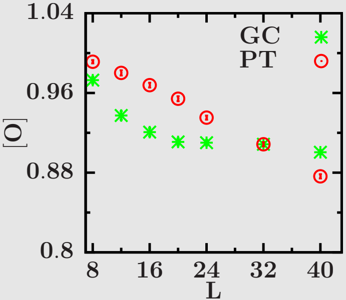
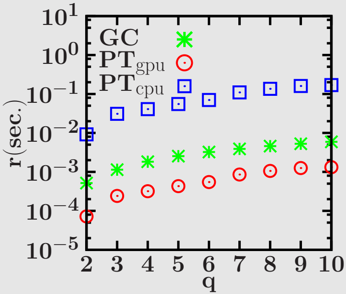
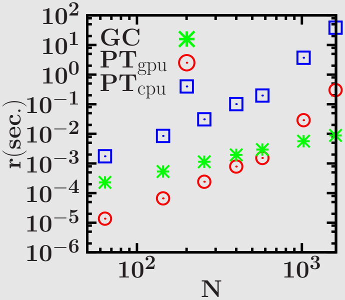




Comparison (4)

Run times





Results: 2D model

RFIM: no long-range order at finite temperatures, but breakup length (Binder, 1982):

$$L_b \sim \exp(A/\Delta) \quad \text{or} \quad \exp(A/\Delta^2)$$

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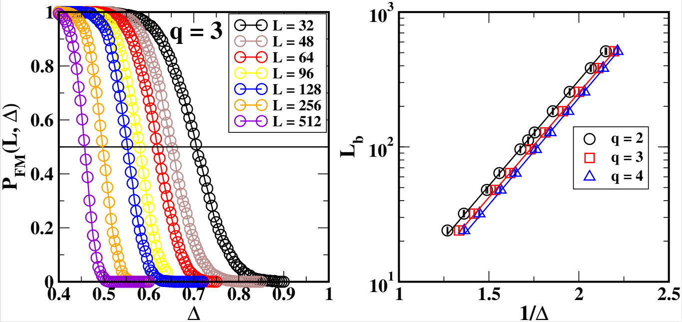

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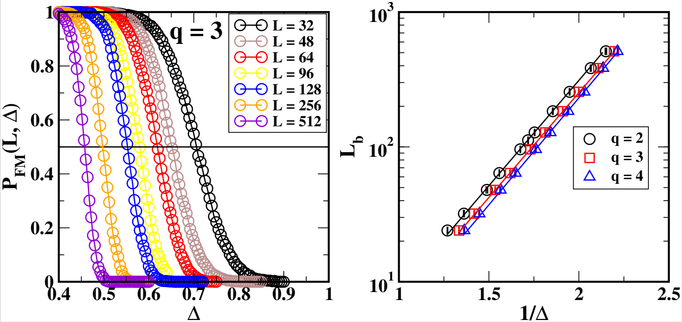

In 2D, $q = 2, 3, 4$ models appear to behave quite similarly.

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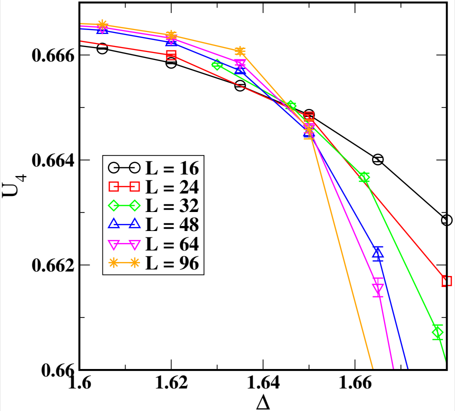

Binder cumulant

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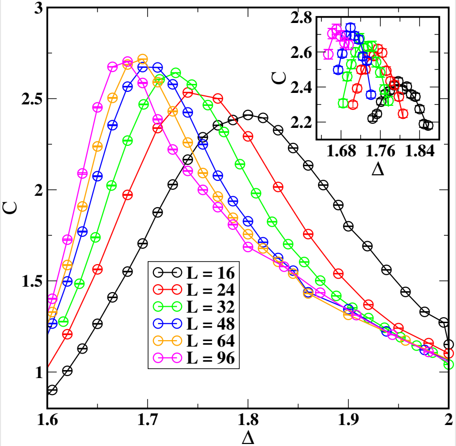

specific heat

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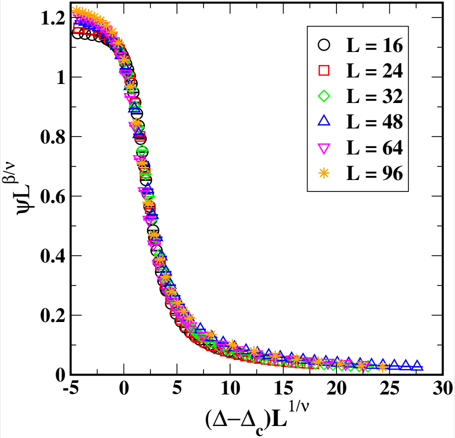

magnetization

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Phys. Rev. E 97, 053307 (2018)

M. Kumar et. al., in preparation