Axiomatic construction of quantum Langevin equations

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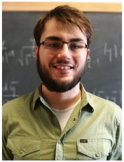

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Overview:

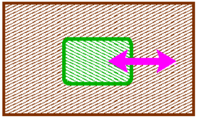
1. Quantum dynamics of open systems
2. Axiomatic construction of a quantum Langevin equation I
3. Axiomatic construction of a quantum Langevin equation II
4. Summary & Discussion

[1] S. Wald, MH, J. Stat. Mech. P07006 (2015) [arxiv:1503.06713]
[2] S. Wald, MH, J. Phys. **A49**, 125001 (2016) IOPSELECT [arxiv:1511.03347]
[3] S. Wald, G.T. Landi, MH, J. Stat. Mech. 013103 (2018) [arxiv:1707.06273]
[4] S. Wald, MH, Int. Transforms Spec. Funct. **29**, 95 (2018) [arxiv:1707.06275]
[5] R. Araújo, S. Wald, MH , [arxiv:1809.08975] . . .

1. Quantum dynamics of open systems

* nature is fundamentally quantum-mechanical
  $\Rightarrow$ classical behaviour is a limit behaviour, when formally $\hbar \to 0$
* any observable system is **open**, i.e. coupled to an 'environment'
  $\Rightarrow$ closed systems are idealisations for very weak coupling

$\Rightarrow$ it is crucial to understand behaviour of open quantum systems



CALDEIRA & LEGGETT 1981, 83

$\Rightarrow$ study '**system**' & '**coupling**' & '**environment**'

? are there effective descriptions of the 'system' ?

? relative importance of thermal and quantum fluctuations ?

* required for studies of "*quantum ageing*" $L(t) \sim t^{1/z}$

quantum fluctuation-dissipation theorem broken through quantum-quenched dynamics

deep quench to $T = 0$: dynamical exponent $z = 2$ (classical) $\to$ $z = 1$ (quantum)

For **classical open systems**: many equivalent descriptions available
common aspect: not to treat the whole environment, but deduce some parameters which describe its effetcs, e.g. temperature T

1. **master equation** $P(\{\sigma\};t) = \text{proba to have configuration } \sigma \text{ at time } t$

$$\partial_t P(\{\sigma\};t) = \sum_{\{\tau\}} [w(\tau \to \sigma)P(\{\tau\};t) - w(\sigma \to \tau)P(\{\sigma\};t)]$$

built-in markov property (no explicit memory-dependence)

averages $\langle X\rangle(t) = \sum_{\{\sigma\}} X(\{\sigma\})P(\{\sigma\};t)$

2. **Langevin equation** here: overdamped limit

$$\partial_t x(t) = \mathscr{F}(x(t)) + \eta(t)$$

random force $\eta(t)$ from interaction of system with many particles of bath

hence $\eta(t)$ **white noise** gaussian random variable (central limit-theorem)

want to find **average** $\langle x(t)\rangle$

? Are these descriptions immediately applicable to quantum systems ?

Example: quantum-mechanical harmonic oscillator CARMICHAEL 99

with 'position' variable x also need conjugate momentum p

write pair of Langevin equations, including friction ($\lambda > 0$) and noise η

$$\partial_t x = \frac{1}{m} p \ \ , \ \ \partial_t p = -m\omega^2 x - \lambda p + \eta \qquad (*)$$

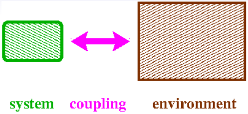
where x, p are operators. Let $\langle [x(t), \eta(t)] \rangle = 0$.

What about the **commutator** $c(t) = [x(t), p(t)]$? Derive eq. of motion

$$\partial_t \langle c(t) \rangle = -\lambda \langle c(t) \rangle \ \ , \ \ c(0) = \mathrm{i}\hbar$$

$\Rightarrow$ canonical commutator decays rapidly $\boxed{\langle c(t) \rangle = \mathrm{i}\hbar\, e^{-\lambda t}}$

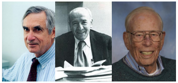
similarly: many-body systems with the above (classical) Langevin equation (*) relax to **classical** (and **not** to the quantum) ground state e.g. WALD & MH 16 concentrate on **Quantum Langevin Equation**



here: case of ohmic dissipation, $\lambda > 0$

$$\partial_t^2 x + \lambda \partial_t x + V'(x) = \zeta$$

Ford, Kac, Mazur 1965
Ford & Kac 1987
Ford, Lewis, O'Connell 1988



use environment of **free harmonic oscillators** (explicitly solvable)
average over initial states of these oscillators
obtain (stationary) **noise correlator**

$k_{\rm B} = 1$

$$\frac{1}{2}\left\langle\left\{\zeta(t),\zeta(0)\right\}\right\rangle = \frac{\lambda}{\pi}\int_0^\infty \mathrm{d}\nu\ \hbar\nu \coth\left(\frac{\hbar\nu}{2T}\right)\cos(\nu t)$$

* for environment of free oscillators, this is **indeed gaussian**

* since $\langle\{\zeta(t),\zeta(0)\}\rangle \not\sim \delta(t)$, this is **not markovian** !

* white noise $\langle\zeta(t)\zeta(0)\rangle = 2\lambda T\delta(t)$ recovered as **classical limit** $\hbar \to 0$

2. Axiomatic construction of quantum Langevin equations I

? are there 'true quantum analogues' of classial Langevin equations ?

? how to be sure that a given description is 'really quantum' and not semi-classical ?

? physical criteria for the identification of 'correct' quantum Langevin equations ?

case study 1: analyse the Bedeaux-Mazur equations,

for a single harmonic oscillator: spin s, momentum p

BEDEAUX & MAZUR 01,02

$$\partial_t s = \frac{1}{m} p + \eta_s \ , \ \ \partial_t p = -m\omega^2 s - \lambda p + \eta_p$$

restrict to ohmic friction $(\lambda > 0)$ and admit two noises η_s, η_p, with

$$\begin{aligned}
\left\langle \eta_p(t)\eta_p(t') \right\rangle &= \lambda m \hbar\omega \coth\left(\frac{\hbar\omega}{2T}\right) \delta(t-t') \\
\left\langle \eta_s(t)\eta_p(t') \right\rangle &= -\left\langle \eta_p(t)\eta_s(t') \right\rangle = \frac{1}{2}\, \mathrm{i}\hbar\lambda\, \delta(t-t') \\
\left\langle \eta_s(t)\eta_s(t') \right\rangle &= 0
\end{aligned}$$

N.B. B & M obtained this from a tedious analysis of the Green functions; is **markovian**

Langevin approach: attempt to absorb all relevant information, on the environment and on the coupling to it, into noise correlators

Minimal criteria for a physically sensible quantum dynamics: AWH 18

(A) canonical equal-time commutators $\langle [s_n(t), p_m(t)] \rangle = i\hbar\delta_{n,m}$
(B) Kubo formula from linear-response theory
(C) Virial theorem from equilibrium statistical mechanics
(D) Quantum fluctuation-dissipation theorem (QFDT)

conditions (A,B) fix noise commutators, conditions (C,D) fix noise anti-commutators

analysis begins with formal solutions of eqs of motion $\Lambda_\pm = \frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} - \omega^2}$

$$
\begin{aligned}
s(t) &= s_+(0)e^{\Lambda_+ t} + s_-(0)e^{\Lambda_- t} \\
&\quad - \frac{1}{\Lambda_+ - \Lambda_-} \int_0^t \mathrm{d}\tau\, e^{-\Lambda_+(t-\tau)} \left[\frac{\eta_p(\tau)}{m} + \Lambda_- \eta_s(\tau) \right] + \frac{1}{\Lambda_+ - \Lambda_-} \int_0^t \mathrm{d}\tau\, e^{-\Lambda_-(t-\tau)} \left[\frac{\eta_p(\tau)}{m} + \Lambda_+ \eta_s(\tau) \right] \\
p(t) &= -m\Lambda_+ s_+(0)e^{\Lambda_+ t} - m\Lambda_- s_-(0)e^{\Lambda_- t} \\
&\quad + \frac{m\Lambda_+}{\Lambda_+ - \Lambda_-} \int_0^t \mathrm{d}\tau\, e^{-\Lambda_+(t-\tau)} \left[\frac{\eta_p(\tau)}{m} + \Lambda_- \eta_s(\tau) \right] - \frac{m\Lambda_-}{\Lambda_+ - \Lambda_-} \int_0^t \mathrm{d}\tau\, e^{-\Lambda_-(t-\tau)} \left[\frac{\eta_p(\tau)}{m} + \Lambda_+ \eta_s(\tau) \right]
\end{aligned}
$$

where $s_\pm(0)$ characterise the initial conditions

Outline of main results:

(A): admit $\langle [\eta_s(t), \eta_p(t')] \rangle = \kappa\, \delta(t-t')$

fix initial condition $\langle [s_+(0), s_-(0)] \rangle = \frac{\kappa/m}{(\Lambda_+ - \Lambda_-)(\Lambda_+ + \Lambda_-)}$

lex parsimoniae OCKHAM

only stationary part remains: $\langle [s(t), p(t)] \rangle = \frac{\kappa}{\lambda} \stackrel{!}{=} \mathrm{i}\hbar$ $\quad\Rightarrow\quad$ $\boxed{\kappa = \mathrm{i}\lambda\hbar}$

(B): perturbation $H \mapsto H - hs$, gives perturbed eqs of motion

$$\partial_t s = \frac{1}{m} p + \eta_s \quad , \quad \partial_t p = -m\omega^2 s - \lambda p + h + \eta_p$$

define **linear response function**

$$R^{(s)}(t,t') := \left. \frac{\delta \langle s(t) \rangle}{\delta h(t')} \right|_{h=0} = \frac{\Theta(t-t')}{m} \frac{1}{\Lambda_+ - \Lambda_-} \left(e^{-\Lambda_-(t-t')} - e^{-\Lambda_+(t-t')} \right)$$

define two-time correlators $\boxed{C_\pm^{(A)}(t,t') := \frac{1}{2} \langle A(t)A(t') \pm A(t')A(t) \rangle}$

can check explicitly the **Kubo formula**

$$R^{(s,p)}(t-t') = \frac{2\mathrm{i}}{\hbar} \Theta(t-t')\, C_-^{(s,p)}(t,t')$$

analogous: perturb $H \mapsto H + kp$, find $R^{(p)}(t-t') = \left. \frac{\delta \langle p(t) \rangle}{\delta k(t')} \right|_{k=0} = m^2 \omega^2 R^{(s)}(t-t')$

(C): admit $\langle\{\eta_p(t),\eta_p(t')\}\rangle = \alpha\,\delta(t-t')$, $\langle\{\eta_s(t),\eta_s(t')\}\rangle = \beta\,\delta(t-t')$

fix α, β from **Virial theorem**:

mean kinetic energy $\langle E_{\rm cin}\rangle = \langle E_{\rm pot}\rangle$ mean potential energy

$$\Rightarrow \langle p^2\rangle = C_+^{(p)}(t,t) \stackrel{!}{=} m^2\omega^2 C_+^{(s)}(t,t) = m^2\omega^2\langle s^2\rangle$$

for $\beta \neq 0$, leads to condition $\lambda^3 \stackrel{!}{=} 4\lambda\omega^2 \Rightarrow \beta = 0$

obtain α from explicit quantum statistical mechanics

$$C_{+,\rm st}^{(s)}(t,t) = \frac{\alpha}{2\lambda m^2\omega^2} \stackrel{!}{=} \frac{\hbar}{2m\omega}\coth\frac{\hbar\omega}{2T} = \left\langle s^2\right\rangle_{\rm eq} \quad\Rightarrow\quad \alpha = \hbar\lambda m\omega\coth\frac{\hbar\omega}{2T}$$

$\Rightarrow$ conditions **(A,B,C)** reproduce the Bedeaux-Mazur equations   AWH 18

(D): observe empirically the stationary fluctuation-dissipation relations

$$\partial_\tau C_{+,\rm st}^{(s)}(\tau) = -\hbar\omega\coth\left(\frac{\hbar\omega}{2T}\right)R^{(s)}(\tau)\ ,\quad \partial_\tau C_{+,\rm st}^{(p)}(\tau) = -\hbar\omega\coth\left(\frac{\hbar\omega}{2T}\right)R^{(p)}(\tau)$$

* if $\hbar \to 0$, reproduce classical FDT, but **no** QFDT if $\hbar > 0$ $\tau = t - t'$

$\Rightarrow$ Bedeaux-Mazur equations describe a **semi-classical dynamics**

3. Axiomatic construction of quantum Langevin equations II

case study 2: reconsider the quantum harmonic oscillator, spin s & momentum p

$$\partial_t s = \frac{1}{m} p + \eta_s \;\;,\;\; \partial_t p = -m\omega^2 s - \lambda p + \eta_p$$

ARAÚJO, WALD, MH 18

fix the noise correlators such that all four conditions **(A,B,C,D)** are satisfied

Environment much larger than the system, hence unaffected by system's behaviour ⇒ environment is at equilibrium

⇒ all noise-correlators are stationary

⇒ simplify analysis by Fourier-transforming into frequency-space

$$(\mathscr{F} s(t))(\nu) = \widehat{s}(\nu) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}t \, e^{-\mathrm{i}\nu t} s(t)$$

and we have the formal (stationary) solutions

$$\widehat{s}(\nu) = \frac{\widehat{\eta}_p(\nu)/m + (\mathrm{i}\nu + \lambda)\widehat{\eta}_s(\nu)}{\omega^2 + \mathrm{i}\lambda\nu - \nu^2} \;\;,\;\; \widehat{p}(\nu) = \frac{\mathrm{i}\nu\widehat{\eta}_p(\nu) - m\omega^2\widehat{\eta}_s(\nu)}{\omega^2 + \mathrm{i}\lambda\nu - \nu^2}$$

recall the basic idea:

Langevin approach: attempt to absorb all relevant information, on the environment and on the coupling to it, into noise correlators

Minimal criteria for a physically sensible quantum dynamics: AWH 18

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- **(B)** Kubo formula from linear-response theory
- **(C)** Virial theorem from equilibrium statistical mechanics
- **(D)** Quantum fluctuation-dissipation theorem (QFDT)

conditions (A,B) fix noise commutators, conditions (C,D) fix noise anti-commutators

Outline of main results:

(B): from perturbed H, find linear responses of momentum and of spin

(same as for classical case, because of linear eqs. of motion)

$$\widehat{R}^{(p)}(\nu) = m^2\omega^2\widehat{R}^{(s)}(\nu) = -(2\pi)^{-1/2}m\omega^2\left(\nu^2 - \mathrm{i}\lambda\nu - \omega^2\right)^{-1}$$

necessary condition for validity of Kubo formulæ

$$\widehat{C}_{-}^{(p)}(\nu) = m^2\omega^2\widehat{C}_{-}^{(s)}(\nu) \qquad (\#)$$

N.B. in frequency-space $\frac{1}{2}\langle[\widehat{s}(\nu),\widehat{s}(\nu')]\rangle = \delta(\nu+\nu')\widehat{C}_{-}^{(s)}(\nu)$ etc.

admit $\langle[\eta_s(t),\eta_p(t')]\rangle = \mathrm{i}\hbar\kappa(t-t')$ and $\langle[\eta_s(t),\eta_s(t')]\rangle = \mathrm{i}\hbar\psi(t-t')$, $\langle[\eta_p(t),\eta_p(t')]\rangle = \mathrm{i}\hbar\chi(t-t')$

$(\#)$ leads to the following condition

$$\begin{aligned}&\nu^2\widehat{\chi}(\nu) + m^2\omega^4\widehat{\psi}(\nu) + \mathrm{i}m\omega^2\nu\left(\widehat{\kappa}(\nu)+\widehat{\kappa}(-\nu)\right)\\ \stackrel{!}{=}\;&\omega^2\widehat{\chi}(\nu) + m^2\omega^2\left(\lambda^2+\nu^2\right)\widehat{\psi}(\nu) + m\omega^2\Big(\mathrm{i}\nu\left(\widehat{\kappa}(\nu)+\widehat{\kappa}(-\nu)\right) + \lambda\left(\widehat{\kappa}(\nu)-\widehat{\kappa}(-\nu)\right)\Big)\end{aligned}$$

$\Rightarrow$ most simple solution (the only one independent of the model parameters ω, m):

$$\boxed{\widehat{\chi}(\nu) = \widehat{\psi}(\nu) = 0 \;\; , \;\; \widehat{\kappa}(\nu) - \widehat{\kappa}(-\nu) = 0}$$

(A): admit $\langle[\eta_s(t), \eta_p(t')]\rangle = \mathrm{i}\hbar\kappa(t-t')$ other commutators vanish

expect $\langle[s(t), p(t')]\rangle = \mathrm{i}\hbar K(t-t')$ with $K(0) \stackrel{!}{=} 1$ and find

$$\widehat{K}(\nu) = \frac{\nu^2 - \mathrm{i}\lambda\nu + \omega^2}{(\nu^2 - \mathrm{i}\lambda\nu - \omega^2)(\nu^2 + \mathrm{i}\lambda\nu - \omega^2)}\,\widehat{\kappa}(\nu)$$

if $\widehat{\kappa}(\nu) = \kappa_0$, then $K(0) = \sqrt{2\pi}\,\kappa_0\lambda^{-1}$ $\Rightarrow$ $\boxed{\sqrt{2\pi}\,\kappa_0 = \lambda}$

this gives the non-vanishing noise commutators **N.B.:** $\widehat{\kappa}(\nu)$ must be symmetric

$$\boxed{\left\langle\left[\eta_s(t), \eta_p(t')\right]\right\rangle = \mathrm{i}\hbar\lambda\delta(t-t') \;\;,\;\; \left\langle\left[\widehat{\eta}_s(\nu), \widehat{\eta}_p(\nu')\right]\right\rangle = \delta(\nu+\nu')\mathrm{i}\hbar\lambda}$$

(B'): from perturbed H, linear responses of momentum and of spin are known

$$\widehat{R}^{(p)}(\nu) = m^2\omega^2\widehat{R}^{(s)}(\nu) = -(2\pi)^{-1/2} m\omega^2\left(\nu^2 - \mathrm{i}\lambda\nu - \omega^2\right)^{-1} \qquad \text{(R)}$$

and also check the **Kubo formulæ** (here in frequency-space)

$$\boxed{\widehat{R}^{(s,p)}(\nu) \doteq -\frac{2}{\hbar\mathrm{i}}\frac{1}{\sqrt{2\pi}}\;\widehat{C}_{-}^{(s,p)}(\nu)}$$

N.B.: general solution depends on two anti-symmetric functions $\psi(t), \chi(t)$,

which contain specific model-parameters m, ω

(C): admit $\langle\{\eta_p(t),\eta_p(t')\}\rangle = 2\alpha(t-t')$, $\langle\{\eta_s(t),\eta_s(t')\}\rangle = 2\beta(t-t')$
and $\langle\{\eta_s(t),\eta_p(t')\}\rangle = 2\gamma(t-t')$

with the **notation**: $\widehat{C}_+^{(s)}(\nu,\nu') = \frac{1}{2}\langle\{\widehat{s}(\nu),\widehat{s}(\nu')\}\rangle = \delta(\nu+\nu')\widehat{C}_+^{(s)}(\nu)$ etc.

the **virial theorem** gives the condition $\boxed{\widehat{C}_+^{(p)}(\nu) \overset{!}{=} m^2\omega^2\widehat{C}_+^{(s)}(\nu)}$. Hence

$$\nu^2\widehat{\alpha}(\nu) + m^2\omega^4\widehat{\beta}(\nu) \overset{!}{=} \omega^2\widehat{\alpha}(\nu) + m^2\omega^2(\lambda^2+\nu^2)\widehat{\beta}(\nu) + \lambda m\omega^2\left(\widehat{\gamma}(\nu)+\widehat{\gamma}(-\nu)\right)$$

The most simple solution is

$$\boxed{\widehat{\alpha}(\nu) = \widehat{\beta}(\nu) = 0 \;\;,\;\; \widehat{\gamma}(\nu)+\widehat{\gamma}(-\nu) = 0}$$

It is the only solution **independent** of the model's parameters m,ω.
$\Rightarrow$ $\widehat{\gamma}(\nu)$ is anti-symmetric ! In that case

$$\widehat{C}_+^{(s)}(\nu) = \frac{2\mathrm{i}}{m}\frac{\nu\widehat{\gamma}(\nu)}{(\nu^2-\mathrm{i}\lambda\nu-\omega^2)(\nu^2+\mathrm{i}\lambda\nu-\omega^2)} = \frac{1}{m^2\omega^2}\widehat{C}_+^{(p)}(\nu) \quad \text{(C)}$$

N.B.: general solution depends on two symmetric functions $\alpha(t),\beta(t)$,
which contain specific model-parameters m,ω

(D): find the anti-symmetric function $\widehat{\gamma}(\nu)$ from the QFDT.

In frequency-space, this requires (**!** independence of observable s, p **!**)

$$\frac{\widehat{C}_+^{(s)}(\nu)}{\widehat{C}_-^{(s)}(\nu)} = \frac{\widehat{C}_+^{(p)}(\nu)}{\widehat{C}_-^{(p)}(\nu)} = -\frac{1}{\sqrt{2\pi}} \coth \frac{\hbar\nu}{2T}$$

Compare the correlator (C) with the response (R) via the Kubo formulæ:

$$\boxed{2\mathrm{i}\sqrt{2\pi}\,\widehat{\gamma}(\nu) = \hbar\lambda \coth \frac{\hbar\nu}{2T}}$$

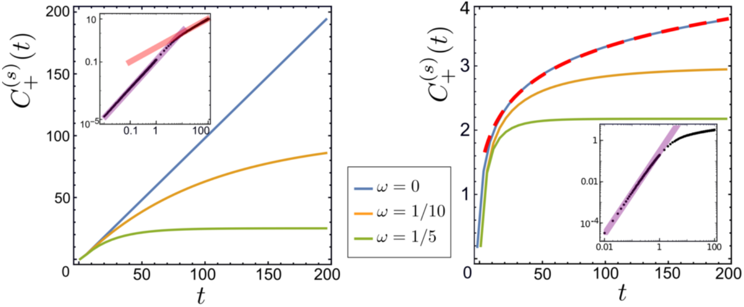
Consequence: the non-vanishing moments are in frequency-space

$$\boxed{\left\langle \left\{ \widehat{\eta}_s(\nu), \widehat{\eta}_p(\nu') \right\} \right\rangle = \frac{\hbar\lambda}{\mathrm{i}} \coth\left(\frac{\hbar\nu}{2T}\right) \delta(\nu+\nu') \; , \; \left\langle \left[\widehat{\eta}_s(\nu), \widehat{\eta}_p(\nu') \right] \right\rangle = \mathrm{i}\hbar\lambda\,\delta(\nu+\nu')}$$

and directly for the times

$$\boxed{\left\langle \left\{ \eta_s(t), \eta_p(t') \right\} \right\rangle = \lambda T \coth\left(\frac{\pi T}{\hbar}(t-t')\right) \; , \; \left\langle \left[\eta_s(t), \eta_p(t') \right] \right\rangle = \mathrm{i}\hbar\lambda\,\delta(t-t')}$$

Illustration: spin-spin correlator $C_+^{(s)}(t) = C_+^{(s)}(t,t)$ several values of ω



$m = 1$,
$\lambda = 1$, $\hbar = 1$

left: $T = 1$

right: $T = 0$

if $\omega > 0$: noisy oscillator, classical and quantum behaviour qualitatively similar

for $T > 0$ saturation of $C_+^{(s)}(t)$ at semi-classical value

if $\omega = 0$: freely diffusing brownian particle, $C_+^{(s)}(t) = \langle x^2(t) \rangle$ **variance**

$$C_+^{(s)}(t) = \langle x^2(t) \rangle \overset{t\to\infty}{\simeq} \begin{cases} t & T > 0 \quad \textbf{classical diffusion} \\ \ln t & T = 0 \quad \textbf{quantum diffusion} \end{cases}$$

Brown 1827
Einstein 1905

Hakim & Ambegaokar 1985

N.B.: quantum diffusion mixes system more slowly than classical diffusion

4. Summary & Discussion

* a **single quantum particle**, with hamiltonian H

* write (a pair of) **quantum Langevin equation**(s) ohmic dissipation

$$\partial_t s = \frac{\mathrm{i}}{\hbar}\left[H, s\right] + \eta_s \ \ , \ \ \partial_t p = \frac{\mathrm{i}}{\hbar}\left[H, p\right] - \lambda p + \eta_p$$

for the operators of **position** (spin) s and conjugate **momentum** p
and with **dissipation rate** $\lambda > 0$

* the **noise operators** η_s, η_p have the non-vanishing second moments

$$\left\langle \left\{ \eta_s(t), \eta_p(t') \right\} \right\rangle = \lambda T \coth\left(\frac{\pi T}{\hbar}(t-t') \right) \ \ , \ \ \left\langle \left[\eta_s(t), \eta_p(t') \right] \right\rangle = \mathrm{i}\hbar\lambda\,\delta(t-t')$$

$\Rightarrow$ **describes the relaxation towards a quantum equilibrium state**
and reproduces classical dynamics in the $\hbar \to 0$ limit

* physical content behind quantum Langevin equation was made explicit (canon. commutator, Kubo formulæ, virial theorem, quantum FDT)

* non-markovianity appears explicitly in $\langle \{ \eta_s(t), \eta_p(t') \} \rangle$

* no time-delay in dissipation required,
  ohmic resistance enough to yield non-markovianity

* **physical link with the quantum FDT** see PASQUALE, RUGGIERO, ZANNETTI 1984
  ⇒ **quantum FDT and Markov property mutually exclusive**

* **mathematical equivalence to the Ford-Kac-Mazur** (FKM) **quantum Langevin equation**

* noise correlators independent of model parameters ⇒ generic result

* noise correlators contain less derivatives than in FKM
  ⇒ useful for numerical studies ?

* formulate treatable models with clear, and physically motivated, ingredients
  → e.g. quantum spherical models → work in progress

* explicit treatment of such systems might help to appreciate better the applicability of more complex procedures (such as Lindblad equations)

A last formal observation

should recover **classical white noise** in the **classical limit** $\hbar \to 0$

* does require care, since singular contributions arise

(usual 'Fourier integral representations' of the noise correlators very strongly divergent !)

* must separate noise correlators into

(i) '**regular**' and (ii) '**singular**' (distributions !) parts

leads to

$$\left\langle \left\{ \eta_s(t), \eta_p(0) \right\} \right\rangle = \lambda T \left[\underbrace{\frac{\exp\left(-\frac{\pi T}{\hbar}|t|\right)}{\sinh\left(\frac{\pi T}{\hbar} t\right)}}_{\text{regular}} + \underbrace{\mathrm{sgn}\, t}_{\textbf{singular}} \right] \quad , \quad \left\langle \left[\eta_s(t), \eta_p(0) \right] \right\rangle = \mathrm{i}\hbar\lambda \underbrace{\delta(t)}$$

N.B. important if derivatives of these correlators are needed