Random transverse-field Ising chain with long-range interactions

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in collaboration with

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Introduction

- Effect of long-range forces in critical phenomena
  - Classical systems (Ising model)

$$\mathscr{H} = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i^z\sigma_j^z$$
$$J_{ij} \sim |i-j|^{-(d+\sigma)}$$

    * d: spatial dimension
    * $\sigma > 0$, to ensure extensivity
  - Different critical regimes
    * $\sigma > \sigma_U$ fast decay $\rightarrow$ short-range critical behaviour
    * $\sigma_L < \sigma < \sigma_U$ non-universal critical behaviour, σ dependent critical exponents
    * $\sigma_L > \sigma$ mean-field critical behaviour
  - Quantum systems at $T = 0$ (quantum Ising model)

$$\mathscr{H} = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i^z\sigma_j^z - \sum_i h\sigma_i^x$$

  - Examples
    * Trapped ions in optical lattices - dipolar interactions $d+\sigma = 3$
    * Quantum Ising chain with ohmic dissipation $\sigma = 1$

      dynamical exponent: $z \approx 2$, $\nu \approx 0.63$
  - In this talk: quantum Ising chain
    * with long-range interactions
    * with quenched random variables

Random transverse-field Ising model

$$\mathcal{H} = -\sum_{i \neq j} \frac{b_{ij}}{r_{ij}^{\alpha}} \sigma_i^x \sigma_j^x - \sum_i h_i \sigma_i^z$$

- with $\alpha = d + \sigma$
- we take $d = 1$ (chain)
- and $\sigma > 0$ (extensive energy)
- **b_{ij} and h_i i.i.d. random numbers**
- in $d = 3$ possible relation with $LiHo_xY_{1-x}F_4$

Methods of investigation

- Strong disorder renormalization group (SDRG)
  - Numerical SDRG analysis
  - Primary model and analytical solution
- Interpretation through extreme value statistics
- Monte Carlo simulation of the contact process

Strong disorder RG approach

(Ma, Dasgupta, Hu 1979, D.S. Fisher 1992, F.I. & Monthus 2005)

- sort the couplings and transverse fields, $\Omega = \max(J_{ij}, h_i)$
- eliminate the largest parameter - reduce the number of spins by one
- generate new effective parameters between the remaining spins
  - $\boxed{\Omega = J_{ij}}$ i and j form a ferromagnetic cluster - aggregation
    in an effective field: $\tilde{h}_{ij} = \frac{h_i h_j}{J_{ij}}$
    having a moment: $\tilde{\mu}_{ij} = \mu_i + \mu_j$
  - $\boxed{\Omega = h_i}$ site i is decimated out - annihilation
    new effective couplings between sites j and k: $\tilde{J}_{jk} = \frac{J_{ij} J_{ik}}{h_i}$
- repeate the transformation
- at the fixed point Ω is reduced to $\Omega^* = 0$.
- final result: set of connected clusters with different masses, μ,
  decimated at different energies, Ω.

Short-range model
Exact results in 1d

Infinite disorder fixed point (IDFP)

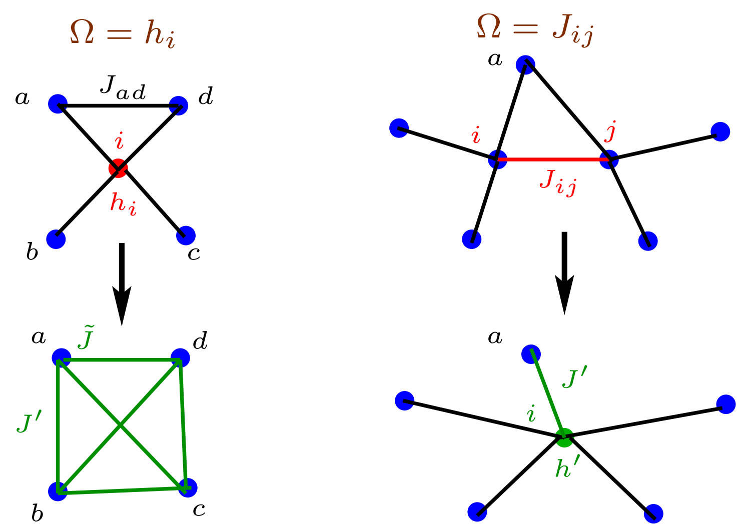
- Critical point: $\overline{\log(J)} = \overline{\log(h)}$
  - control parameter: $\theta = \overline{\log(h)} - \overline{\log(J)}$, $\theta_c = 0$
  - distribution of the effective parameters is logarithmically broad
  - asymptotically exact decimation steps
  - strongly anisotropic scaling $\ln \Omega_L \sim L^{\psi}$, $\boxed{\psi = 1/2}$
  - large effective spin clusters
    * size: $\xi \sim |\theta - \theta_c|^{-\nu}$ $\boxed{\nu = 2}$
    * moment: $\mu \sim [\ln(\Omega_0/\Omega)]^{\phi} \sim L^{d_f}$
      $d_f = \phi\psi = d - x$
      $\boxed{\phi = \frac{\sqrt{5}+1}{2}}$, $\boxed{x = \frac{3-\sqrt{5}}{4}}$

- In the Griffiths phase: $\theta - \theta_c > 0$
  - non-singular static behaviour: $\xi < \infty$
  - singular dynamical behaviour:
    $\Omega \sim L^{-z}$ dynamical exponent: $z = z(\theta)$
    $\chi(T) \sim T^{-1+d/z}$, $\quad C_v(T) \sim T^{d/z}$

Numerical implementation of the RG procedure for $D>1$

Phys. Rev. B**83**, 174207 (2011), J. Phys.: Cond. Mat. **23**, 404204 (2011)

- differences with the 1D procedure
  - change in the topology
  - application of the maximum rule (valid at IDFP)



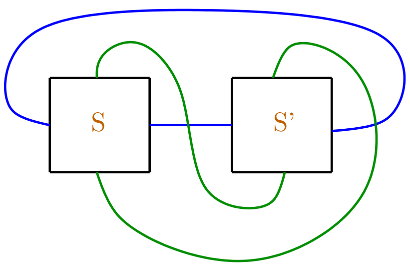
$J' = J_{ai}J_{bi}/h_i$
$\tilde{J} = \max(J_{ad}, J_{ai}J_{di}/h_i)$

$J' = \max(J_{ai}, J_{aj})$
$h' = h_i h_j / J_{ij}$

- problems with the naïve implementation
  - h-decimations induce several new bonds
  - the lattice transforms to a fully connected cluster
  - slow algorithm: for N sites works in $\mathcal{O}(N^3)$ time
- improved algorithm
  - concept of local maxima - which can be decimated independently
  - concept of optimal RG trajectory - along which the time is minimal
  - filtering out irrelevant bonds - getting rid of latent couplings
  - improved algorithm works in $\mathcal{O}(N \log N)$ time

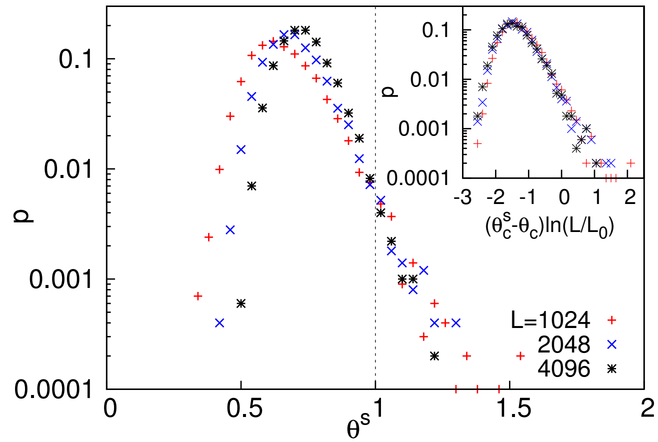
Numerical SDRG study of the long-range model

- Finite-size critical points - $\theta_c(S,L)$
  - two-copies of the same sample (S and S') are coupled together



  - continuously increase θ and monitor the clusters, which are built of identical sites in the copies
  - at $\theta_c(S,L)$ the last correlated cluster disappears, thus for $\theta > \theta_c(S,L)$ we are in the paramagnetic phase

- **Distribution of pseudocritical points**



- Finite-size scaling
  - shift of the mean: (conv. scaling)

$$\left|\theta_c - \overline{\theta_c}(L)\right| \sim 1/\ln L \qquad (L^{-1/\nu_s})$$

  - width of the distribution:

$$\Delta\theta_c(L) \sim 1/\ln L \qquad (L^{-1/\nu_w})$$

  - KT-like scaling of the correlation length

$$\boxed{\xi \sim \exp\left(\text{const}/|\theta-\theta_c|\right)} \qquad \xi \sim |\theta-\theta_c|^{-\nu}$$

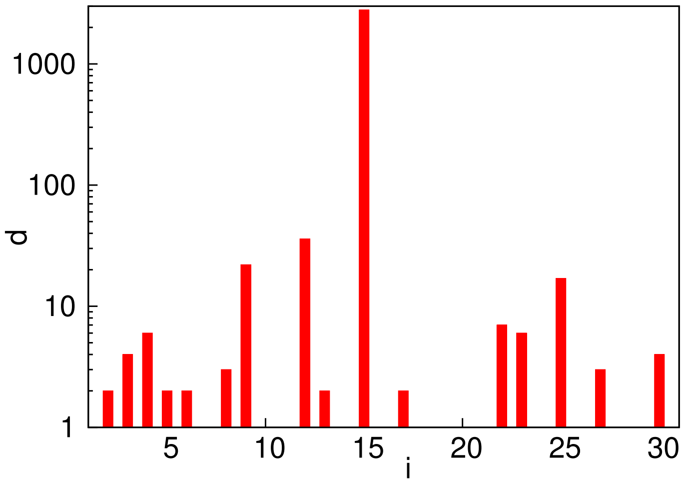
Numerical SDRG study

- ratio of the frequency of the bond and field decimations at the critical point:

$$r_{\theta_c}(L) = \frac{N^{\#}_{bond}(L)}{N^{\#}_{field}(L)} \sim 1/\ln^2(L)$$

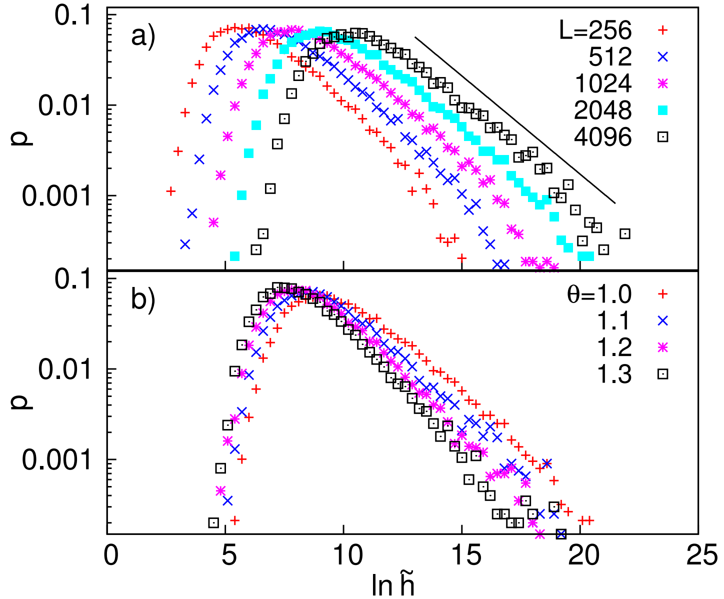
- average magnetic moment of the last remaining cluster is

$$\boxed{\mu(L) \sim \ln^2 L}$$



structure of the largest non-decimated cluster: $\mu = 32$ and $L = 8192$

- energy scaling



$$\varepsilon \sim L^{-z(\theta)}$$

- strong disorder fixed point with $\boxed{z_c \equiv z(\theta_c) \simeq \alpha}$

- in the Griffiths phase $z(\theta) < z_c$

Analytical derivation of the results - Primary model

- observations in the RG procedure

  1. almost always transverse fields are decimated
  2. after a field decimation, the maximum rule leads almost always to $\tilde{J}_{jk} = J_{jk}$
  3. the extension w_i of (non-decimated) clusters are typically much smaller than the distances between them

- construction of the primary model

  – take $b_{ij} = b = 1$, but let h_i random

  – according to $2)$ we have from the additivity of the bond lengths:

$$\tilde{J}_{i-1,i+1}^{-1/\alpha} = J_{i-1,i}^{-1/\alpha} + J_{i,i+1}^{-1/\alpha} + w_i$$

  and w_i is neglected due to $3)$.

  – Using reduced variables

$$\zeta = \left(\frac{\Omega}{J}\right)^{1/\alpha} - 1$$

$$\beta = \frac{1}{\alpha} \ln \frac{\Omega}{h}$$

  – the RG equations reads:

$$\tilde{\zeta} = \zeta_{i-1,i} + \zeta_{i,i+1} + 1$$

$$\tilde{\beta} = \beta_i + \beta_{i+1}$$

- equivalent to a $1d$ disordered $O(2)$ quantum rotor model ($1d$ disordered bosons) (E. Altman, Y. Kafri, A. Polkovnikov, and G. Refael (2004)) with

  – grain charging energy $U_i \leftrightarrow J_{i,i+1}^{1/\alpha}$

  – Josephson coupling $\mathscr{J}_{i,i+1} \leftrightarrow h_i^{1/\alpha}$

- Note, that site and bond variables are interchanged in the two problems.

Solution of the primary model

- Let us change the log-energy-scale: $\Gamma \equiv \frac{1}{\alpha} \ln \frac{\Omega_0}{\Omega} \to \Gamma + \mathrm{d}\Gamma$

- the distributions $g_\Gamma(\beta)$ and $f_\Gamma(\zeta)$ will follow the equations:

$$\frac{\partial g_\Gamma(\beta)}{\partial \Gamma} = \frac{\partial g_\Gamma(\beta)}{\partial \beta} + f_0(\Gamma) \int d\beta' g_\Gamma(\beta') g_\Gamma(\beta - \beta') + g_\Gamma(\beta)[g_0(\Gamma) - f_0(\Gamma)]$$

$$\frac{\partial f_\Gamma(\zeta)}{\partial \Gamma} = (\zeta + 1) \frac{\partial f_\Gamma(\zeta)}{\partial \zeta} + g_0(\Gamma) \int d\zeta' f_\Gamma(\zeta') g_\Gamma(\zeta - \zeta' - 1) + f_\Gamma(\zeta)[f_0(\Gamma) + 1 - g_0(\Gamma)],$$

- the fixed-point solutions ($\Gamma \to \infty$) are:

$$g_\Gamma(\beta) = g_0(\Gamma) e^{-g_0(\Gamma)\beta},$$
$$f_\Gamma(\zeta) = f_0(\Gamma) e^{-f_0(\Gamma)\zeta}$$

- which satisfy the ordinary differential equations

$$\frac{dg_0(\Gamma)}{d\Gamma} = -f_0(\Gamma) g_0(\Gamma),$$
$$\frac{df_0(\Gamma)}{d\Gamma} = f_0(\Gamma)(1 - g_0(\Gamma)),$$

- thus $f_0(\Gamma) = g_0(\Gamma) - \ln g_0(\Gamma) - 1 + \varepsilon$
  $\varepsilon = -a + \ln(1 + a)$

- The boundary conditions in the $\Gamma \to \infty$ limit:

$$f_0(\Gamma) \to 0,$$
$$g_0(\Gamma) \to 1 + a$$

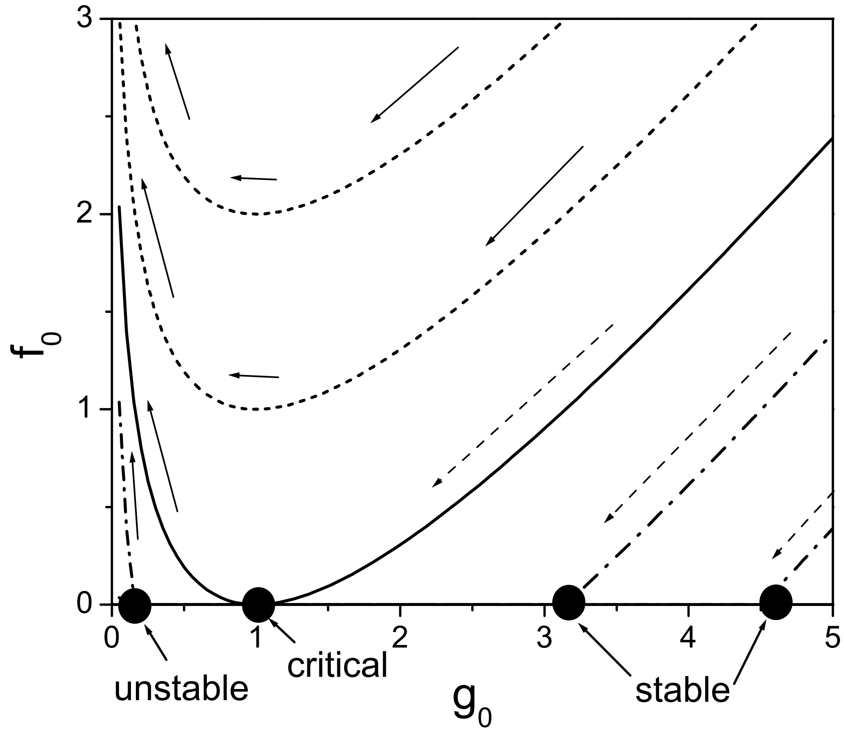
  - paramagnetic phase $a > 0$
  - critical point $a = 0$

- The solutions in leading order in Γ:

$$g_0(\Gamma) \simeq 1 + a\coth[(\Gamma + C)a/2],$$
$$f_0(\Gamma) \simeq \frac{a^2}{2\sinh^2[(\Gamma + C)a/2]},$$

Solution of the primary model

RG-flow



- the fraction of non-decimated sites n satisfies

$$\frac{dn}{d\Gamma} = -n(g_0 + f_0)$$

- the length scale, $l = 1/n$ is given by:

$$l \simeq e^{\Gamma} a^{-2} \sinh^2[(\Gamma + C)a/2] \sim \left(\frac{\Omega_0}{\Omega}\right)^{\frac{1+a}{\alpha}}$$

- the dynamical exponent

$$z = \alpha/(1+a)$$

- z continuously varies with a
- maximal, at the critical point: $z_c = \alpha$.
- in the vicinity of the critical point: $\Gamma a = C'$
- the characteristic length scale $\xi \sim \exp(\Gamma) \sim \exp(C'/a)$
- explanation of the magnetization and the entanglement entropy

Interpretation through extreme value statistics (EVS)

- chain of length L, renormalized to a cluster of μ spins

- its effective field is given by:

$$\tilde{h} \sim \prod_{i=1}^{\mu} h_i / \prod_{i=1}^{\mu-1} J_i, \qquad J_i = b_i r_i^{-\alpha}$$

- limit distribution of the fields

$$g(h) \sim h^{-1+(1+a)/\alpha}$$

- h_i is the *smallest* out of r_i variables

- according to EVS $\rightarrow$ $h_i \simeq \kappa_i r_i^{-\alpha/(1+a)}$

- κ_i follow Fréchet statistics

$$P(\kappa) = \alpha^{-1} \kappa^{1/\alpha - 1} \exp(-\kappa^{1/\alpha})$$

- the asymptotic behavior of $\tilde{h}$ is different
  - $\overline{\ln h} > \overline{\ln J}$ (paramagnet)
  - $\overline{\ln h} < \overline{\ln J}$ (ferromagnet)

- criticality: $\boxed{a = 0}$ and $\boxed{\overline{\ln b} = \overline{\ln \kappa}}$

- at the critical point:
  - $\tilde{h} \sim \prod_{i=1}^{\mu} \kappa_i / \prod_{i=1}^{\mu-1} b_i \sim \exp(-c\mu^{1/2})$ from the central limit theorem
  - from energy scaling: $\tilde{h} \sim L^{-\alpha}$ which implies $\boxed{\mu \sim \ln^2 L}$

- in the paramagnetic phase $0 < a \ll 1$
  - $\xi(a)$ is the length of the longest decimated bond, r_l
  - $J_l / h_l \sim (b_l r_l^{-\alpha}) / (r_l^{-\alpha/(1+a)} \kappa_l) \sim r_l^{-\alpha a} / \kappa_l > 1$
  - thus the smallest value: $\kappa_l < \xi^{-\alpha a}$

$$\mathrm{Prob}(\kappa_l < \xi^{-\alpha a}) = \int_0^{\xi^{-\alpha a}} P(\kappa) \mathrm{d}\kappa$$

$$= 1 - e^{-\xi^{-a}} \sim \xi^{-a} = e^{-C'} = \mathcal{O}(1)$$

  - $\boxed{\xi \sim \exp(C'/a) \sim \exp\left(\mathrm{const}/|\theta - \theta_{\mathrm{c}}|\right)}$

Equivalent dynamical process
Long-range epidemic spreading in a random environment

- Contact process (CP) with long-range activation (SIS model)

- agents at a lattice
  - $\sigma_i = S$ susceptible (healthy)
  - $\sigma_i = I$ infected (ill)

- processes
  - $I_i S_j \to I_i I_j$, with rate $\lambda_{ij} = \Lambda_{ij} |i-j|^{-(d+\sigma)}$
  - $I_i \to S_i$, with rate μ_i

- stationary phases
  - $\overline{\ln \mu} - \overline{\ln \Lambda} = \theta < \theta_c$ active phase
  - $\overline{\ln \mu} - \overline{\ln \Lambda} = \theta > \theta_c$ inactive phase

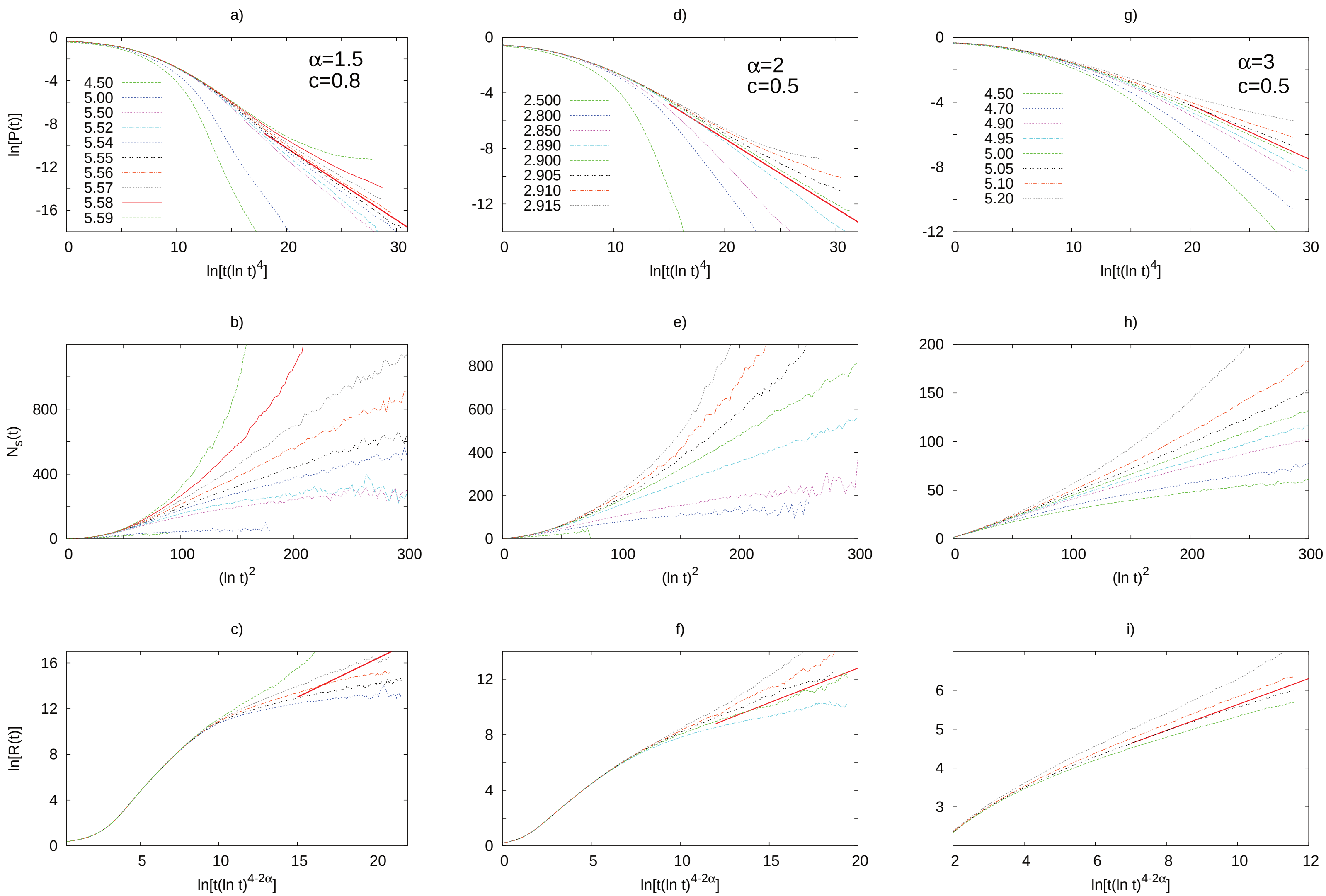
- SDRG procedure
  - $\Omega = \lambda_{ij} \to \tilde{\mu}_{ij} \simeq \frac{2\mu_i \mu_j}{\Omega}$
  - $\Omega = \mu_i \to \tilde{\lambda}_{jk} \simeq \frac{\lambda_{ji}\lambda_{ik}}{\Omega}$

- Relation with the quantum Ising model
  - $I_i, S_i \to \sigma_i^z = \pm 1$
  - $\lambda_{ij} \to J_{ij}$
  - $\mu_i \to h_i$
  - the extra factor 2 is irrelevant
  - Identical critical behavior for short-range interactions

Numerical simulation of the random CP

- Simulation details
  - diluted lattice: $0 < c < 1$ fraction of sites removed
  - active site with prob. $1/(1+\lambda)$ made inactive
  - with prob. $\lambda/(1+\lambda)$ a random variable r is taken from $f(r) = (\alpha - 1)r^{-\alpha}$
  - if a site at distance r is inactive, it will be activated
  - $\lambda > \lambda_c$ ($\lambda < \lambda_c$) active (inactive) phase

- Starting from a single active site we measure
  - $P(t) \sim t^{-d/z}$ average surviving probability
  - $R(t) \sim t^{1/z_c}$ spatial extension of the infected cluster
  - $N_s(t) \sim (\ln t)^2$ average number of infected sites in surviving trials
  - extra multiplicative logarithmic corrections at the critical point


a)
α=1.5
c=0.8
4.50
5.00
5.50
5.52
5.54
5.55
5.56
5.57
5.58
5.59
ln[P(t)]
ln[t(ln t)^4]
d)
α=2
c=0.5
2.500
2.800
2.850
2.890
2.900
2.905
2.910
2.915
g)
α=3
c=0.5
4.50
4.70
4.90
4.95
5.00
5.05
5.10
5.20
b)
e)
h)
N_s(t)
(ln t)^2
c)
f)
i)
ln[R(t)]
ln[t(ln t)^{4-2α}]

Conclusions

- critical behaviour is controlled by a strong disorder fixed point
  - dynamical exponent $z_c = \alpha$
  - KT-like scaling: $\xi \sim \exp(\mathrm{const}/|\theta - \theta_\mathrm{c}|)$
  - critical cluster is a logarithmic fractal: $\mu_L \sim (\ln L)^2$
- Griffiths region in the inactive phase with $z < \alpha$
- identical behaviour in models with a discrete order parameter (CP, Potts model, etc.)
  - numerical verification for the LR random contact process
- Further questions:
  - What is expected in higher dimensions?
  - Disorder dependent cross-over effects?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