Boundary driven open XXZ quantum spin chain

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Boundary driven open XXZ quantum spin chain

Physical setup and mathematical description

The physical setup

The physical setup

Interaction

$H_s$

$H_1$

$T_1$

$H_2$

$T_2$
Open XXZ chain

**Anisotropic Heisenberg XXZ spin chain** with boundary fields

The Hamiltonian

\[ H = \sum_{k=1}^{N-1} h_{k,k+1} + (g_1^L + g_N^R) \]

with the two-spin interaction

\[ h = \frac{1}{2} [\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta \sigma^z \otimes \sigma^z] \]

and for the boundary fields:

\[ g^L = f^L \sigma_u , \quad g^R = f^R \sigma_v \]
Dynamics of the open XXZ chain

Non-unitary dynamics governed by the Boundary Lindblad equation

\[
\frac{d}{dt} \rho = -i [H, \rho] + \mathcal{D}^L(\rho) + \mathcal{D}^R(\rho)
\]

with left and right boundary dissipators

\[
\mathcal{D}^{L,R}(\rho) = D^{L,R} \rho D^{L,R+} - \frac{1}{2} \left\{ \rho, D^{L,R+} D^{L,R} \right\}
\]

Remark: \( \mathcal{D}^{L,R} \) local dissipative terms which force a relaxation of the leftmost and the rightmost spins towards fully polarized target states (\( \mathcal{D}^{L,R}(\rho^{L,R}) = 0 \))

\[
\rho^L = |\uparrow_u\rangle\langle \uparrow_u|, \quad \rho^R = |\downarrow_v\rangle\langle \downarrow_v|
\]
The non-equilibrium steady state (NESS) $\rho_\infty = \lim_{t \to \infty} \rho(t)$ satisfying the fixed point equation:

$$-i[H, \rho_\infty] + D^L(\rho_\infty) + D^R(\rho_\infty) \equiv \mathcal{L}(\rho_\infty) = 0$$

is unique.

Exact MPA NESS: Main idea

Stationary density matrix $\rho$ satisfying
\[ i[H, \rho] = D^L(\rho) + D^R(\rho) \]

Matrix Product Ansatz (MPA): $\rho = SS^\dagger / \text{tr}(SS^\dagger)$ with $S \in \mathbb{C}^{2^\otimes N}$

\[ S = \langle \phi | \Omega^{\otimes N} | \psi \rangle, \quad \Omega \equiv \sigma^s \otimes A_s = \begin{pmatrix} A_1 & A_+ \\ A_- & A_2 \end{pmatrix}; \quad \Omega \in \mathbb{C}^{2} \otimes \mathbb{R} \]

Demand for local divergence condition

\[ [h, \Omega \otimes \Omega] = \Xi \otimes \Omega - \Omega \otimes \Xi \]

with $\Xi = \sigma^s \otimes E_s$, such that the effect of the unitary bulk dynamics is expelled at the boundaries

\[ [H, \Omega^{\otimes N}] = \underbrace{\Xi \otimes \Omega^{\otimes N-1}}_{\text{left}} - \underbrace{\Omega^{\otimes N-1} \otimes \Xi}_{\text{right}} \]
The local divergence condition

\[ [ h, \Omega \otimes \Omega ] = \Xi \otimes \Omega - \Omega \otimes \Xi \]

turns out to an algebraic conditioning on the auxiliary operators \( \Rightarrow U_q(SU(2)) \) quantum algebra

\[ A_{\pm} =: i \alpha S_{\pm}, \quad Q =: \lambda q^{S_z} \]

with

\[ [ S_{+}, S_{-} ] = \frac{q^{2S_z} - q^{-2S_z}}{q - q^{-1}} , \quad q^{S_z} S_{\pm} = q^{\pm 1} S_{\pm} q^{S_z} \]
Irreducible representation of the auxiliary algebra

Irreducible representation

\[
S_z = \sum_{k=0}^{\infty} (p - k) |k \rangle \langle k |
\]

\[
S_+ = \sum_{k=0}^{\infty} [k + 1]_q |k \rangle \langle k + 1 |
\]

\[
S_- = \sum_{k=0}^{\infty} [2p - k]_q |k + 1 \rangle \langle k |
\]

where \( p \in \mathbb{C} \) is an arbitrary complex parameter (finite-dimensional at \( 2p \in \mathbb{N} \)).

With \([x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}\)
Example: isotropic point \((\Delta = (q + q^{-1})/2 = 1) \Rightarrow SU(2)\)

\[
q = 1 \quad \Rightarrow \quad \Omega = i \begin{pmatrix} S_z & S_+ \\ S_- & -S_z \end{pmatrix}, \quad \Xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

The local divergence condition implies

\[
[H, \Omega \otimes N] = \Xi_1 \otimes \Omega \otimes \Xi^{N-1} - \Omega \otimes \Xi^{N-1} \otimes \Xi
\]

The stationary eq. \(\mathcal{D}(SS^\dagger) = i[H, SS^\dagger]\) can thus be split into two parts

\[
\mathcal{D}_L(SS^\dagger) = i(\Xi_1 S^\dagger - S \Xi_1^\dagger) \quad (= i[H, SS^\dagger]|_{Left})
\]

and

\[
\mathcal{D}_R(SS^\dagger) = -i(\Xi_N S^\dagger - S \Xi_N^\dagger) \quad (= i[H, SS^\dagger]|_{Right})
\]
Explicit solution at the isotropic point

Twisting Polarization with angle $\theta$

Take the polarization

$$\vec{n}_L = (1, 0, 0) \quad \vec{n}_R = (\cos \theta, \sin \theta, 0),$$

Representation parameter

$$p = \frac{i}{\Gamma - 2ih}$$

and with the coherent state

$$|R_\theta(p)\rangle = \sum_{n=0}^{\infty} \frac{(-\cot \frac{\theta}{2})^n (S_-)^n}{n!} |0\rangle = \sum_{n=0}^{\infty} \frac{(-\cot \frac{\theta}{2})^n}{n!} \binom{2p}{n} |n\rangle$$

Solution is

$$\rho = SS^\dagger / tr(SS^\dagger) \text{ with } S = \langle 0 | \Omega^{\otimes N} | R_\theta(p) \rangle.$$
Local observables

It is convenient to rewrite $\rho$ by doubling the auxiliary space:

$$\rho = \langle 0, 0 | \Omega(p)^{\otimes N} | R_\theta, R_\theta^* \rangle$$

where $\Omega(p) = \Omega(p) \otimes_{au} \Omega^T(-p)$, lives in $\mathbb{C}^2 \otimes \mathcal{R} \otimes \mathcal{R}$ and transposition $\Omega^T$ is done in the physical space only.

Let us introduce polarization operators defined on the auxiliary space $\mathcal{R} \otimes \mathcal{R}$:

$$B_\alpha(p) = Tr(\sigma^\alpha \Omega(p))$$
Local observables

Observables are expressed in terms of the polarization operators $B_\alpha$

In terms of the $B$-operators, the normalization factor becomes

$$Z(N) \equiv Tr(\rho) = Tr \langle 0, 0 | \Omega(p)^{\otimes N} | R_\theta, R_\theta^* \rangle = \langle 0, 0 | (Tr(\Omega(p)))^N | R_\theta, R_\theta^* \rangle$$

$$= \langle 0, 0 | (Tr(\sigma^0 \Omega(p)))^N | R_\theta, R_\theta^* \rangle = \langle 0, 0 | B_0^N | R_\theta, R_\theta^* \rangle$$

Magnetization

$$M_{k,N}^\alpha = \langle \sigma_k^\alpha \rangle = \frac{\langle 0, 0 | B_0^{k-1} B_\alpha B_0^{N-k} | R_\theta, R_\theta^* \rangle}{Z(N)}.$$  

Magnetization flux

$$J_k^\alpha = 2 \varepsilon_{\alpha\beta\gamma} \langle \sigma_k^\beta \sigma_{k+1}^\gamma \rangle = 2 \varepsilon_{\alpha\beta\gamma} \frac{\langle 0, 0 | B_0^{k-1} B_\beta B_\gamma B_0^{N-k-1} | R_\theta, R_\theta^* \rangle}{Z(N)}.$$
Using the \( SU(2) \) commutation rules for \( S_\alpha \) one obtains the exact equation

\[
(-2M_{k+1,N+1}^\alpha + M_{k+2,N+1}^\alpha + M_{k,N+1}^\alpha) \frac{Z(N+1,\theta)}{Z(N,\theta)} + 2(M_{k,N}^\alpha + M_{k+1,N}^\alpha)
\]

\[
= 8p^2 M_{k,N-1}^\alpha \frac{Z(N-1,\theta)}{Z(N,\theta)} .
\]

In the large \( \Gamma \) and size limit and \( \Gamma \gg \frac{\theta^2}{N} \), one has in the continuum limit

\[
\partial_x^2 M^\alpha(x) + \theta^2 M^\alpha(x) = 0 ,
\]

and the boundary conditions \( M^\alpha(0) = \sigma^\alpha_{\text{target}(L)} \), \( M^\alpha(1) = \sigma^\alpha_{\text{target}(R)} \) where \( \sigma^a_{\text{target}(L,R)} \) are the targeted boundary magnetizations.
Magnetization flux

In terms of the MPA the steady magnetization currents are given by

\[ j^x(N) = -8ip \frac{Z(N - 1, \theta)}{Z(N, \theta)}, \quad j^y(N, \theta) = -\cot \frac{\theta}{2} \times j^x(N, \theta) \]

For sufficiently large \( \Gamma, N \)

\[ j^x(N) \bigg|_{\Gamma \gg \frac{\theta^2}{N}, N \gg 1} = \frac{2 \theta^2}{\Gamma N^2} + O \left( N^{-3} \right), \]
Magnetization currents

Different scaling for the $z$-component $j^z(N, \theta)$ leading to

$$j^z(N)\Big|_{\Gamma \gg \theta^2/N, N \gg 1} = \frac{2\theta}{N} + O\left(\frac{1}{N^2}\right)$$

$$NJ^z_{NESS}(N)$$

Notice that the limits $N \to \infty$ and $\theta \to \pi$ do not commute $(NJ^z(N, \pi) = 0 \ \forall N)$
Conclusion

- Explicit expressions for one- and two-point observables (magnetization currents and magnetization profiles) in the MPA steady state.
- Easy to compute numerically (linear growth of the space size instead of exponential)
- The magnetization flux is qualitatively different in the direction parallel to the twisting plane, and in the orthogonal direction.
- Spin valve like effect when the boundary fields are competing with the reservoir polarizations (with Gabriel Landi)