An overview of recent numerical results on the random-field Ising model

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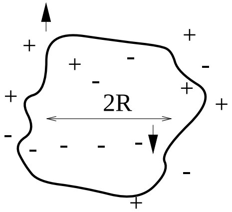
The random-field Ising model (RFIM)

$$\mathcal{H} = -J \sum_{\langle x,y \rangle} S_x S_y - \sum_x h_x S_x; \quad S_x = \pm 1; \quad J > 0$$

$\{h_x\}$ independent random magnetic fields with zero mean and dispersion σ.

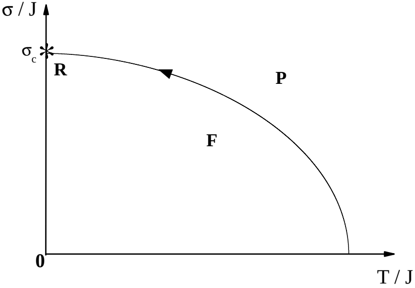
Experimental relevance: Diluted antiferromagnets in a homogeneous external field, fluids in porous media, and many others.

Imry-Ma argument ($\sigma \ll J$): $\mathcal{O}(JR^{D-1})$ vs. $\mathcal{O}(\sigma R^{D/2}) \to D_{\mathrm{l}} = 2$.



RG suggests $D_{\mathrm{u}} = 6 \to$ physically relevant dimensions $3 \le D < 6$.

RG fixed-point and critical exponents



$C_{xy}^{(\mathrm{con})} = \overline{\frac{\partial \langle S_x \rangle}{\partial h_y}} \sim r^{-(D-2+\eta)}$ and $C_{xy}^{(\mathrm{dis})} = \overline{\langle S_x S_y \rangle} \sim r^{-(D-4+\bar{\eta})} \rightarrow \{\nu,\, \eta,\, \bar{\eta}\}$.

$F_\xi \approx k_b T + \sigma \xi^{\theta} \rightarrow 2 - \alpha = (D-\theta)\nu$, where $\theta = 2 - \bar{\eta} + \eta$.

Dimensional reduction

Mapping to the pure Ising model (IM)

- Parisi and Sourlas in 1979 proposed the famous conjecture of *dimensional reduction*: The critical behavior of the RFIM at dimension D is the same as that of the pure IM at dimension $D-2$ ($\eta = \overline{\eta}$).

- Unfortunately, we know today that the RFIM orders in $D=3$ while the IM in $D=1$ does not.

- Is there an intermediate dimension $D_{\mathrm{int}} < D_{\mathrm{u}}$ such that the dimensional reduction is accurate for $D > D_{\mathrm{int}}$?

- Previous numerical works offered inconclusive and in many cases contradictory results. One of the main reasons is that scaling corrections (the Achilles' heel in the random-field problem) were neglected.

- An exception is the functional RG work by Tissier and Tarjus that proposed $D_{\mathrm{int}} \approx 5.1$, see PRL **107**, 041601 (2011).

Targets of our work from 2011 and on

($D = 3$, 4, and $D = 5$)

1. Examine claims of universality violation by comparing different random-field distributions.

2. Provide high-accuracy estimates for the complete set of critical exponents and for other universal ratios.

3. Check the validity of the fundamental scaling relations.

4. Test the conjecture of dimensional reduction $\mathrm{RFIM}^{(D)} \to \mathrm{IM}^{(D-2)}$.

Computational scheme

1. **Optimization algorithms**: We work at $T = 0$ taking advantage of the zero-T RG fixed point using efficient graph-theoretical algorithms that calculate exact ground states of the model in polynomial time.

2. **Fluctuation-dissipation formalism**: We compute both *connected* and *disconnected* correlation functions and the respective correlation lengths.

3. **Re-weighting extrapolation**: From a single simulation we extrapolate the mean value of observables to nearby parameters of the disorder distribution.

4. **Random-field distributions**:
   - $\mathcal{P}^{(\mathrm{G})}(h_x, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{h_x^2}{2\sigma^2}}$
   - $\mathcal{P}^{(\mathrm{P})}(h_x; \sigma) = \frac{1}{2|\sigma|} e^{-|h_x|/\sigma}$

5. **Disorder averaging**: 10^7 samples.

6. **System sizes**: $L_{\mathrm{max}}^D = \{192^3, 60^4, 28^5\}$.

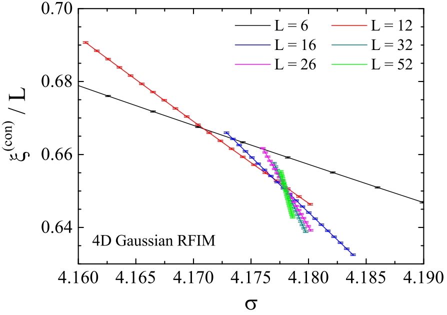
N.G. Fytas and V. Martín-Mayor, PRE **93**, 063308 (2016).

Finite-size scaling within the quotients method

We compare *dimensionless* observables g at pairs of $(L, 2L)$, $g_{2L}/g_L = 2$.

$g = \xi^{(\mathrm{con})}/L$, $\xi^{(\mathrm{dis})}/L$, and the Binder ratio $U_4 = \overline{\langle m^4 \rangle}/\overline{\langle m^2 \rangle}^2$.

$g^{\mathrm{cross}}_{(L);(2L)} = g^* + \mathcal{O}(L^{-\omega})$.

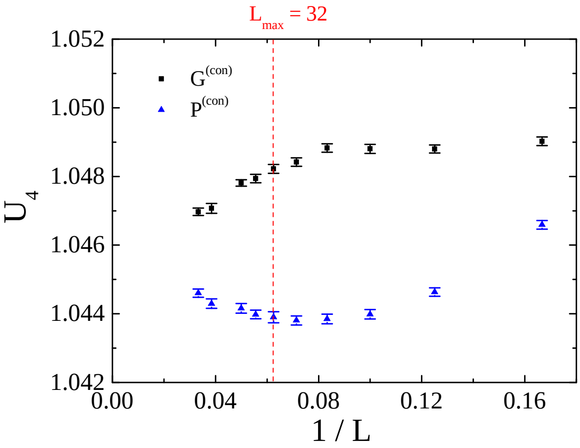


We also have *dimensionful* quantities O: derivatives of $\xi^{(\mathrm{con})}$ and $\xi^{(\mathrm{dis})} \to \nu$; derivatives of $\chi^{(\mathrm{con})}$ and $\chi^{(\mathrm{dis})} \to \eta$ and $\bar{\eta}$ (also $\chi^{(\mathrm{dis})}/[\chi^{(\mathrm{con})}]^2 \to 2\eta - \bar{\eta}$).

$(O_{2L}/O_L)^{\mathrm{cross}} = 2^{x_O/\nu} + \mathcal{O}(L^{-\omega})$.

Non-monotonic behavior

Possible explanation of previously reported universality violations



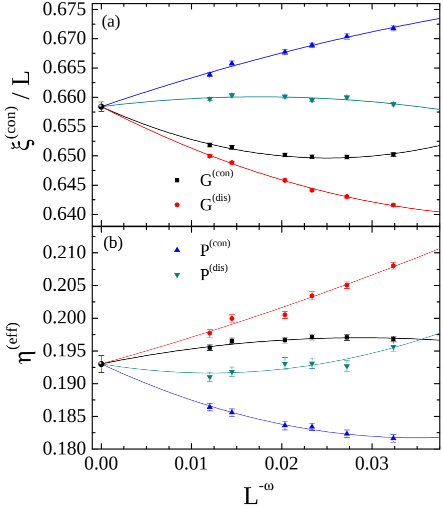
Higher-order corrections are necessary: $g_L = g^* + a_1 L^{-\omega} + a_2 L^{-2\omega} + a_3 L^{-3\omega}$.

Fitting scheme

1. We restrict to data with $L \geq L_{\min}$. To determine an acceptable $L_{\min}$ we employ the χ^2-test, computed using the **complete covariance matrix**.
2. We consider a fit as being fair only if $10\% < p < 90\%$, where p is the probability of finding a χ^2 value which is even larger than the one actually found from our data.
3. As a rule, we keep the lowest order for which an acceptable $L_{\min}$ can be found. Having decided the order of $L^{-\omega}$, we also keep the smallest possible $L_{\min}$.
4. We fit **simultaneously** several data sets for the 2 field distributions and 3 crossing points.
5. We use a two-step approach:
   - Estimation of ω using joint fits for several magnitudes.
   - Individual extrapolation of all other observables fixing ω.

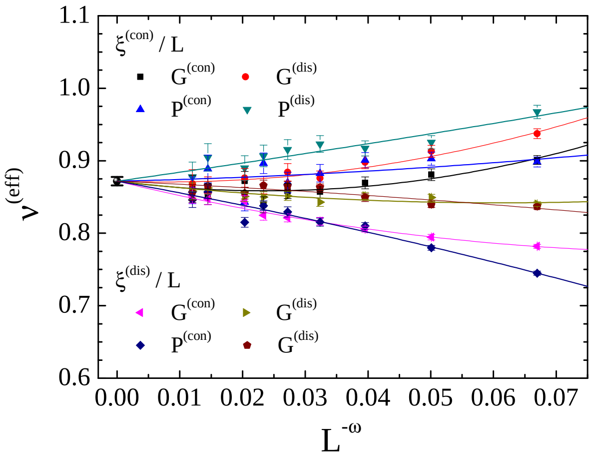
Universality in the 4D RFIM

$\omega = 1.30(9)$
$\xi^{(\mathrm{con})}/L = 0.6584(8)$
$\eta = 0.1930(13) \neq 0.25 = \eta^{(\mathrm{2D\ IM})}$



Critical exponent ν of the 4D RFIM

$\nu = 0.8718(58) \neq 1 = \nu^{(2D\ IM)}$



Summary of results for the 4D RFIM

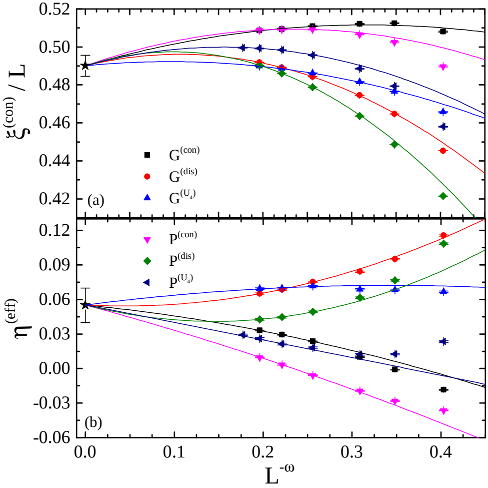
	QF	χ^2/DOF
ω	1.30(9)	
$\xi^{(\mathrm{con})}/L$	0.6584(8)	27.85/29
η	0.1930(13)	
$\sigma_{\mathrm{c}}(G)$	4.17749(4)(2)	5.6/7
$\sigma_{\mathrm{c}}(P)$	3.62052(3)(8)	8.85/11
U_4	1.04471(32)(14)	10/11
$\xi^{(\mathrm{dis})}/L$	2.4276(36)(34)	16/15
ν	0.8718(58)(19)	62.9/55
$2\eta - \bar{\eta}$	0.0322 (23)(1)	16.0/19

$2\eta \neq \bar{\eta}$ (clear case).

N.G. Fytas, V. Martín-Mayor, M. Picco, and N. Sourlas, PRL **116**, 227201 (2016).

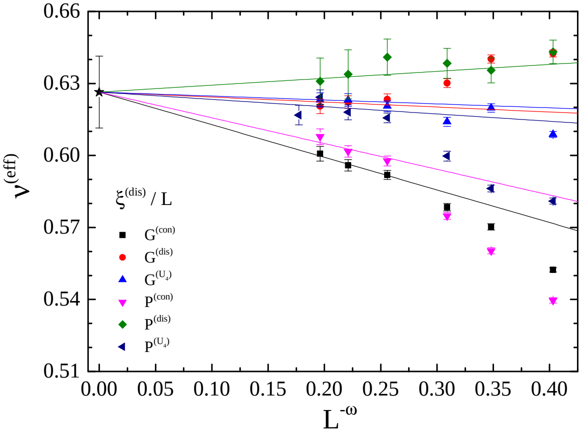
Universality in the 5D RFIM

$\omega = 0.66(15) \sim 0.82966(9) = \omega^{(\mathrm{3D\ IM})}$
$\xi^{(\mathrm{con})}/L = 0.4901(55)$
$\eta = 0.055(15) \sim 0.036298(2) = \eta^{(\mathrm{3D\ IM})}$



Critical exponent ν of the 5D RFIM

$\nu = 0.626(15) \approx 0.629971(4) = \nu^{(3D\ IM)}$



Summary of results for the 5D RFIM

Extrapolation to $L \to \infty$	χ^2/DOF	$L_{\min}$	order in $L^{-\omega}$
$\xi^{(\mathrm{con})}/L = 0.4901(55)$	11.3/10	8	second
$\eta = 0.055(15)$			
$\omega = 0.66(+15/-13)$			
$\xi^{(\mathrm{dis})}/L = 1.787(8)[+30/-82]$	5.3/9	6	second
$U_4 = 1.103(16)[+18/-43]$	1.9/6	6	third
$2\eta - \bar{\eta} = 0.058(7)[+1/-2]$	3.8/6	10	first
$\nu = 0.626(15)[+2/-3]$	8.3/6	10	first
$\sigma_c(\mathrm{G}) = 6.02395(7)[+2/-7]$	0.1/2	8	second
$\sigma_c(\mathrm{P}) = 5.59038(16)[+9/-13]$	2.7/3	8	second

N.G. Fytas, V. Martín-Mayor, M. Picco, and N. Sourlas, PRE **95**, 042117 (2017).

A statistical test

We make the null-hypothesis of *equality* of the two universality classes.
Within our accuracy the two universality classes are indistinguishable.

Observable	Extrapolation to $L \to \infty$	χ^2/DOF	p-value	$L_{\min}$	order in $L^{-\omega}$
$\xi^{(\mathrm{con})}/L$	$0.4972(+16/-35)$				
η	$0.0453(+19/-44)$	13.37/11	27%	8	second
ω	0.82966 (fixed)				
η	0.036298 (fixed)	15.82/12	20%	8	second
ω	0.82966 (fixed)				
$\xi^{(\mathrm{dis})}/L$	1.8184(52)	13.08/9	16%	6	second
U_4	1.123(8)	2.76/6	84%	6	third
$2\eta - \bar{\eta}$	0.036298 (fixed)	4.15/7	76%	8	second
ν	0.629971 (fixed)	3.43/7	84%	8	second
$\sigma_c(\mathrm{G})$	6.02393(18)	0.95/2	62%	8	second
$\sigma_c(\mathrm{P})$	5.59028(13)	2.01/3	57%	8	second

Conclusions

An overview of results for the RFIM at $3 \leq D < 6$

	3D RFIM	4D RFIM	5D RFIM	2D IM	3D IM	MF
ν	1.38(10)	0.8718(58)	0.626(15)	1	0.629971 (4)	1/2
η	0.5153(9)	0.1930(13)	0.055(15)	0.25	0.036298(2)	0
$\bar{\eta}$	1.028(2)	0.3538(35)	0.052(30)	0.25	0.036298(2)	0
$\Delta_{\eta,\bar{\eta}} = 2\eta - \bar{\eta}$	0.0026(9)	0.0322(23)	0.058(7)	0.25	0.036298(2)	0
β	0.019(4)	0.154(2)	0.329(12)	0.125	0.326419(3)	1/2
γ	2.05(15)	1.575(11)	1.217(31)	1.875	1.237075(10)	1
θ	1.487(1)	1.839(3)	2.00(2)	2	2	2
α	-0.16(35)	0.12(1)	-	-	-	-
α (from hyperscaling)	-0.09(15)	0.12(1)	0.12(5)	0	0.110087 (12)	0
$\alpha + 2\beta + \gamma$	2.00(31)	2.00(3)	2.00(11)	2	2.000000 (28)	2
$\sigma_c(G)$	2.27205(18)	4.17749(6)	6.02395(7)	-	-	-
$\sigma_c(P)$	1.7583(2)	3.62052(11)	5.59038(16)	-	-	-
U_4	1.0011(18)	1.04471(46)	1.103(16)			
$\xi^{(\mathrm{con})}/L$	1.90(12)	0.6584(8)	0.4901(55)			
$\xi^{(\mathrm{dis})}/L$	8.4(8)	2.4276(70)	1.787(8)			
ω	0.52(11)	1.30 (9)	0.66(+15/-13)		0.82966(9)	0

A short review:
N.G. Fytas, *et al.*, J. Stat. Phys. **172**, 665 (2018).