Specific heat and partition function zeros for the dimer model on the checkerboard B lattice: Finite-size effects

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Izmailian, Hu and Kenna, Phys. Rev. E 91, 062139 (2015)
Izmailian, Wu and Hu, Phys. Rev. E94, 052141 (2016)
Izmailian, Wu, Chen and Hu, submitted to Phys. Rev. E

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Outline

- Dimer model
- Dimer model on finite 2M x 2N checkerboard lattice.
- Specific heat and partition function zeros:

  Finite size analysis
- Summary

Dimer model

The dimer problem originated from investigation of the thermodynamic properties of a system of diatomic molecules (called dimers) absorbed on the surface of a crystal.

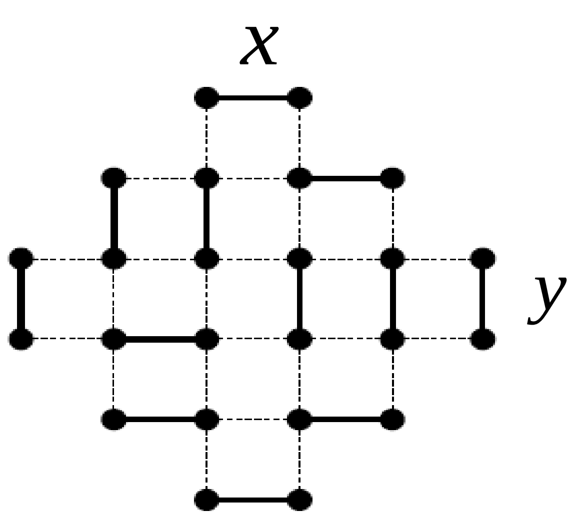
The first papers dealing with the dimer statistics were published by Fowler and Rushbrooke (1937), Chang (1939) and Flory (1942).

A "dimer" is a two-atom molecule. Dimer system is specified by a lattice G consisting of vertices (sites) connected by bonds. Dimer can be placed on the bonds of G so that no vertex has more than one dimer. The "dimer problem" is to determine the number of ways of covering of a given lattice with dimers, so that all sites are occupied and no two dimers overlap. The number of ways of covering a given lattice with dimers is given by partition function $Z_G(x, y)$

$$Z_G(x, y) = \sum_{\text{all d.c.}} x^{n_1} y^{n_2}$$

$$x = e^{-\beta\varepsilon_x}$$

$$y = e^{-\beta\varepsilon_y}$$



Equivalence to other statistical models

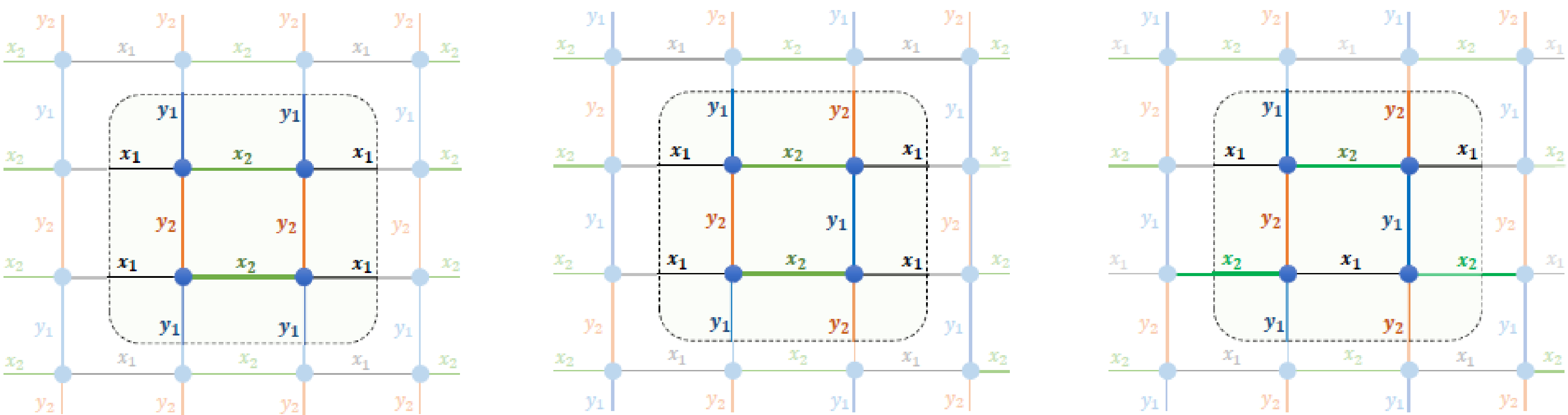
➢ Ising model	Fisher (1966)	Dimer model on 4 x 8 lattice (bathroom-tile lattice)
➢ 6 - Vertex model	Wu (1968)	Dimer model
➢ Biomembranes	Nagle (1973)	on honeycomb lattice
➢ Polymer	Nagle (1974)	
➢ Spanning tree	Temperley (1974)	Dimer model
➢ Sandpile model	Majumdar and Dhar (1992)	on square lattice

Checkerboard lattice

The checkerboard lattice is a simple rectangular lattice with alternated anisotropic dimer weights in horizontal (x_1, x_2), and vertical (y_1, y_2) directions.

There are three possible classifications of the dimer weights on the bonds of the lattice and they denoted as checkerboard A, B, and C lattices.

Dimer model on the checkerboard lattices A, B, and C has different critical behaviour.



(a) Checkerboard A (b) Checkerboard B (c) Checkerboard C

Exact solution of the dimer model on checkerboard lattice

(*N.I., C.-K. Hu, R. Kenna, Phys. Rev. 91 (2015) 2139*)

Partition function for dimer model on checkerboard lattices under periodic b.c.

$$Z = \frac{1}{2}\left(-Z_{0,0}{}^{2} + Z_{0,\frac{1}{2}}{}^{2} + Z_{\frac{1}{2},0}{}^{2} + Z_{\frac{1}{2},\frac{1}{2}}{}^{2}\right)$$

A: $$Z_{\alpha,\beta}{}^{2} = \prod_{n=0}^{N-1}\prod_{m=0}^{M-1}\left(\left|x_1 e^{i\,\varphi_{\alpha,n}} - x_2 e^{-i\,\varphi_{\alpha,n}}\right|^2 + \left|y_1 e^{i\,\theta_{\beta,m}} - y_2 e^{-i\,\theta_{\beta,m}}\right|^2\right)$$

B: $$Z_{\alpha,\beta}{}^{2} = \prod_{n=0}^{N-1}\prod_{m=0}^{M-1}\left|\left|x_1 e^{i\,\varphi_{\alpha,n}} - x_2 e^{-i\,\varphi_{\alpha,n}}\right|^2 - \left(y_1 e^{i\,\theta_{\beta,m}} - y_2 e^{-i\,\theta_{\beta,m}}\right)^2\right|$$

C: $$Z_{\alpha,\beta}{}^{2} = \prod_{n=0}^{N-1}\prod_{m=0}^{M-1}\left|\left(x_1 e^{i\,\varphi_{\alpha,n}} - x_2 e^{-i\,\varphi_{\alpha,n}}\right)^2 + \left(y_1 e^{i\,\theta_{\beta,m}} - y_2 e^{-i\,\theta_{\beta,m}}\right)^2\right|$$

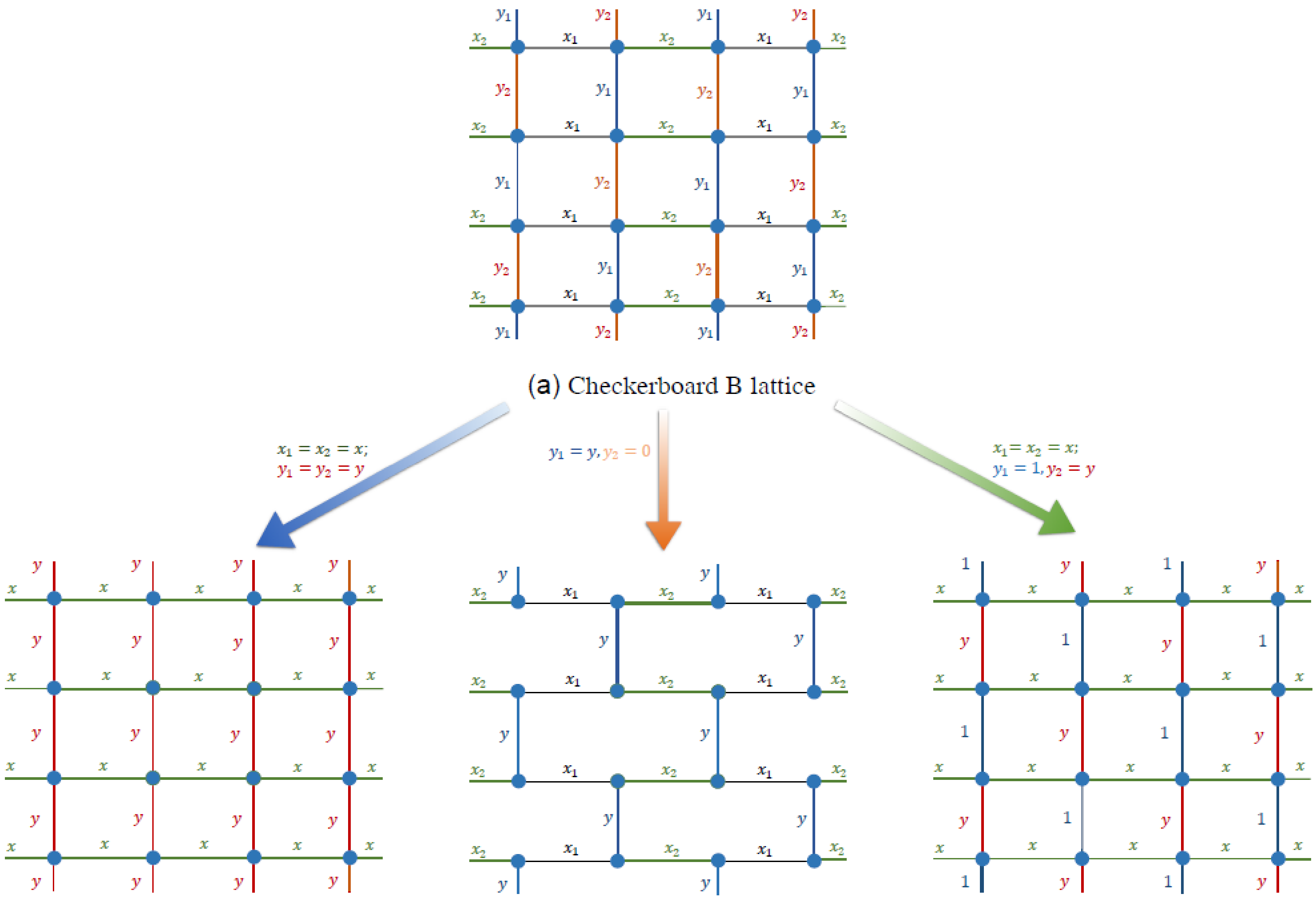
$$\varphi_{\alpha,n} = \frac{\pi(n+\alpha)}{N}, \theta_{\beta,m} = \frac{\pi(m+\beta)}{M}$$

Square lattice:
$x_1 = x_2 = x$
$y_1 = y_2 = y$

$$Z_{\alpha,\beta}{}^{2} = \prod_{n=0}^{N-1}\prod_{m=0}^{M-1}\left(x^2\left|e^{i\,\varphi_{\alpha,n}} - e^{-i\,\varphi_{\alpha,n}}\right|^2 + y^2\left|e^{i\,\theta_{\beta,m}} - e^{-i\,\theta_{\beta,m}}\right|^2\right)$$

P.W. Kasteleyn, Physica 27 (1961) 27; M.E. Fisher Phys. Rev. 124 (1961), 1664

Checkerboard B lattice



(a) Checkerboard B lattice

(b) Rectangular lattice

(c) Honeycomb lattice

(d) Generalized K-model

Partition function for dimer model on checkerboard B lattice under periodic b.c.

$$Z_{2M,2N} = \frac{1}{2}\left(-Z_{0,0}{}^{2} + Z_{0,\frac{1}{2}}{}^{2} + Z_{\frac{1}{2},0}{}^{2} + Z_{\frac{1}{2},\frac{1}{2}}{}^{2}\right)$$

$$Z_{\alpha,\beta}{}^{2} = \prod_{n=0}^{N-1}\prod_{m=0}^{M-1}\left|\left|x_1 e^{i\,\varphi_{\alpha,n}} - x_2 e^{-i\,\varphi_{\alpha,n}}\right|^2 - \left(y_1 e^{i\,\theta_{\beta,m}} - y_2 e^{-i\,\theta_{\beta,m}}\right)^2\right|$$

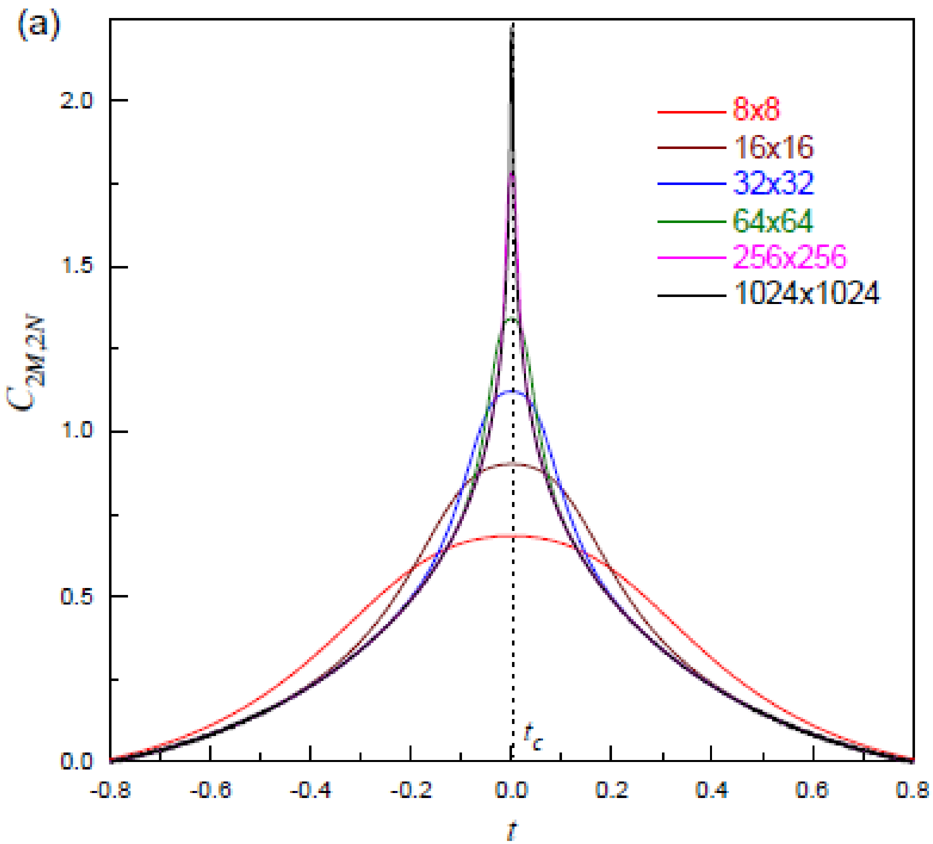
$$\varphi_{\alpha,n} = \frac{\pi(n+\alpha)}{N}, \theta_{\beta,m} = \frac{\pi(m+\beta)}{M}$$

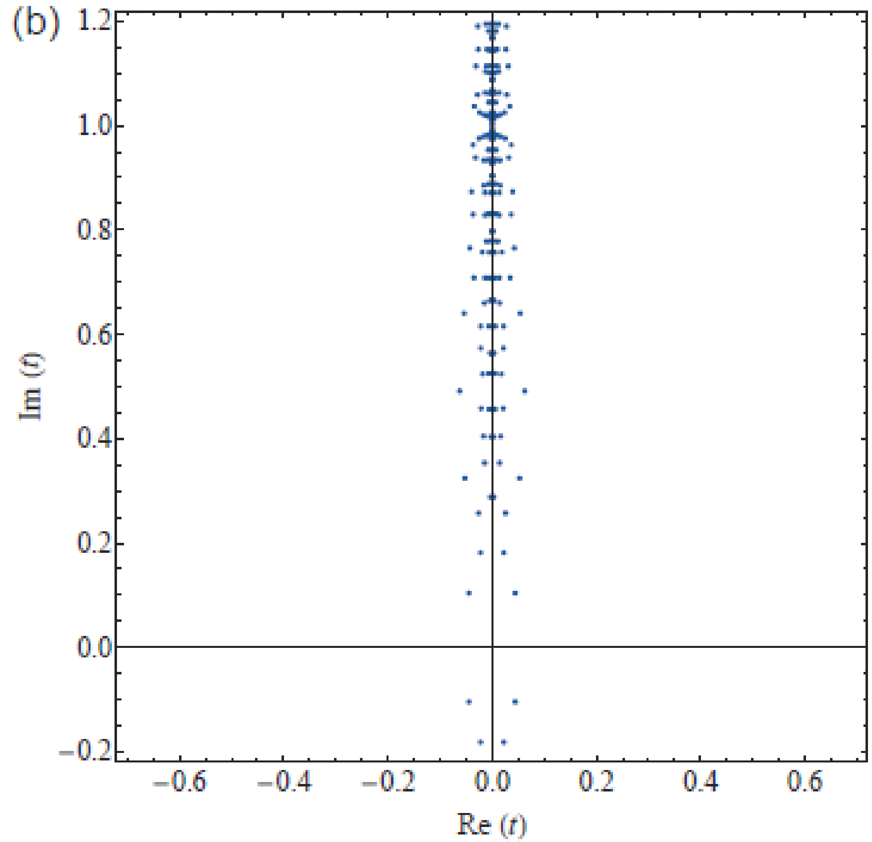
Square lattice limit: $x_1 = x_2 = x, \quad y_1 = y_2 = y.$

Generalized K model limit: $x_1 = x_2 = x, \quad y_1 = 1, \quad y_2 = y.$

Honeycomb lattice limit: $x_1, \quad x_2, \quad y_1 = y, \quad y_2 = 0.$

Dimer model on square lattice



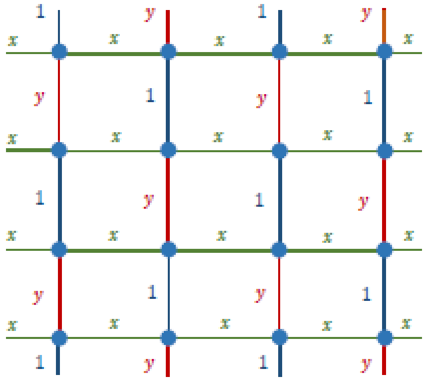


(a) The specific heat on the 2N x 2N square lattice as a function of t.
Here $t^{\,2} = (x_1 - x_2)^2 / 4x_1 x_2$ and $y_1 = y_2$.

(b) The partition function zeros for the lattice of 2N = 32.

Generalized K model
$(x_1 = x_2 = x; \;\; y_1 = 1; \;\; y_2 = y)$

Nagle, J. Chem. Phys. 58, 252 (1973); Bhattacharjee, Nagle, Phys. Rev. A, 3199 (1985)



Partition function for generalized K model

$$Z_{2M,2N} = \frac{1}{2}\left(-Z_{0,0}{}^2 + Z_{0,\frac{1}{2}}{}^2 + Z_{\frac{1}{2},0}{}^2 + Z_{\frac{1}{2},\frac{1}{2}}{}^2\right)$$

$$Z_{\alpha,\beta}{}^2 = \prod_{n=0}^{N-1}\prod_{m=0}^{M-1}\left|4x^2 \sin^2 \varphi_{\alpha,n} - \left(e^{i\,\theta_{\beta,m}} - y\, e^{-i\,\theta_{\beta,m}}\right)^2\right|$$

$$\boxed{\begin{aligned}\varphi_{\alpha,n} &= \frac{\pi(n+\alpha)}{N},\\ \theta_{\beta,m} &= \frac{\pi(m+\beta)}{M}\end{aligned}}$$

Phase diagram

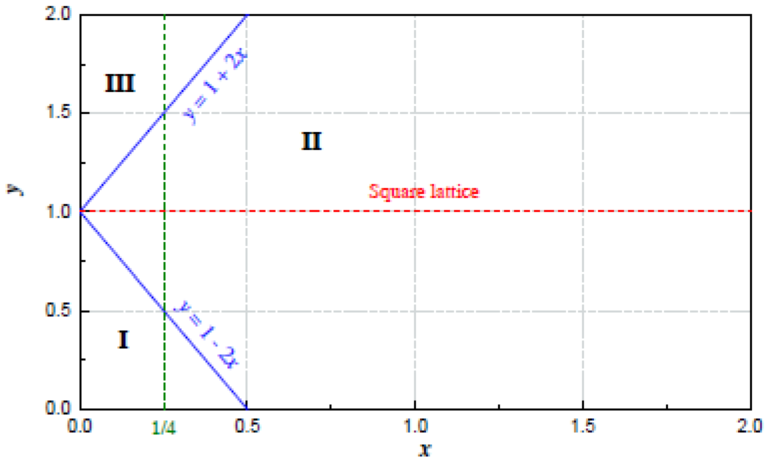


FIG. 6: (Color online) The phase diagram of the generalized K-model.

Region **I** separated from the region **II** by critical line **y = 1 – 2 x.**

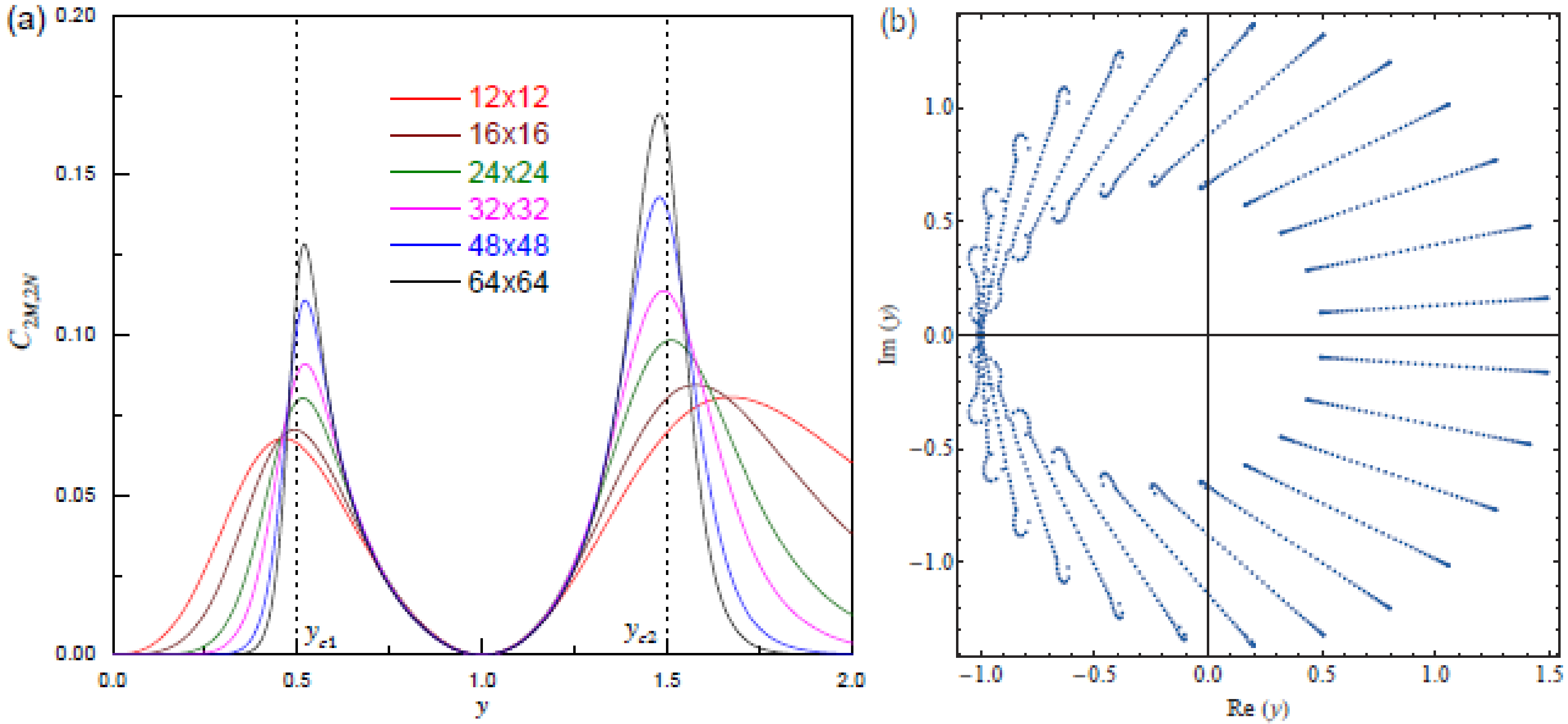
Region **III** separated from the region **II** by critical line **y = 1 + 2 x**

In region **I**, the system frozen in the ground state, where the dimers are on the edges of activity **1**.

Region **III** is also a frozen ground state, where the dimers are on the edges of activity **y**.

Region **II** is the disorder phase.

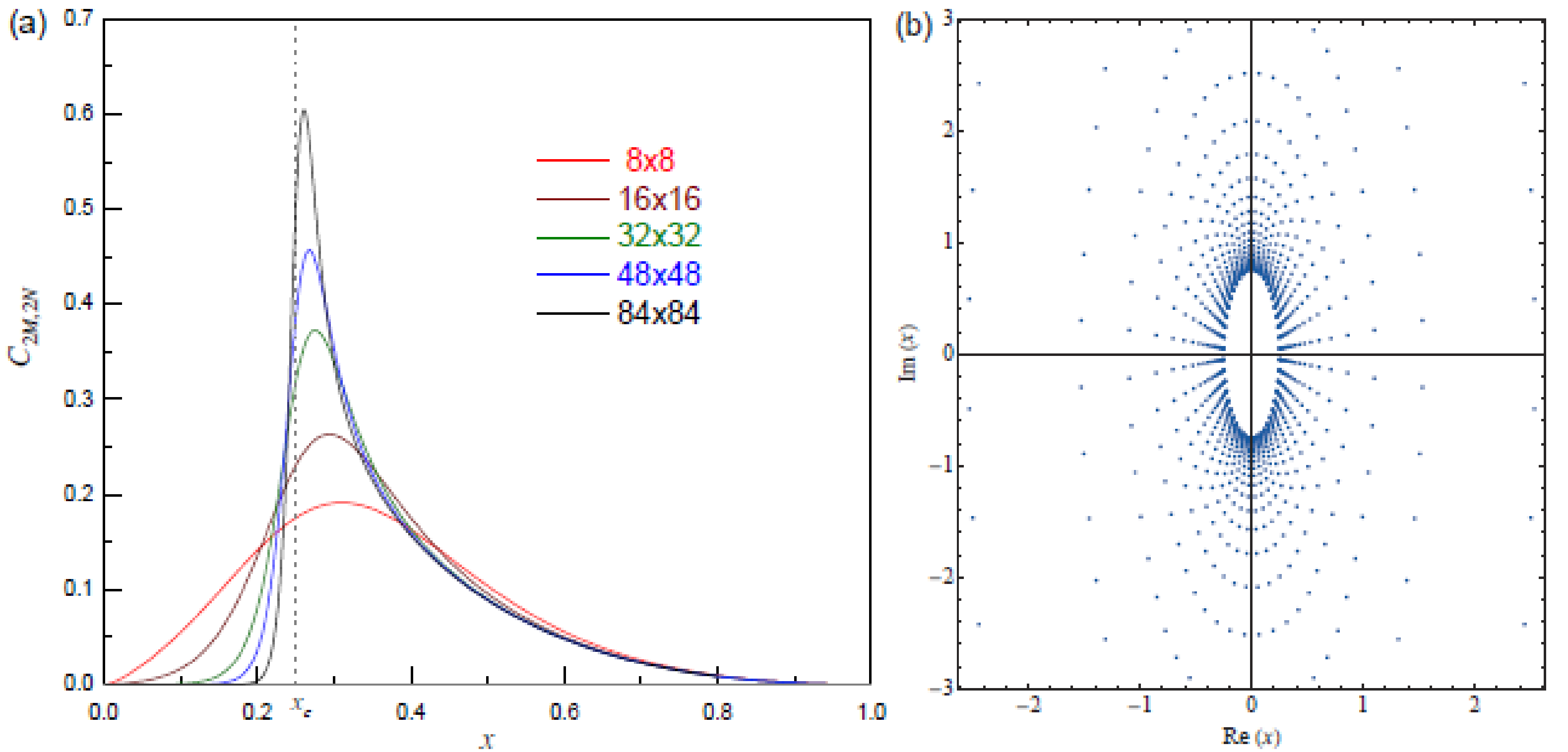
Specific heat and partition function zeros for x =1/4.
The critical points at $y_c = 1/2$ and $y_c = 3/2$



(a) The specific heat as a function of y for the generalized K-model in with x = 1/4. Here $y_{c1} = 1/2$ and $y_{c2} = 3/2$.

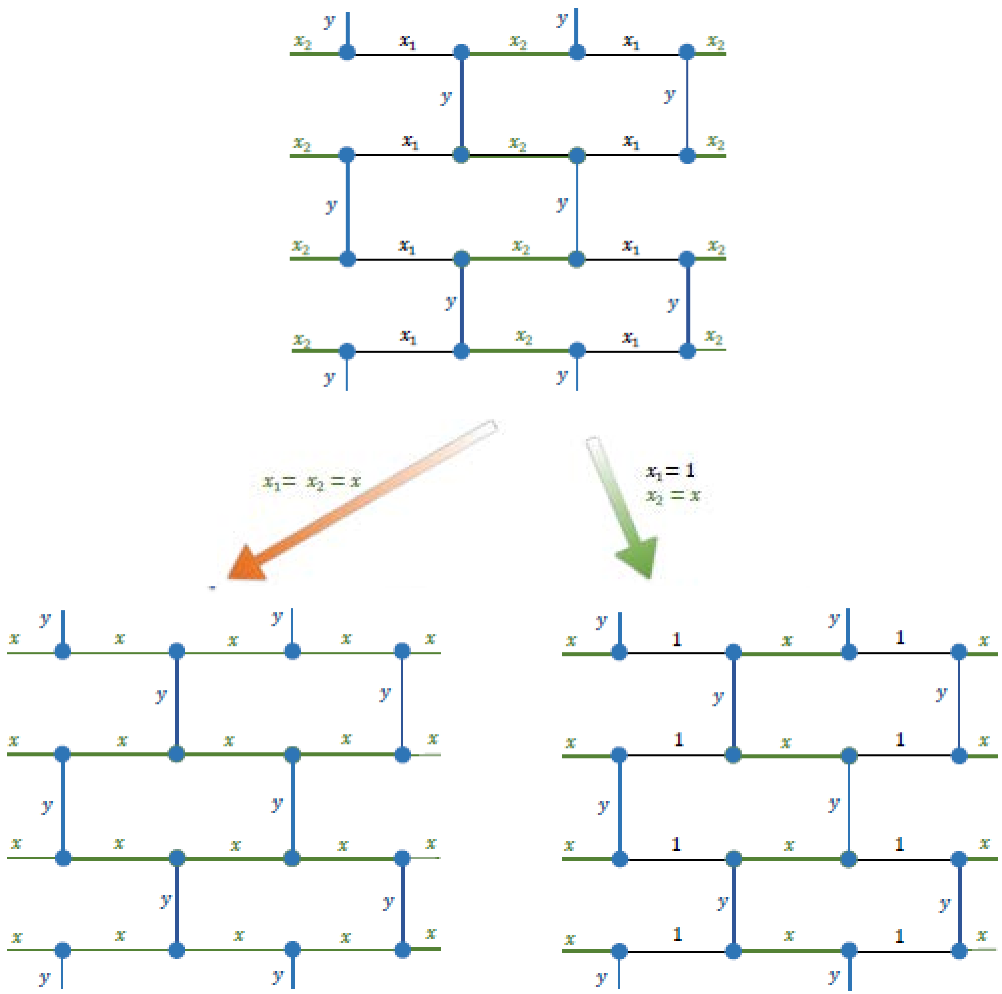
(b) (b) The partition function zeros for the lattice of 2M = 2N = 48.

Specific heat and partition function zeros for y =1/2.
The critical points at $x_c = 1/4$



(a) The specific heat as a function of x for the generalized K-model with y = 1/2. Here $x_c = 1/4$.
(b) The partition function zeros for the lattice of 2M = 2N = 48.

Kasteleyn K_2 – model $(y_1 = 0;\ y_2 = y)$

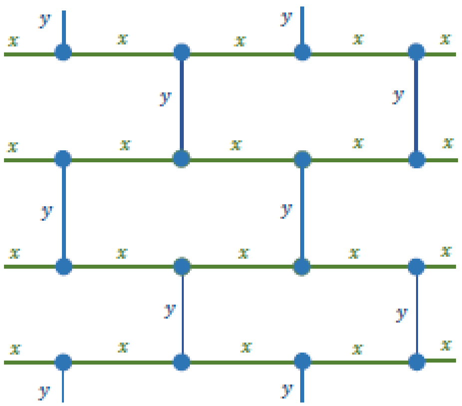


Honeycomb lattice I

Honeycomb lattice II

Honeycomb I model

$(x_1 = x_2 = x, \ y_1 = y, \ y_2 = 0)$



The critical line **y=2x** separated the frozen region **I** from disordered region **II.**

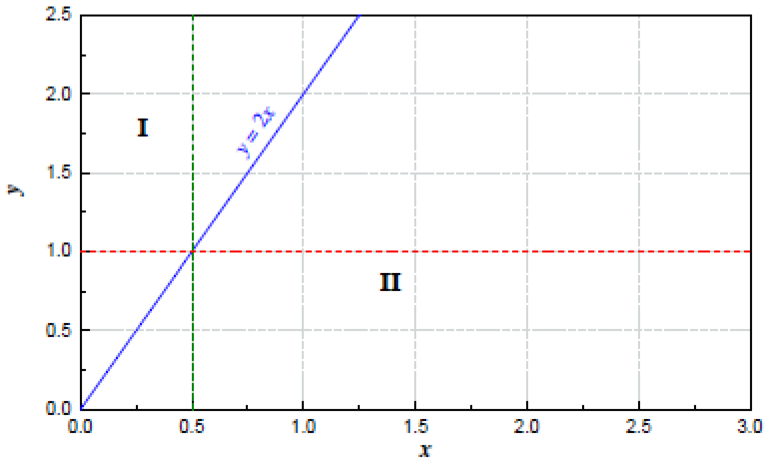
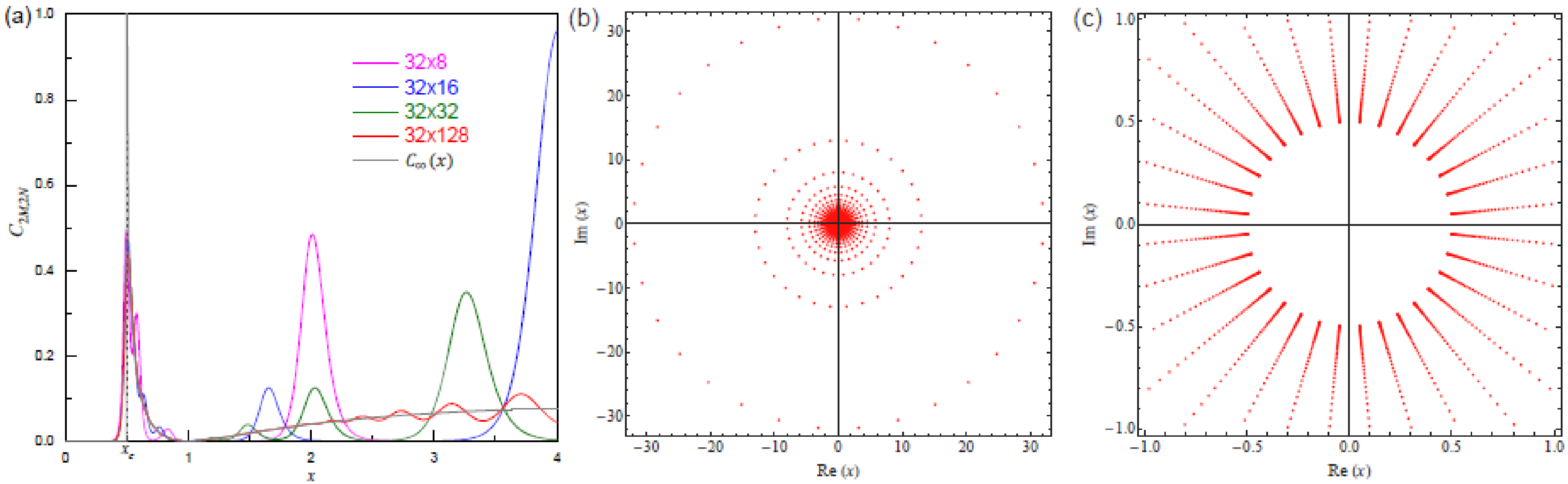


FIG. 9: (Color online) The phase diagram of the honeycomb lattice K_2-model with $x_1 = x_2 = x, y_1 = y, y_2 = 0$

Honeycomb I model

y = 1 and fixed M = 16



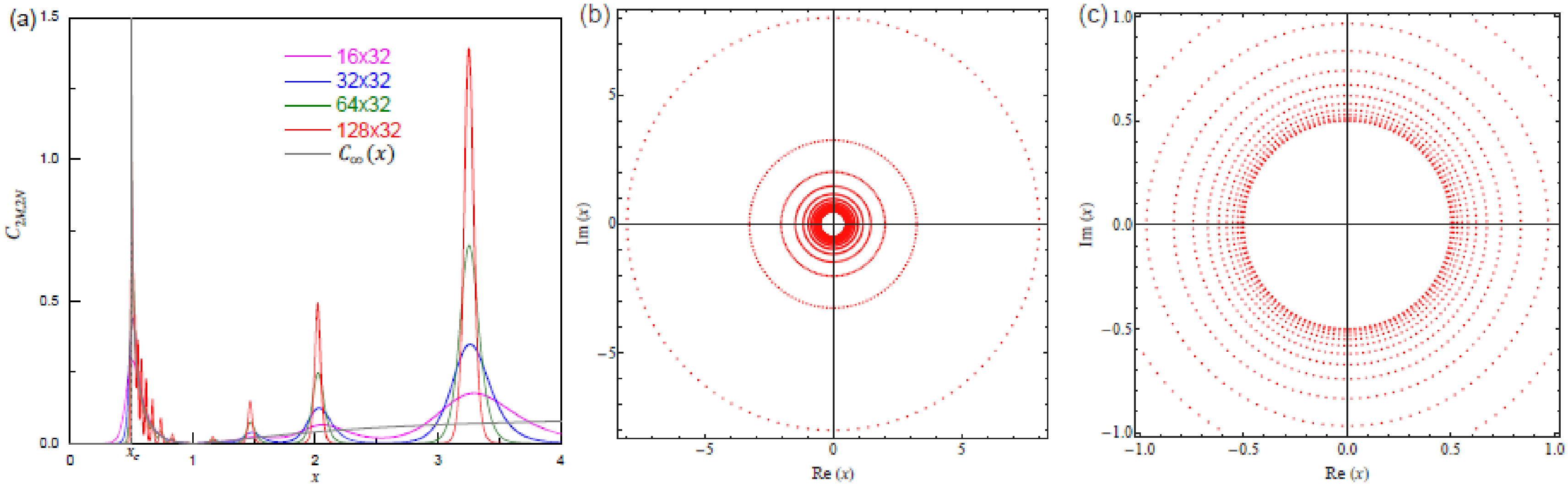
(a) The specific heat of the honeycomb lattice I and y = 1, as a function of x. Here $x_c = 1/2$

(b) The partition function zeros for the lattice of $2M \times 2N = 32 \times 128$.

(c) The zoom-in of (b).

Honeycomb I model

y = 1 and fixed 2N = 32



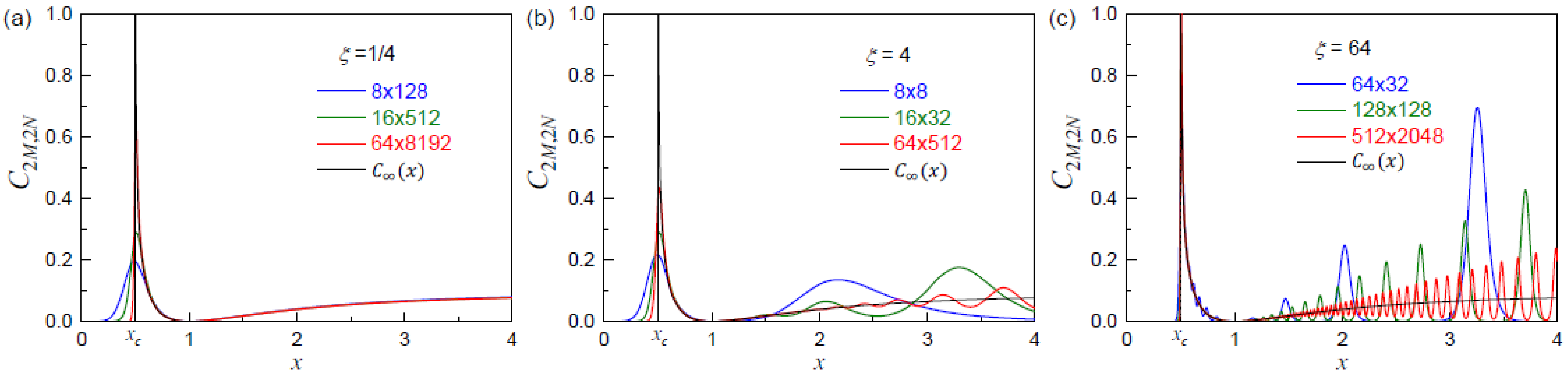
(a) The specific heat of the honeycomb lattice I and y = 1, as a function of x. Here $x_c = 1/2$

(b) The partition function zeros for the lattice of 2M × 2N = 128 × 32.

(c) The zoom-in of (b)

Honeycomb I model

y = 1 and different shape factor $\xi = M^2/N$



The specific heat for the honeycomb lattice I and y = 1, as a function of x for the different shape factor ξ

(a) $\xi = 1/4$,

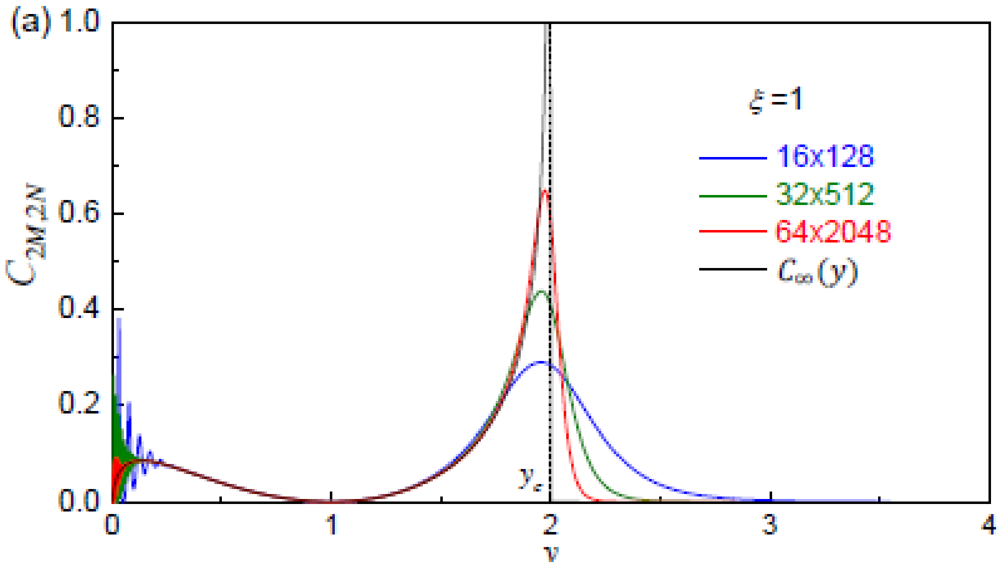
(b) $\xi = 4$,

(c) $\xi = 64$.

Here $x_c = 1/2$

Honeycomb I model

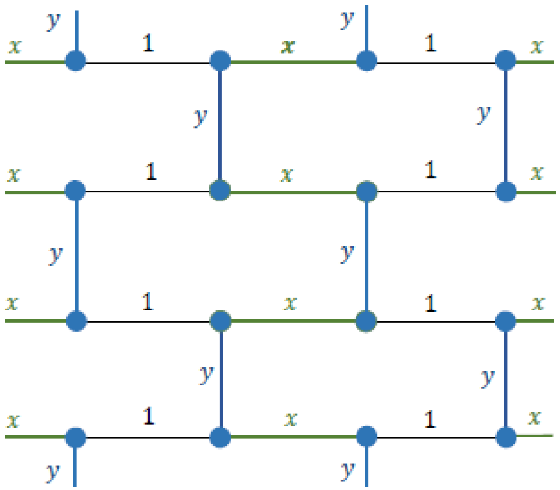
x = 1 and shape factor $\xi = M^2/N = 1$



The specific heat for the honeycomb I model for x = 1 and $\xi = 1$, as a function of y. Here $y_c = 2$

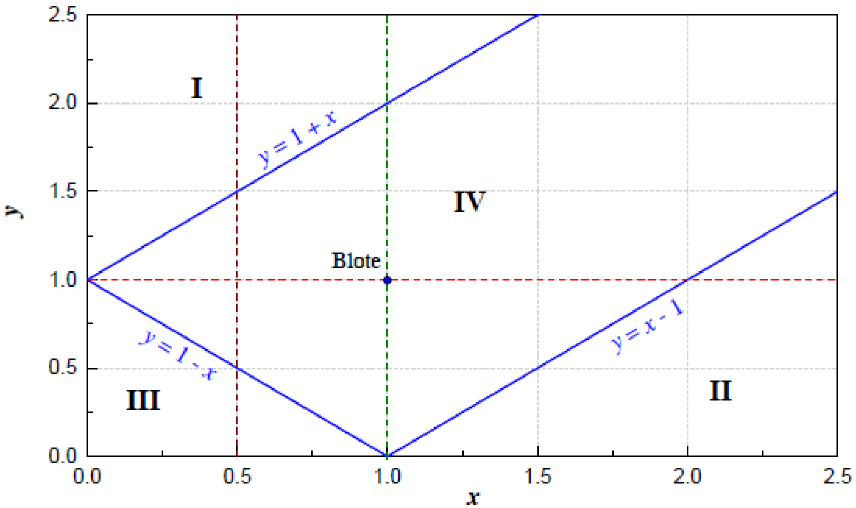
Honeycomb II model

$$(x_1 = 1, x_2 = x,\ y_1 = y,\ y_2 = 0)$$



(d) Honeycomb lattice II

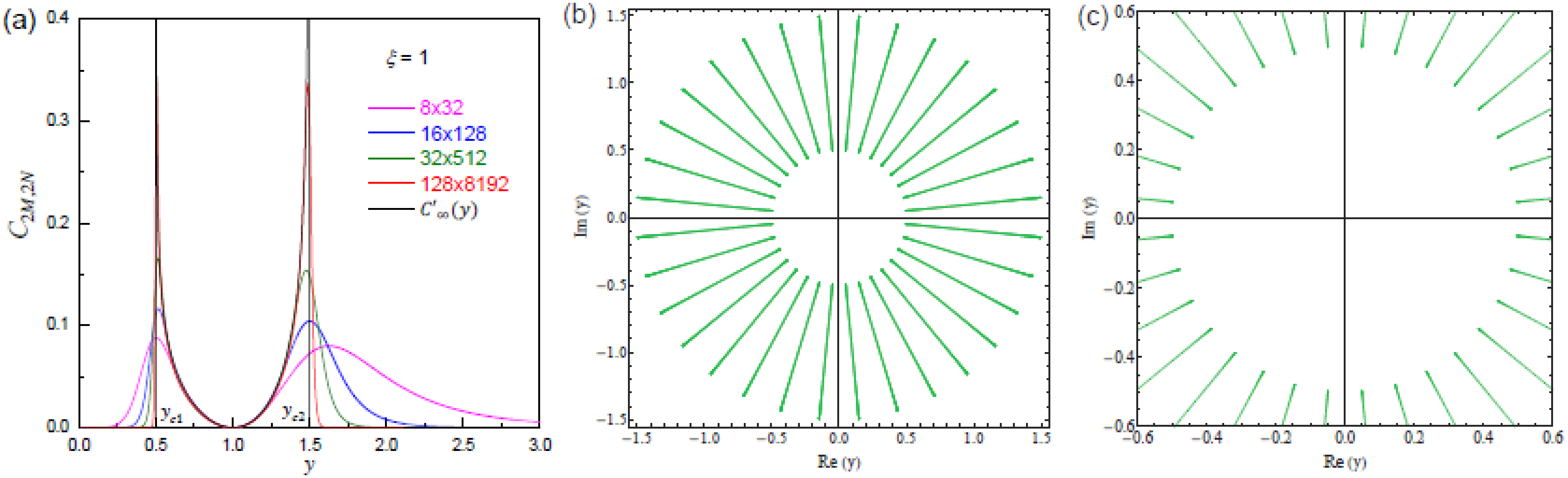
The phase diagram of the dimer model on the honeycomb II lattice



In regions I, II and III the system frozen in the ground state, where the dimers are on the edges of activity 1, x, and y respectively. Region IV is the disorder phase.

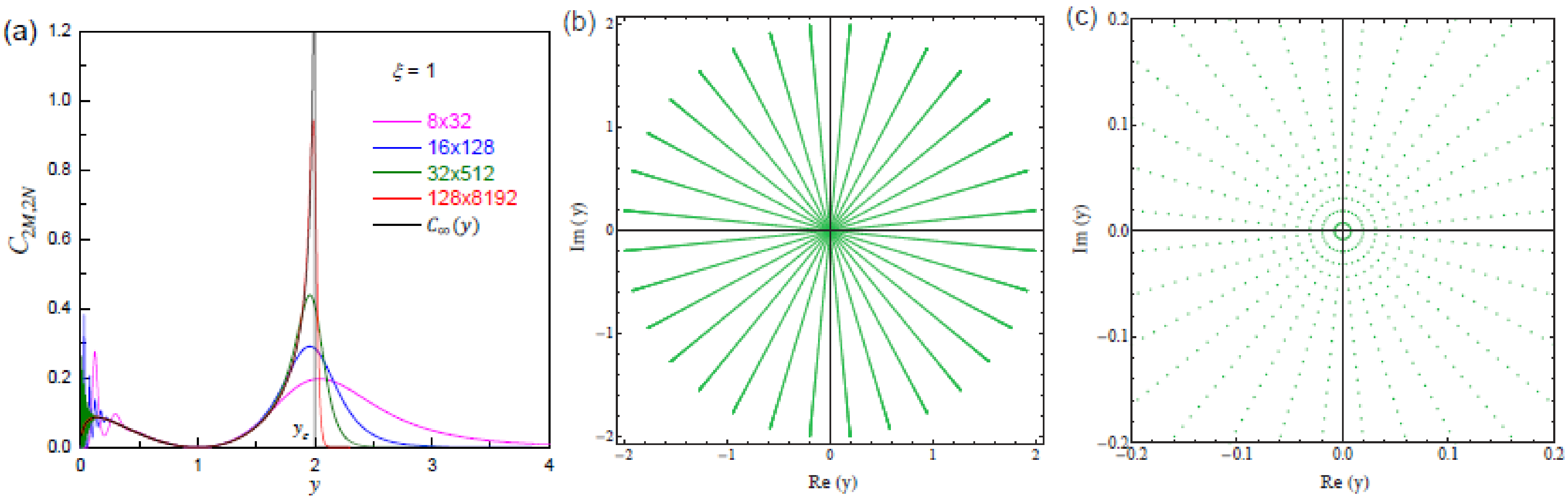
The region I separated from the region IV by critical line y=1+x,
the region II separated from the region IV by critical line y=x-1,
the region III separated from the region IV by critical line y=1-x.

Honeycomb II model at x = 1/2:
The specific heat and partition function zeros



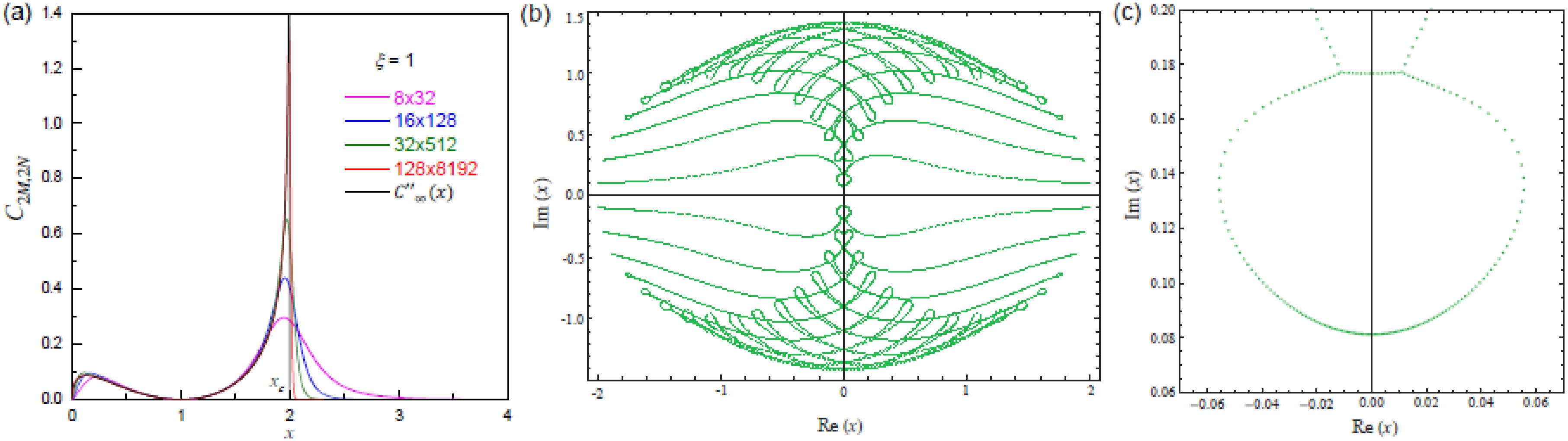
(a) The specific heat of the honeycomb II model at x = 1/2 as a function of y. Here y_{c1} = 1/2 and y_{c2} = 3/2
(b) The partition function zeros for the lattice of 2M x 2N = 32 x 512.
(c) The zoom-in of (b)

Honeycomb II model at x = 1:
The specific heat and partition function zeros



(a) The specific heat of the honeycomb II model at x = 1as a function of y.
Here $y_{c1} = 0$ and $y_{c2} = 2$
(b) The partition function zeros for the lattice of 2M x 2N = 32 x 512
(c) The zoom-in of (b).

Honeycomb II model at y = 1 and ξ = 1: The specific heat and partition function zeros



(a) The specific heat for honeycomb II model at y = 1 with ξ = 1 as a function of x. Here $x_c = 2$

(b) The partition function zeros for the lattice of 2M × 2N = 32 × 512.

(c) The zoom-in of (b)

Thank You!