Boundary driven open XXZ quantum spin chain

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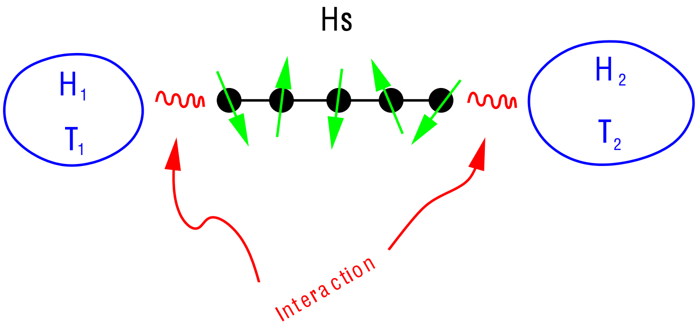
Summary

The physical setup



Open XXZ chain

Anisotropic Heisenberg XXZ spin chain with boundary fields
The Hamiltonian

$$H = \sum_{k=1}^{N-1} h_{k,k+1} \quad +(g_1^L + g_N^R)$$

with the two-spin interaction

$$h = \frac{1}{2}[\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta\sigma^z \otimes \sigma^z]$$

and for the boundary fields:

$$g^L = f^L \sigma_u \ , \qquad g^R = f^R \sigma_v$$

Dynamics of the open XXZ chain

Non-unitary dynamics governed by the Boundary Lindblad equation

$$\frac{d}{dt}\rho = -i\,[H,\rho] + \mathcal{D}^{L}(\rho) + \mathcal{D}^{R}(\rho)$$

with left and right boundary dissipators

$$\mathcal{D}^{L,R}(\rho) = D^{L,R}\rho D^{L,R+} - \frac{1}{2}\left\{\rho, D^{L,R+}D^{L,R}\right\}$$

Remark: $\mathcal{D}^{L,R}$ local dissipative terms which force a relaxation of the leftmost and the rightmost spins towards fully polarized target states ($\mathcal{D}^{L,R}(\rho^{L,R}) = 0$)

$$\rho^{L} = |\uparrow_u\rangle\langle\uparrow_u|\,, \qquad \rho^{R} = |\downarrow_v\rangle\langle\downarrow_v|$$

Open XXZ chain

The Non-equilibrium steady state (NESS) $\rho_\infty = \lim_{t\to\infty} \rho(t)$ satisfying the fixed point equation:

$$-i\,[H,\rho_\infty] + \mathcal{D}^L(\rho_\infty) + \mathcal{D}^R(\rho_\infty) \equiv \mathcal{L}(\rho_\infty) = 0$$

is unique.

Since the generators $(H, D^L, D^{L+}, D^R, D^{R+})$ generate, under multiplication and addition, the entire Pauli algebra $B(Fn)$ of the spin chain on N sites there is a unique NESS. See D. E. Evans, Comm. Math. Phys. 54, 293 (1977), H. Spohn, Lett. Math. Phys. 2, 33 (1977) and T. Prosen, Phys. Rev. Lett. 106, 217206 (2011).

Exact MPA NESS: Main idea

Stationary density matrix ρ satisfying[1] $\quad i\,[H,\rho] = \mathcal{D}^L(\rho) + \mathcal{D}^R(\rho)$

Matrix Product Ansatz (MPA): $\quad \rho = SS^\dagger / tr(SS^\dagger)$ with $S \in \mathbb{C}^{2^{\otimes N}}$

$$S = \langle\,\phi\,|\Omega^{\otimes N}|\,\psi\,\rangle\ , \qquad \Omega \equiv \sigma^s \otimes A_s = \begin{pmatrix} A_1 & A_+ \\ A_- & A_2 \end{pmatrix}\ ; \quad \Omega \in \mathbb{C}^2 \otimes \mathfrak{R}$$

Demand for local divergence condition

$$[\,h\,,\,\Omega \otimes \Omega\,] = \Xi \otimes \Omega - \Omega \otimes \Xi$$

with $\Xi = \sigma^s \otimes E_s$, such that the effect of the unitary bulk dynamics is expelled at the boundaries

$$[\,H\,,\,\Omega^{\otimes N}\,] = \underbrace{\Xi \otimes \Omega^{\otimes N-1}}_{left} - \underbrace{\Omega^{\otimes N-1} \otimes \Xi}_{right}$$

[1] T. Prosen, Phys. Rev. Lett. 107, 137201 (2011).
D. Karevski, V. Popkov and G. Schuetz, Phys. Rev. Lett. 110, 047201 (2013).

Auxiliary algebra

The local divergence condition

$$[\, h\, ,\, \Omega \otimes \Omega\,] = \Xi \otimes \Omega - \Omega \otimes \Xi$$

turns out to an algebraic conditioning on the auxiliary operators $\Rightarrow U_q(SU(2))$ quantum algebra

$$A_\pm =: i\alpha S_\pm , \quad Q =: \lambda q^{S_z}$$

with

$$[\, S_+\, ,\, S_-\,] = \frac{q^{2S_z} - q^{-2S_z}}{q - q^{-1}}\, , \qquad q^{S_z} S_\pm = q^{\pm 1} S_\pm q^{S_z}$$

Irreducible representation of the auxiliary algebra

Irreducible representation

$$S_z = \sum_{k=0}^{\infty} (p-k) |\, k \,\rangle\langle\, k \,|$$

$$S_+ = \sum_{k=0}^{\infty} [k+1]_q |\, k \,\rangle\langle\, k+1 \,|$$

$$S_- = \sum_{k=0}^{\infty} [2p-k]_q |\, k+1 \,\rangle\langle\, k \,|$$

where $p \in \mathbb{C}$ is an arbitrary complex parameter (finite-dimensional at $2p \in \mathbb{N}$).

With $[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$

Example: isotropic point ($\Delta = (q+q^{-1})/2 = 1$) $\Rightarrow SU(2)$

$$q = 1 \quad \Rightarrow \qquad \Omega = i \begin{pmatrix} S_z & S_+ \\ S_- & -S_z \end{pmatrix}, \quad \Xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The local divergence condition implies

$$[\, H \, , \, \Omega^{\otimes N} \,] = \underbrace{\Xi \otimes \Omega^{\otimes N-1}}_{\Xi_1} - \underbrace{\Omega^{\otimes N-1} \otimes \Xi}_{\Xi_N}$$

The stationary eq. $\mathcal{D}(SS^\dagger) = i[H, SS^\dagger]$ can thus be split into two parts

$$\mathcal{D}^L(SS^\dagger) = i(\Xi_1 S^\dagger - S\Xi_1^\dagger) \qquad (= i[H, SS^\dagger]\big|_{Left})$$

and

$$\mathcal{D}^R(SS^\dagger) = -i(\Xi_N S^\dagger - S\Xi_N^\dagger) \qquad (= i[H, SS^\dagger]\big|_{Right})$$

Explicit solution at the isotropic point

Twisting Polarization with angle θ

Take the polarization

$$\vec{n}_L = (1, 0, 0) \qquad \vec{n}_R = (\cos\theta, \sin\theta, 0) \ ,$$

Representation parameter

$$p = \frac{i}{\Gamma - 2ih}$$

and with the coherent state

$$|R_\theta(p)\rangle = \sum_{n=0}^{\infty} \frac{(-\cot\frac{\theta}{2})^n (S_-)^n}{n!} |0\rangle = \sum_{n=0}^{\infty} \frac{(-\cot\frac{\theta}{2})^n}{n!} \binom{2p}{n} |n\rangle$$

Solution is $\rho = SS^\dagger / tr(SS^\dagger)$ with $S = \langle 0 | \Omega^{\otimes N} | R_\theta(p) \rangle$.

Local observables

It is convenient to rewrite ρ by doubling the auxiliary space:

$$\rho = \langle 0, 0| \, \mathbf{\Omega}(p)^{\otimes N} \, |R_\theta, R_\theta^*\rangle$$

where $\mathbf{\Omega}(p) = \Omega(p) \otimes_{au} \Omega^T(-p)$, lives in $\mathbb{C}^2 \otimes \mathfrak{R} \otimes \mathfrak{R}$ and transposition Ω^T is done in the physical space only.
Let us introduce polarization operators defined on the auxiliary space $\mathfrak{R} \otimes \mathfrak{R}$:

$$B_\alpha(p) = Tr(\sigma^\alpha \mathbf{\Omega}(p))$$

Local observables

Observables are expressed in terms of the polarization operators B_α

In terms of the B-operators, the normalization factor becomes

$$Z(N) \equiv Tr(\rho) = Tr \langle 0,0| \mathbf{\Omega}(p)^{\otimes N} |R_\theta, R_\theta^*\rangle = \langle 0,0| \left(Tr(\mathbf{\Omega}(p))\right)^N |R_\theta, R_\theta^*\rangle$$
$$= \langle 0,0| \left(Tr(\sigma^0 \mathbf{\Omega}(p))\right)^N |R_\theta, R_\theta^*\rangle = \langle 0,0| B_0^N |R_\theta, R_\theta^*\rangle$$

Magnetization

$$M_{k,N}^\alpha = \langle \sigma_k^\alpha \rangle = \frac{\langle 0,0| B_0^{k-1} B_\alpha B_0^{N-k} |R_\theta, R_\theta^*\rangle}{Z(N)}.$$

Magnetization flux

$$J_k^\alpha = 2\varepsilon_{\alpha\beta\gamma} \langle \sigma_k^\beta \sigma_{k+1}^\gamma \rangle = 2\varepsilon_{\alpha\beta\gamma} \frac{\langle 0,0| B_0^{k-1} B_\beta B_\gamma B_0^{N-k-1} |R_\theta, R_\theta^*\rangle}{Z(N)}$$

Magnetization

Using the $SU(2)$ commutation rules for S_α one obtains the exact equation

$$(-2M^\alpha_{k+1,N+1} + M^\alpha_{k+2,N+1} + M^\alpha_{k,N+1})\frac{Z(N+1,\theta)}{Z(N,\theta)} + 2(M^\alpha_{k,N} + M^\alpha_{k+1,N}) = 8p^2 M^\alpha_{k,N-1}\frac{Z(N-1,\theta)}{Z(N,\theta)} \ .$$

In the large Γ and size limit and $\Gamma \gg \frac{\theta^2}{N}$, one has in the continuum limit

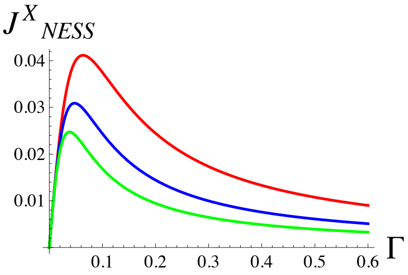
$$\partial_x^2 M^\alpha(x) + \theta^2 M^\alpha(x) = 0 \ ,$$

and the boundary conditions $M^\alpha(0) = \sigma^\alpha_{\text{target(L)}}$, $M^\alpha(1) = \sigma^\alpha_{\text{target(R)}}$ where $\sigma^a_{\text{target(L,R)}}$ are the targeted boundary magnetizations.

Magnetization flux

In terms of the MPA the steady magnetization currents are given by

$$j^x(N) = -8ip\frac{Z(N-1,\theta)}{Z(N,\theta)}\,, \qquad j^y(N,\theta) = -\cot\frac{\theta}{2}\times j^x(N,\theta)$$



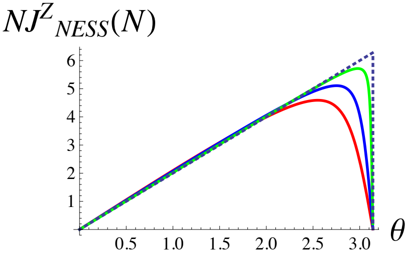
For sufficiently large Γ, N

$$j^x(N)\big|_{\Gamma\gg\frac{\theta^2}{N},N\gg1} = \frac{2}{\Gamma}\frac{\theta^2}{N^2} + O\left(N^{-3}\right),$$

Magnetization currents

Different scaling for the z-component $j^z(N,\theta)$ leading to

$$j^z(N)\big|_{\Gamma \gg \frac{\theta^2}{N}, N \gg 1} = \frac{2\theta}{N} + O\left(\frac{1}{N^2}\right)$$



Notice that the limits $N \to \infty$ and $\theta \to \pi$ do not commute ($Nj^z(N,\pi) = 0 \;\; \forall N$)

Conclusion

- Explicit expressions for one- and two-point observables (magnetization currents and magnetization profiles) in the MPA steady state.
- Easy to compute numerically (linear growth of the space size instead of exponential)
- The magnetization flux is qualitatively different in the direction parallel to the twisting plane, and in the orthogonal direction.
- Spin valve like effect when the boundary fields are competing with the reservoir polarizations (with Gabriel Landi)