Exact solutions to plaquette Ising models with free and periodic boundaries

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Motivation

- your *first Monte Carlo simulation* of spin-lattices was (is) erroneous almost surely
- compare to enumeration, exact solutions for *finite lattices*

Exact solutions:

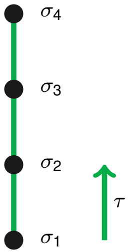
- $1d$ Ising model: {free, fixed, (anti)periodic}-boundary conditions
- $2d$ Ising model: {(anti)periodic, Brascamp-Kunz,...}-boundary conditions, *no solution* for free boundaries

Spin-Bond transformation: solving the $1d$ Ising chain

- for free boundary conditions:

$$H = -\sum_{i=1}^{L-1} \sigma_i \sigma_{i+1}, \qquad \sigma_i \in \{+1, -1\},$$

$$Z_{1d,\,\mathrm{free}} = \sum_{\{\sigma\}} \exp\left(\beta \sum_{i=1}^{L-1} \sigma_i \sigma_{i+1}\right)$$



- spin-bond transformation:

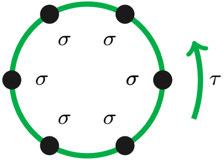
$$\{\sigma_1, \sigma_2, \dots \sigma_L\} \to \{\tau_1, \tau_2, \dots \tau_L\}$$

where $\tau_1 = \sigma_1\sigma_2,\ \tau_2 = \sigma_2\sigma_3,\ \dots, \tau_{L-1} = \sigma_{L-1}\sigma_L$ and setting $\tau_L = \sigma_L$, the mapping $\{\sigma\} \to \{\tau\}$ with an inverse relation of the form $\sigma_i = \tau_L\,\tau_{L-1}\,\tau_{L-2}\cdots\tau_i$ is *one-to-one*

- partition function factorises:

$$Z_{1d,\,\mathrm{free}} = \sum_{\{\tau\}} \exp\left(\beta \sum_{i=1}^{L-1} \tau_i\right) = 2 \prod_{i=1}^{L-1} \sum_{\tau_i = \pm 1} \exp\left(\beta \tau_i\right) = 2^L \,\mathrm{ch}(\beta)^{L-1}$$

Spin-Bond transformation: solving the $1d$ Ising chain (again)



- for periodic boundary conditions:

$$H = -\sum_{i=1}^{L} \sigma_i \sigma_{i+1}, \qquad \sigma_i \in \{+1, -1\}$$

- spin-bond transformation:

$$\{\sigma_1, \sigma_2, \ldots \sigma_L\} \to \{\tau_1, \tau_2, \ldots \tau_L\}$$

$\tau_1 = \sigma_1\sigma_2,\ \tau_2 = \sigma_2\sigma_3,\ \ldots, \tau_L = \sigma_L\sigma_{L+1} = \sigma_L\sigma_1$, with an inverse relation of the form $\sigma_i = \sigma_1 \times \tau_1\,\tau_2\,\tau_3 \cdots \tau_{i-1}$, mapping is *two-to-one* and we have the *constraint*

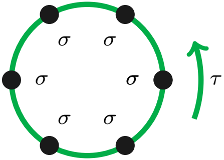
$$\prod_{i=1}^{L} \tau_i = \prod_{i=1}^{L} \sigma_i^2 = 1$$

Spin-Bond transformation: solving the 1d Ising chain (again), cont'd

- partition function:

$$Z_{1d,\ \mathrm{periodic}} = \sum_{\{\sigma\}} \exp\left(\beta \sum_{i=1}^{L} \sigma_i \sigma_{i+1}\right)$$

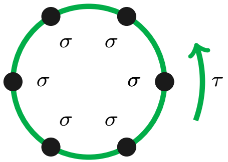
$$= 2 \sum_{\{\tau\}} \exp\left(\beta \sum_{i=1}^{L} \tau_i\right) \delta\left(\prod_{i=1}^{L} \tau_i, 1\right)$$



Spin-Bond transformation: solving the 1d Ising chain (again), cont'd

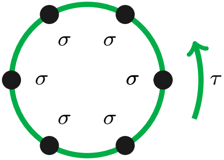
- partition function:

$$
\begin{aligned}
Z_{1d,\ \mathrm{periodic}} &= \sum_{\{\sigma\}} \exp\left(\beta \sum_{i=1}^{L} \sigma_i \sigma_{i+1}\right) \\
&= 2 \sum_{\{\tau\}} \exp\left(\beta \sum_{i=1}^{L} \tau_i\right) \delta\left(\prod_{i=1}^{L} \tau_i, 1\right) \\
&= \sum_{\{\tau\}} \exp\left(\beta \sum_{i=1}^{L} \tau_i\right) \left(1 + \prod_{i=1}^{L} \tau_i\right)
\end{aligned}
$$



Spin-Bond transformation: solving the 1*d* Ising chain (again), cont'd

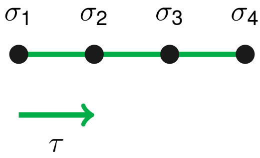
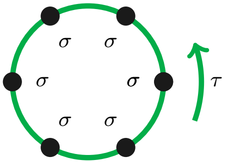
- partition function:



$$
\begin{aligned}
Z_{1d,\,\mathrm{periodic}} &= \sum_{\{\sigma\}} \exp\left(\beta \sum_{i=1}^{L} \sigma_i \sigma_{i+1}\right) \\
&= {\color{red}2} \sum_{\{\tau\}} \exp\left(\beta \sum_{i=1}^{L} \tau_i\right) {\color{blue}\delta\left(\prod_{i=1}^{L} \tau_i, 1\right)} \\
&= \sum_{\{\tau\}} \exp\left(\beta \sum_{i=1}^{L} \tau_i\right) \left(1 + \prod_{i=1}^{L} \tau_i\right) \\
&= \left[\prod_{i=1}^{L} \sum_{\tau_i = \pm 1} \exp\left(\beta \tau_i\right) + \prod_{i=1}^{L} \sum_{\tau_i = \pm 1} \tau_i \exp\left(\beta \tau_i\right)\right] \\
&= 2^L \operatorname{ch}(\beta)^L \left[1 + \operatorname{th}(\beta)^L\right] \ .
\end{aligned}
$$

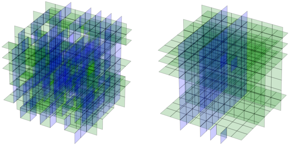
Spin-Bond transformation, highlights

solving the $1d$ Ising chain

	free	periodic
		
last spin, σ_L cause of “2”	remains untransformed summing over σ_L	is transformed two-to-one transformation additional constraint
Z	$2^L \, \mathrm{ch}(\beta)^{L-1}$	$2^L \, \mathrm{ch}(\beta)^L \left[1 + \mathrm{th}(\beta)^L\right]$

Plaquette model (in 3*d*)

$$\mathcal{H} = -\sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$



- particular limit of 3*d* model of the gonihedric string

$$\mathcal{H} = -2\kappa \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \frac{\kappa}{2} \sum_{\langle\langle i,j \rangle\rangle} \sigma_i \sigma_j - \frac{1-\kappa}{2} \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

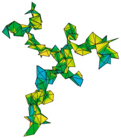


Fig. 7.1. A typical collapsed triangulated surface resulting from a simulation of the Gaussian Hamiltonian in (7.3)

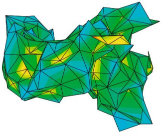
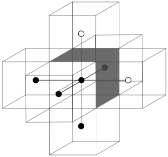


Fig. 7.2. A typical uncollapsed triangulated surface resulting from a simulation of the Gaussian plus extrinsic curvature Hamiltonian in (7.2) with $\lambda = 1.1$



D. A. Johnston, A. Lipowski, and R. P. K. C. Malmini, in *Rugged Free Energy Landscape,s*, Vol. 736 of *Lecture Notes in Physics, Berlin Springer Verlag*, edited by W. Janke (2008), pp. 173–199.

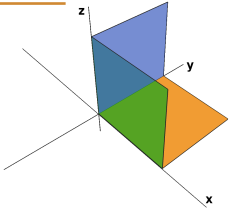
Anisotropic plaquette model

$$H_{\text{aniso}}(\{\sigma\}) = \underbrace{-J_x \sum_{\square_{yz}} \sigma\sigma\sigma\sigma}\ \underbrace{-J_y \sum_{\square_{zx}} \sigma\sigma\sigma\sigma}\ \underbrace{-J_z \sum_{\square_{xy}} \sigma\sigma\sigma\sigma}$$

$$H_{\text{aniso}}^{J_x=J_y=0}(\{\sigma\}) = -J_z \sum_{z=1}^{L_z} \left[\sum_{2d\ \square} \sigma\sigma\sigma\sigma \right]$$

$$Z_{\text{aniso}}^{J_x=J_y=0} = \sum_{\{\sigma\}} \exp\left(-\beta H_{\text{aniso}}^{J_x=J_y=0}(\{\sigma\})\right)$$

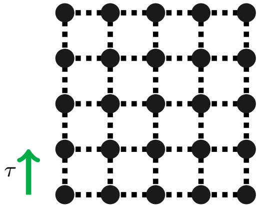
$$= \left(Z_{2d,\ \text{gonihedric}}\right)^{L_z}$$



Two dimensional plaquette model: free boundaries in y-direction

- Spin-bond-transformation in y-direction, $\tau_{x,y} = \sigma_{x,y}\sigma_{x,y+1}$, with the condition $\tau_{x,L_y} = \sigma_{x,L_y}$
- partition function factorises:

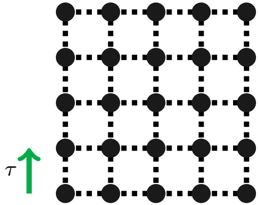
$$Z_{2d,\text{ gonihedric, free}}$$
$$= \sum_{\{\sigma\}} \exp\left(\beta \sum_{x=1}^{L_x-1}\sum_{y=1}^{L_y-1} \sigma_{x,y}\sigma_{x,y+1}\sigma_{x+1,y}\sigma_{x+1,y+1}\right)$$
$$= \sum_{\{\tau\}} \exp\left(\beta \sum_{x=1}^{L_x-1}\sum_{y=1}^{L_y-1} \tau_{x,y}\tau_{x+1,y}\right)$$
$$= 2^{L_x}\left(Z_{1d,\text{ Ising}}\right)^{L_y-1}$$



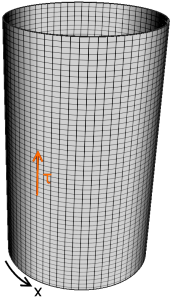
- the factor 2^{L_x} comes from the L_x sums over $\tau_{x,L_y} = \sigma_{x,L_x} = \pm 1$ which do not appear in the exponent

Two dimensional plaquette model: mixed boundary conditions

$$2^{L_x L_y}\,\mathrm{ch}\,(\beta)^{(L_x-1)(L_y-1)}$$



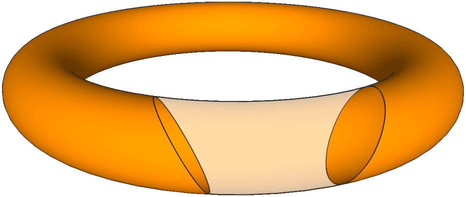
$$2^{L_x L_y}\,\mathrm{ch}\,(\beta)^{L_x(L_y-1)}\left(1+\mathrm{th}\,(\beta)^{L_x}\right)^{L_y-1}$$



Two dimensional plaquette model: periodic boundaries

- Consider periodic boundary conditions in y-direction: $\sigma_{x,L_y+1} = \sigma_{x,1}$, here also in x-direction $\sigma_{L_x+1,y} = \sigma_{1,y}$
- Spin-bond-transformation in y-direction, $\tau_{x,y} = \sigma_{x,y}\sigma_{x,y+1}$ is *two-to-one* and imposes L_x constraints $\prod_y \tau_{x,y} = 1$

$$Z_{2d,\ \mathrm{gonihedric},\ \mathrm{periodic}} = 2^{L_x} \sum_{\{\tau\}} \exp\left(\beta \sum_{x=1}^{L_x} \sum_{y=1}^{L_y} \tau_{x,y}\tau_{x+1,y}\right) \prod_{x=1}^{L_x} \delta\left(\prod_{y=1}^{L_y} \tau_{x,y}, 1\right)$$



- the funny "trick" of rewriting the δ-constraints leads to complicated products $\to$ we go straight to the high-temperature representation

Two dimensional plaquette model: periodic boundaries

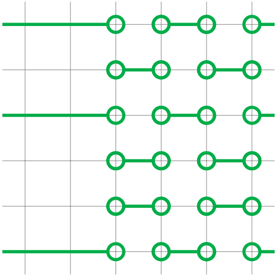
- high-temperature representation

$$
\begin{aligned}
&Z_{2d,\ \text{gonihedric},\ \text{periodic}} \\
&\quad = 2^{L_x} \sum_{\{\tau\}} \exp\left(\beta \sum_{x=1}^{L_x} \sum_{y=1}^{L_y} \tau_{x,y}\tau_{x+1,y} \right) \prod_{x=1}^{L_x} \delta\left(\prod_{y=1}^{L_y} \tau_{x,y}, 1 \right) \\
&\quad = 2^{L_x}\, \mathrm{ch}(\beta)^{L_x L_y} \sum_{\{\tau\}} \left[\prod_{y=1}^{L_y} \prod_{x=1}^{L_x} \left(1 + \mathrm{th}\left(\beta\right) \tau_{x,y}\tau_{x+1,y}\right) \right] \prod_{x=1}^{L_x} \delta\left(\prod_{y=1}^{L_y} \tau_{x,y}, 1 \right)
\end{aligned}
$$

- similar to counting loops in the $2d$ Ising model, but simpler: only coupling in x-direction

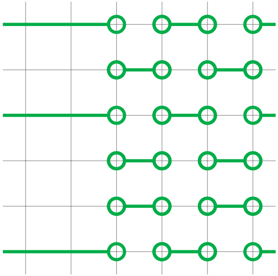
Two dimensional plaquette model: periodic boundaries

$$2^{L_x}\,\mathrm{ch}(\beta)^{L_x L_y}\sum_{\{\tau\}}\left[\prod_{y=1}^{L_y}\prod_{x=1}^{L_x}\left(1+\mathrm{th}\left(\beta\right)\tau_{x,y}\tau_{x+1,y}\right)\right]\prod_{x=1}^{L_x}\delta\left(\prod_{y=1}^{L_y}\tau_{x,y},1\right)$$



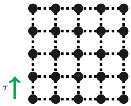
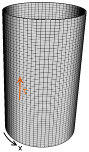
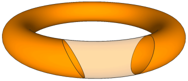
Two dimensional plaquette model: periodic boundaries

$$2^{L_x} \operatorname{ch}(\beta)^{L_x L_y} \sum_{\{\tau\}} \left[\prod_{y=1}^{L_y} \prod_{x=1}^{L_x} \left(1 + \operatorname{th}(\beta)\, \tau_{x,y} \tau_{x+1,y}\right) \right] \prod_{x=1}^{L_x} \delta \left(\prod_{y=1}^{L_y} \tau_{x,y}, 1 \right)$$



$$\left(\frac{1}{2}\right) 2^{L_x L_y} \operatorname{ch}(\beta)^{L_x L_y} \sum_{v=0}^{L_x} \binom{L_x}{v} \left(\operatorname{th}(\beta)^{v} + \operatorname{th}(\beta)^{L_x - v} \right)^{L_y}$$

Two dimensional plaquette model

solving the $2d$ plaquette model			
	free-free	periodic-free	periodic-periodic
			
top line, σ_{x,L_y} cause of "2^{L_x}"	not transformed summing over top row		transformed two-to-one additional constraint

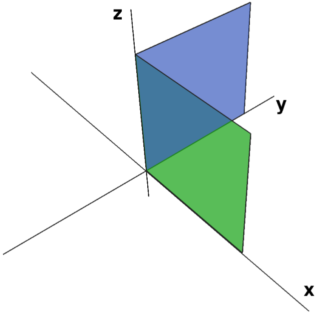
$$Z_{2d,\ \mathrm{gonihedric,\ free}} = 2^{L_x L_y}\,\mathrm{ch}\,(\beta)^{(L_x-1)(L_y-1)}$$

$$Z_{2d,\ \mathrm{gonihedric,\ mixed}} = 2^{L_x L_y}\,\mathrm{ch}\,(\beta)^{L_x(L_y-1)}\left(1+\mathrm{th}\,(\beta)^{L_x}\right)^{L_y-1}$$

$$Z_{2d,\ \mathrm{gonihedric,\ periodic}} = \left(\frac{1}{2}\right) 2^{L_x L_y}\,\mathrm{ch}(\beta)^{L_x L_y}\sum_{v=0}^{L_x}\binom{L_x}{v}\left(\mathrm{th}(\beta)^{v}+\mathrm{th}(\beta)^{L_x-v}\right)^{L_y}$$

Anisotropic plaquette model (again) - "fuki-nuke"

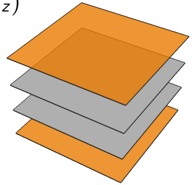
$$H_{\text{fuki-nuke}}(\{\sigma\}) \quad = \quad \underline{-J_x \sum_{\Box_{yz}} \sigma\sigma\sigma\sigma} \quad \underline{-J_y \sum_{\Box_{zx}} \sigma\sigma\sigma\sigma}$$



Three dimensional plaquette model: free boundaries in z-direction

- Spin-bond-transformation in z-direction $\tau_{x,y,z} = \sigma_{x,y,z}\sigma_{x,y,z+1}$ in a cuboidal $L \times L \times L_z$, for *one-to-one* correspondence: equality on one plane $\tau_{x,y,L_z} = \sigma_{x,y,L_z}$
- partition function factorises:

$$H_{\text{fuki-nuke}}(\{\tau\}) = -\sum_{x=1}^{L}\sum_{y=1}^{L}\sum_{z=1}^{L_z-1}\left(\tau_{x,y,z}\tau_{x+1,y,z} + \tau_{x,y,z}\tau_{x,y+1,z}\right)$$



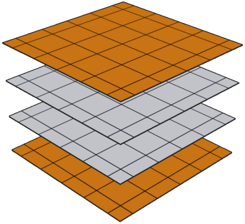
$$Z_{\text{fuki-nuke}} = \sum_{\{\tau\}} \exp\left(-\beta H_{\text{fuki-nuke}}(\{\tau\})\right)$$

$$= 2^{L^2}\left(Z_{2d \text{ Ising}}\right)^{L_z-1}$$

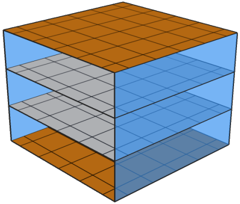
- the factor 2^{L^2} comes from the $L \times L$ sums over $\tau_{x,y,L_z} = \sigma_{x,y,L_z} = \pm 1$ which do not appear in the exponent
- free energy contributions

$$\beta f_{\text{fuki-nuke}} \equiv -\lim_{L\to\infty}\frac{1}{L^2 L_z}\ln Z_{\text{fuki-nuke}} = \beta f_{2d \text{ Ising}} - \frac{\ln 2 + \beta f_{2d \text{ Ising}}}{L_z}$$

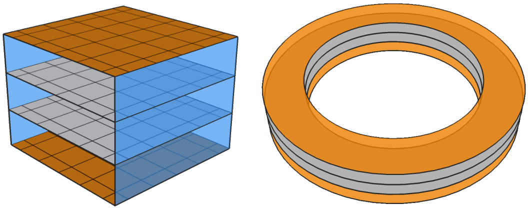
Interlude: topology



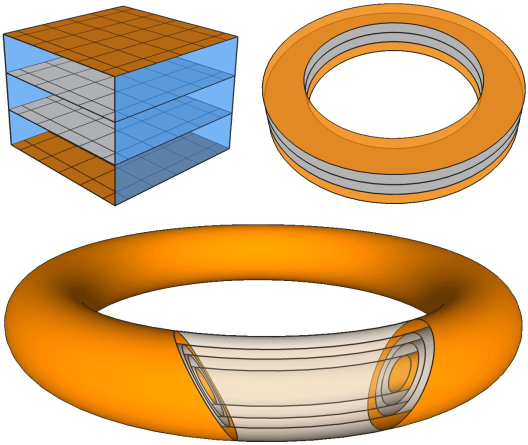
Interlude: topology



Interlude: topology



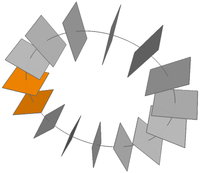
Interlude: topology



Three dimensional plaquette model: periodic boundaries in z-direction

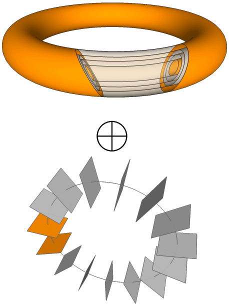
- Spin-bond-transformation in z-direction $\tau_{x,y,z} = \sigma_{x,y,z}\sigma_{x,y,z+1}$ is *two-to-one* and imposes $L \times L$ constraints $\prod_z \tau_{x,y,z} = 1$

$$H_{\text{fuki-nuke}}(\{\tau\}) = -\sum_{x=1}^{L}\sum_{y=1}^{L}\sum_{z=1}^{L_z}\left(\tau_{x,y,z}\tau_{x+1,y,z} + \tau_{x,y,z}\tau_{x,y+1,z}\right) \quad ,$$



$$\begin{aligned} Z_{\text{fuki-nuke}} &= \quad 2^{L^2}\sum_{\{\tau\}}\exp\left(-\beta H_{\text{fuki-nuke}}(\{\tau\})\right)\prod_{x=1}^{L}\prod_{y=1}^{L}\delta\left(\prod_{z=1}^{L_z}\tau_{x,y,z}, 1\right) \\ &= \quad \sum_{\{\tau\}}\exp\left(-\beta H_{\text{fuki-nuke}}(\{\tau\})\right)\prod_{x=1}^{L}\prod_{y=1}^{L}\left(1 + \prod_{z=1}^{L_z}\tau_{x,y,z}\right) \end{aligned}$$

Three dimensional plaquette model: periodic boundaries in z-direction



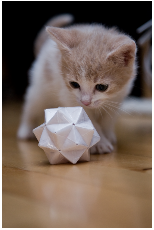
$$
\begin{aligned}
&Z_{\text{fuki-nuke}} \\
&= \sum_{\{\tau\}} \exp\left(-\beta H_{\text{fuki-nuke}}(\{\tau\})\right) \left(1 + \sum_{x=1}^{L}\sum_{y=1}^{L}\prod_{z=1}^{L_z} \tau_{x,y,z} + \mathcal{O}\left(\tau\tau\right)\right) \\
&= \left(Z_{2d\ \text{Ising}}\right)^{L_z} \left(1 + \sum_{x=1}^{L}\sum_{y=1}^{L} \left(\langle \tau_{x,y}\rangle_{Z_{2d\ \text{Ising}}}\right)^{L_z} + \mathcal{O}(\tau\tau)\right)
\end{aligned}
$$

- assuming translational invariance in each layer (2d periodic Ising model)

$$
Z_{\text{fuki-nuke}} \quad = \quad \left(Z_{2d,\ \text{Ising}}\right)^{L_z} \left(1 + L^2 C_1^{L_z} + \mathcal{O}\left(\tau\tau\right)\right)
$$

- $C_1 = \langle \tau_{1,1}\rangle_{Z_{2d,\ \text{Ising}}}$ is the normalized one-point function (magnetization)
- $\mathcal{O}\left(\tau\tau\right) =$
  $\frac{1}{2}\left(\sum_{x_1=1}^{L}\sum_{y_1=1}^{L}\sum_{x_2=1}^{L}\sum_{y_2=1}^{L}\left(\langle \tau_{x_1,y_1}\tau_{x_2,y_2}\rangle_{Z_{2d\ \text{Ising}}}\right)^{L_z} - 1\right) + \mathcal{O}\left(\tau\tau\tau\right)$

Three dimensional plaquette model: periodic boundaries in z-direction



$$
\begin{aligned}
Z_{\text{fuki-nuke}} \\
&= \sum_{\{\tau\}} \exp\left(-\beta H_{\text{fuki-nuke}}(\{\tau\})\right) \left(1 + \sum_{x=1}^{L} \sum_{y=1}^{L} \prod_{z=1}^{L_z} \tau_{x,y,z} + \mathcal{O}(\tau\tau)\right) \\
&= \left(Z_{2d\ \text{Ising}}\right)^{L_z} \left(1 + \sum_{x=1}^{L} \sum_{y=1}^{L} \left(\langle \tau_{x,y} \rangle_{Z_{2d\ \text{Ising}}}\right)^{L_z} + \mathcal{O}(\tau\tau)\right)
\end{aligned}
$$

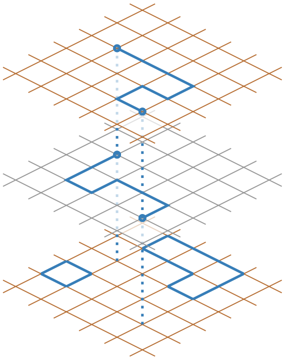
- assuming translational invariance in each layer ($2d$ periodic Ising model)

$$
Z_{\text{fuki-nuke}} = \left(Z_{2d,\ \text{Ising}}\right)^{L_z} \left(1 + L^2 C_1^{L_z} + \mathcal{O}(\tau\tau)\right)
$$

- $C_1 = \langle \tau_{1,1} \rangle_{Z_{2d,\ \text{Ising}}}$ is the normalized one-point function (magnetization)
- $\mathcal{O}(\tau\tau) =$
  $\frac{1}{2}\left(\sum_{x_1=1}^{L} \sum_{y_1=1}^{L} \sum_{x_2=1}^{L} \sum_{y_2=1}^{L} \left(\langle \tau_{x_1,y_1} \tau_{x_2,y_2} \rangle_{Z_{2d\ \text{Ising}}}\right)^{L_z} - 1\right) + \mathcal{O}(\tau\tau\tau)$

Fuki-Nuke: full-periodic

$$\left(Z_{2d\ \mathrm{Ising}}\right)^{L_z}\left(1+\sum_{x=1}^{L}\sum_{y=1}^{L}\left(\langle\tau_{x,y}\rangle_{Z_{2d\ \mathrm{Ising}}}\right)^{L_z}+\mathcal{O}(\tau\tau)\right)$$

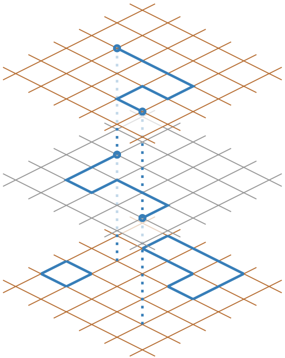


Fuki-Nuke: full-periodic

$$\left(Z_{2d\ \mathrm{Ising}}\right)^{L_z} \left(1 + \sum_{x=1}^{L} \sum_{y=1}^{L} \left(\langle \tau_{x,y} \rangle_{Z_{2d\ \mathrm{Ising}}}\right)^{L_z} + \mathcal{O}(\tau\tau)\right)$$

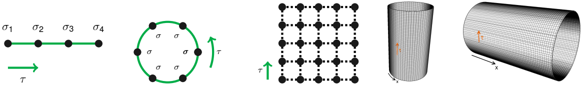
- without the power L_z in $\mathcal{O}(\tau\tau)$ $\rightarrow$ (high-temperature) susceptibility of the $2d$ Ising model, no closed-form expression
- too late, discovered in loop-matrix calculations already

T. Jonsson and G. K. Savvidy, Phys. Lett. B **449** (1999) 253;
T. Jonsson and G. K. Savvidy, Nucl. Phys. B **575** (2000) 661;
G. K. Savvidy, J. High Energy Phys. **09** (2000) 44;
G. K. Savvidy, Mod. Phys. Lett. B **29** (2015) 1550203.

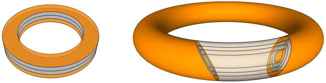


Conclusion

- identical spin-bond transformation can be treated explicitly for the 1*d* Ising and 2*d* plaquette models



- the 3*d* fuki-nuke model: explicit closed-form solution, as long as one boundary is free and 2*d* Ising model boundary is known



- the 3*d* fuki-nuke model: fully-periodic lattice creates sum over non-trivial *n*-point correlation functions
- the (full) 3*d* plaquette model: to be investigated (or maybe not)

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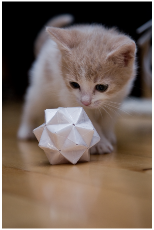

This work is, with full references (but less pictures), in press at Nucl. Phys. B,
Exact solutions to plaquette Ising models with free and periodic boundaries
http://dx.doi.org/10.1016/j.nuclphysb.2016.11.005
arXiv:1601.03997

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