UNIVERSITÄT DUISBURG ESSEN

*Offen* im Denken

# Casimir force scaling functions in 2d Ising systems with open boundaries: The importance of corner contributions

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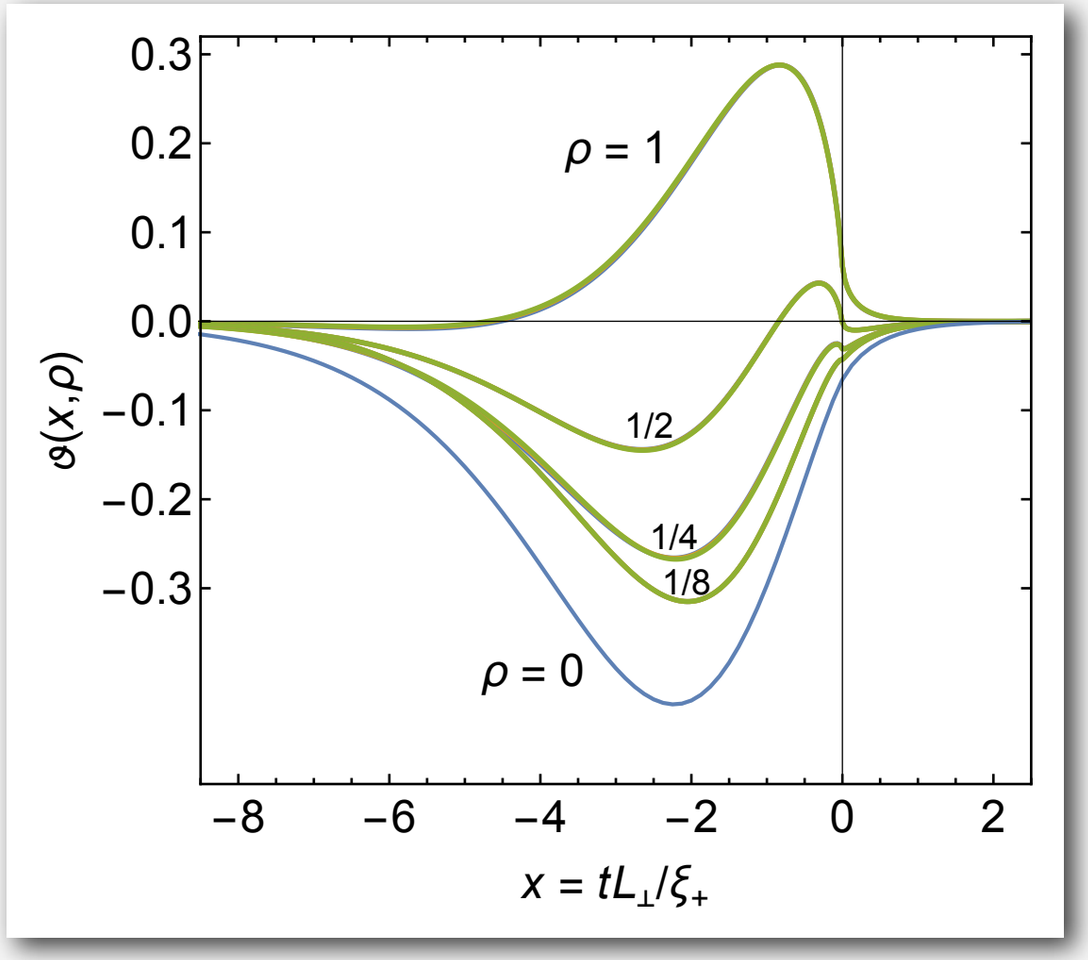
- Free energy contributions and Casimir forces
- Finite size quantities
- Bulk quantities
- *q*-Products
- Results

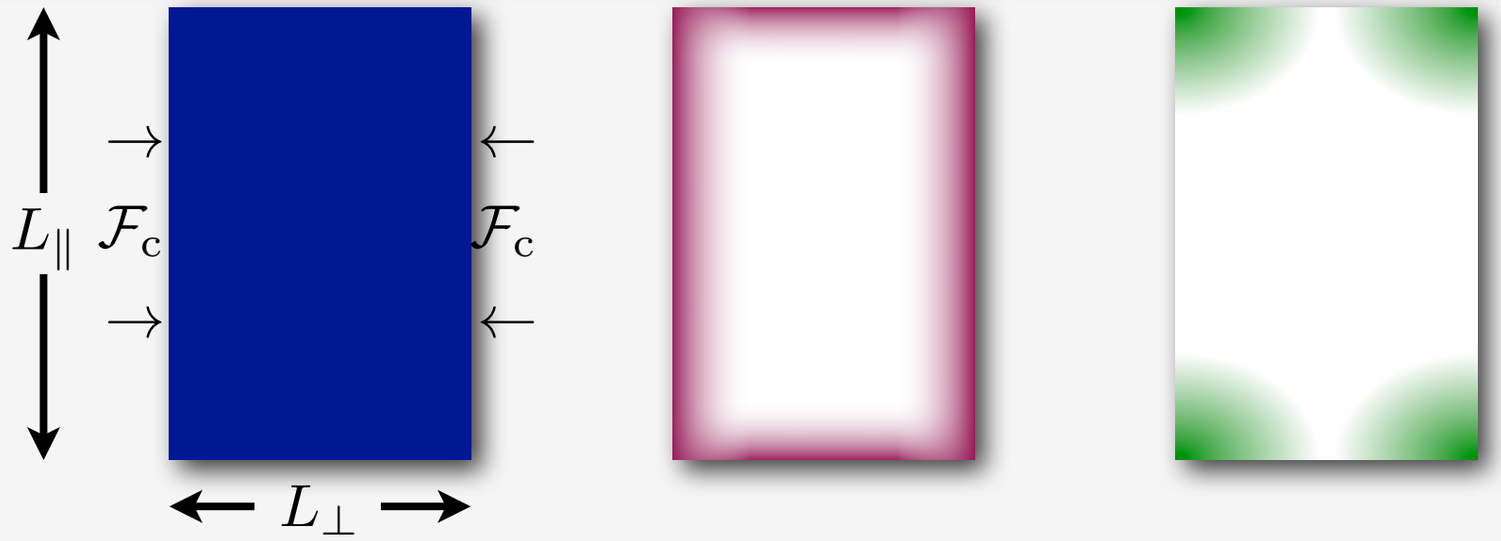



Collaboration: Felix M. Schmidt (UDE), Jesper L. Jacobsen (ENS, Paris)

# Free energy of finite 2D Ising model with OBC

$$F(t, L_\perp, L_\parallel) = \overbrace{L_\parallel L_\perp f_{\rm b}(t) + (L_\perp + L_\parallel) f_{\rm s}(t) + f_{\rm c}(t)}^{F_\infty(t, L_\perp, L_\parallel)} + \delta F(t, L_\perp, L_\parallel)$$

$t \simeq \frac{T}{T_{\rm c}} - 1$




residual free energy

$$\delta F(\cdot) = F(\cdot) - F_\infty(\cdot)$$

critical Casimir force
(per area, in $\perp$ direction)

$$\beta \mathcal{F}_{\rm C}(\cdot) = -\frac{1}{L_\parallel} \frac{\partial\, \delta F(\cdot)}{\partial L_\perp} = -\frac{1}{L_\parallel} \frac{\partial F(\cdot)}{\partial L_\perp} + f_{\rm b}(t) + \frac{f_{\rm s}(t)}{L_\parallel}$$

universal scaling fns
with scaling variables

$$\delta F(\cdot) \simeq \Theta(x, \rho) \quad \beta F_{\rm C}(\cdot) \simeq L_\perp^{-d} \vartheta(x, \rho)$$
$$x = tL/\xi_+ \quad \rho = L_\perp / L_\parallel$$

# Finite size part $F(T, L_\perp, L_{||})$

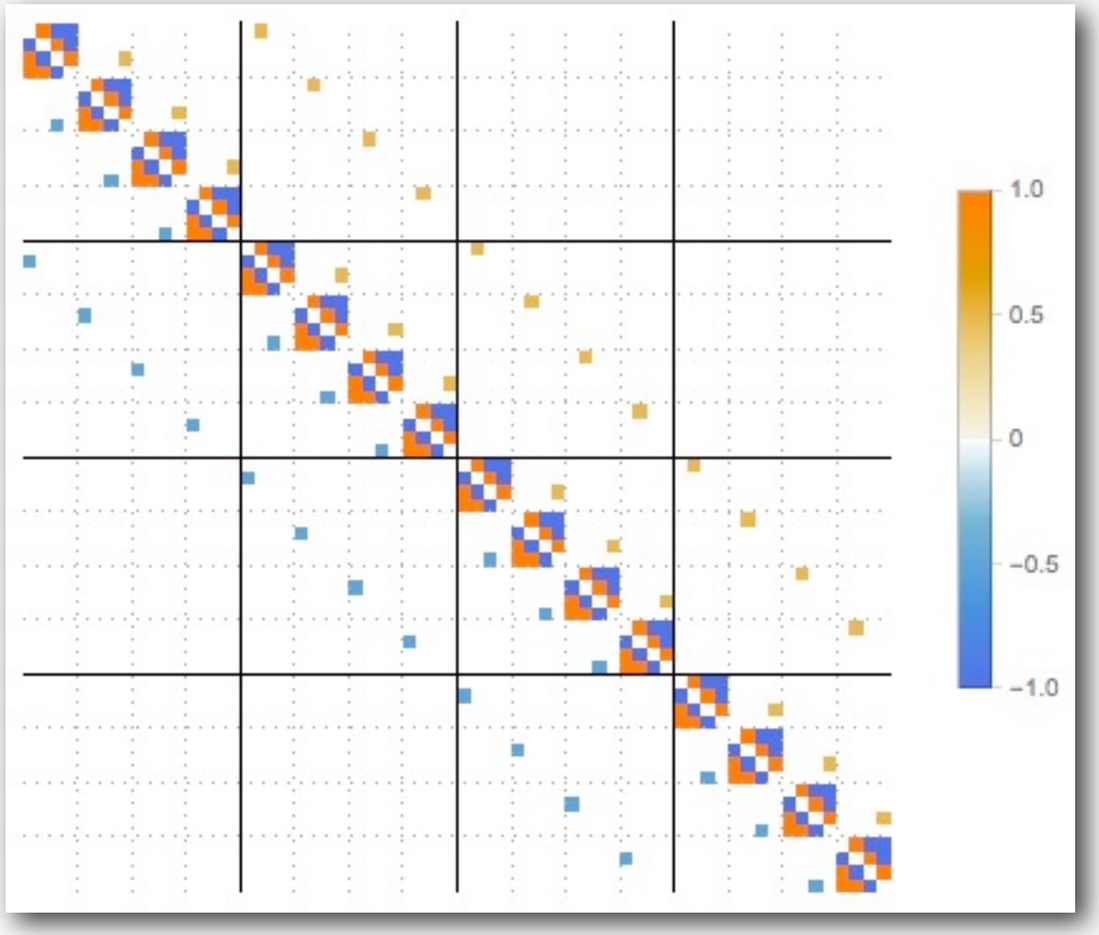
- no closed form expression known due to lack of translational invariance
- use dimer mapping of McCoy & Wu [$Z \sim \mathrm{Pf}(\mathbf{A})$] and division free Pfaffian algorithm for sparse matrices (four elements ≠ 0 per row)
- 4x4 spins: $64^2$ matrix

$$\mathrm{Pf}(\mathbf{A}) = 1 + 9z^4 + 12z^6 + 50z^8 + 92z^{10} + 158z^{12} + 116z^{14} + 69z^{16} + 4z^{18} + z^{20}$$

with $z = \tanh K$

- 25x25 spins: $2500^2$ matrix
  $\mathrm{Pf}(\mathbf{A})$: $24^2$-th degree polynomial in $z^2$, largest coefficient 173 digits

$\mathbf{A} =$




$\mathrm{Pf}(\mathbf{A}) = 1 + \ldots + 113675851990029329109814319606681535835858799673627033758890549639359542745334076689587380611279570253328150101033132836884170254135189927376689466194656775763621895726822 40\, z^{600} + \ldots + 8z^{1152}$

# Well known bulk quantities of the 2D Ising model

- bulk free energy density ($k = 2\tanh 2K/\cosh 2K$, regular part red)

$$f_{\rm b} = -\ln(2\cosh 2K) + \frac{k^2}{16}\,{}_4F_3\left(\begin{matrix}1,\frac{3}{2},\frac{3}{2},1\\2,2,2\end{matrix}\middle|\,k^2\right)$$

- surface free energy density ($z = \tanh K$)

$$f_{\rm s} = -\frac{1}{4}\ln(1-z^2) - \frac{1}{4\pi}\int_{-\pi}^{\pi} {\rm d}\theta \ln\left[\frac{1}{2}\left(1 - \frac{\sqrt{z^4-1+2z(1+z^2)\cos\theta-4iz^2\sin\theta}}{\sqrt{z^4-1+2z(1+z^2)\cos\theta+4iz^2\sin\theta}}\right)\right]$$

- corner free energy density not so well known, but [1]...

$$f_{\rm c}^{<} = -\ln\left[2\prod_{k=1}^{\infty}\frac{1}{1-q^{8k-4}}\frac{(1-q^{8k-6})^{4k}}{(1-q^{8k-2})^{4k-4}}\left(\frac{1-q^{4k-1}}{1-q^{4k-3}}\right)^{8k-4}\right]$$

with $e^{-2K} = \sqrt{q}\prod_{k=1}^{\infty}\frac{(1-q^{8k-7})(1-q^{8k-1})}{(1-q^{8k-5})(1-q^{8k-3})}$

[1] E. Vernier and J. L. Jacobsen, *Corner free energies ... for Ising ... models on the square ... lattices*, J. Phys. A: Math. Theor. 45, 045003 (2012)

# $q$-Products

- general form with *elliptic nome* $q$:

$$P(\{c_k\};q) = \prod_{k=1}^{\infty}(1-q^k)^{c_k}$$

- observation/conjecture (Jacobsen [2]):
  partition functions $Z_{\mathrm{b}}^{\gtrless}$, $Z_{\mathrm{s}}^{\gtrless}$, $Z_{\mathrm{c}}^{\gtrless}$ can be written as $q$-products,
  with **periodic** $c_k$ and short period (4,8,12,16) for many integrable models (Ising, Potts, Chromatic polynomial, FPL$^2$)

- $q$ is a natural variable, e.g., Ising: $z = \sqrt{q}\,\dfrac{(1-q)(1-q^7)(1-q^9)(1-q^{15})(1-q^{17})(1}{(1-q^3)(1-q^5)(1-q^{11})(1-q^{13})(1-q^{19})(}$

- also holds for order parameter, but not for susceptibility (Baxter)

- introduce handy notation:

$$\Pi(\mathbf{C}|q) = \prod_{k=1}^{\infty}(1-q^k)^{c_k} \quad \text{with} \quad c_k = \sum_{j=0}^{m-1} C_{j,k \bmod n}\, k^j$$

- then, e.g.: $z = \sqrt{q}\,\Pi\left(\begin{array}{cccccccc} 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{array}\middle| q\right)$

[2] J. L. Jacobsen J. Phys. A: Math. Theor. 43, 315002 (2010), arXiv:1005.3609

# Shut up and calculate

- use finite lattice method [3] to find series expansion for $Z_{\mathrm{b}}^{\gtrless}, Z_{\mathrm{s}}^{\gtrless}, Z_{\mathrm{c}}^{\gtrless}$

$$Z_{\mathrm{c}}^{>} = 1 + z^4 + 4z^6 + 22z^8 + 116z^{10} + 616z^{12} + 3284z^{14} + 17617z^{16} + 95076z^{18} + \ldots$$

- rewrite as a function of $q$

$$Z_{\mathrm{c}}^{>} = 1 + q^2 + 4q^4 + q^6 + 7q^8 + 3q^{10} + 19q^{12} + 9q^{14} + 27q^{16} + 5q^{18} + 56q^{20} + 35q^{22} + \ldots$$

- use inverse Euler transform [4] to get product

$$\prod_{k=1}^{n}(1-q^k)^{c_k}$$

$$Z_{\mathrm{c}}^{>} = \frac{(1-q^6)^3}{(1-q^2)^1(1-q^4)^3}\frac{(1-q^{14})^7}{(1-q^{10})^5(1-q^{12})^3}\frac{(1-q^{22})^{11}}{(1-q^{18})^9(1-q^{20})^3}\cdots$$

$$\begin{array}{rcccccccc} c_k = & 0, & -1, & 0, & -3, & 0, & 3, & 0, & \\ & 0, & 0, & -5, & 0, & -3, & 0, & 7, & 0, \\ & 0, & 0, & -9, & 0, & -3, & 0, & 11, & 0, \\ & 0, & 0, & -13, & 0, & -3, & 0, & 15, & 0, \quad \ldots \end{array}$$

- guess educated...

$$Z_{\mathrm{c}}^{>} = e^{-f_{\mathrm{c}}^{>}} = \Pi\left(\begin{array}{cccccccc|c} 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & q \end{array}\right)$$


[3] I. G. Enting, J.Phys.A:Math.Gen. 11, 563 (1978)
[4] http://mathworld.wolfram.com/EulerTransform.html

# Exact $q$-Products for 2d Ising model

$$\tanh K^{>} = e^{-2K^{<}} = \sqrt{q}\,\Pi\left(\begin{array}{cccccccc} 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{array}\middle| q\right)$$

$$\Pi(\mathbf{C}|q) = \prod_{k=1}^{\infty}(1-q^k)^{c_k}$$

**Bulk**

$$e^{-f^{>}_{\mathrm{b,sing}}} = \Pi\left(\begin{array}{cccccccc} 0 & 2 & -4 & 2 & 0 & 2 & -4 & 2 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\middle| q\right) = \Pi\left(\begin{array}{cccc} 0 & 2 & -4 & 2 \\ 0 & -1 & 0 & 1 \end{array}\middle| q\right)$$

$$e^{-f^{<}_{\mathrm{b,sing}}} = \frac{1}{\sqrt{q}}\Pi\left(\begin{array}{cccccccc} 0 & 0 & -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\middle| q\right)$$

**Surface**

$$e^{-f^{>}_{\mathrm{s,sing}}} = \Pi\left(\begin{array}{cccccccc} 0 & \frac{1}{4} & 1 & -\frac{1}{4} & -2 & -\frac{1}{4} & 1 & \frac{1}{4} \\ 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{array}\middle| q\right)$$

$$e^{-f^{<}_{\mathrm{s,sing}}} = \Pi\left(\begin{array}{cccccccccccccc} 0 & -\frac{1}{2} & \frac{3}{4} & \frac{1}{2} & -1 & \frac{1}{2} & -\frac{3}{4} & -\frac{1}{2} & 2 & -\frac{1}{2} & -\frac{3}{4} & \frac{1}{2} & -1 & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{8} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{8} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{8} & \frac{1}{2} \end{array}\middle| \sqrt{q}\right)$$

**Corner**

$$e^{-f^{>}_{\mathrm{c}}} = \Pi\left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \end{array}\middle| q\right) = \Pi\left(\begin{array}{cccc} 0 & 0 & -3 & 0 \\ 0 & -1 & 0 & 1 \end{array}\middle| q^2\right)$$

$$e^{-f^{<}_{\mathrm{c}}} = 2\Pi\left(\begin{array}{cccccccc} 0 & -2 & 3 & -2 & -1 & -2 & 3 & -2 \\ 0 & -2 & \frac{1}{2} & 2 & 0 & -2 & -\frac{1}{2} & 2 \end{array}\middle| q\right)$$

# Critical values for $q \longrightarrow 1$ aka. $t \longrightarrow 0$

- critical bulk free energy (well known)

$$f^{=}_{\mathrm{b,sing}} = \ln 2 - \frac{2C}{\pi}$$

- critical surface free energy (not known yet)

$$f^{=}_{\mathrm{s,sing}} = \frac{1}{4}\ln\left[4(1+\sqrt{2})\right] - 2\left[\zeta^{(1,0)}(-1,\tfrac{1}{8}) + \zeta^{(1,0)}(-1,\tfrac{3}{8}) - \zeta^{(1,0)}(-1,\tfrac{5}{8}) - \zeta^{(1,0)}(-1,\tfrac{7}{8})\right]$$

- critical corner free energy

$$f^{<}_{\mathrm{c,sing}} \simeq \frac{\pi^2}{16\ln q} - \frac{2C}{\pi} - \ln 2 \simeq \frac{\ln(-t)}{8} + const$$

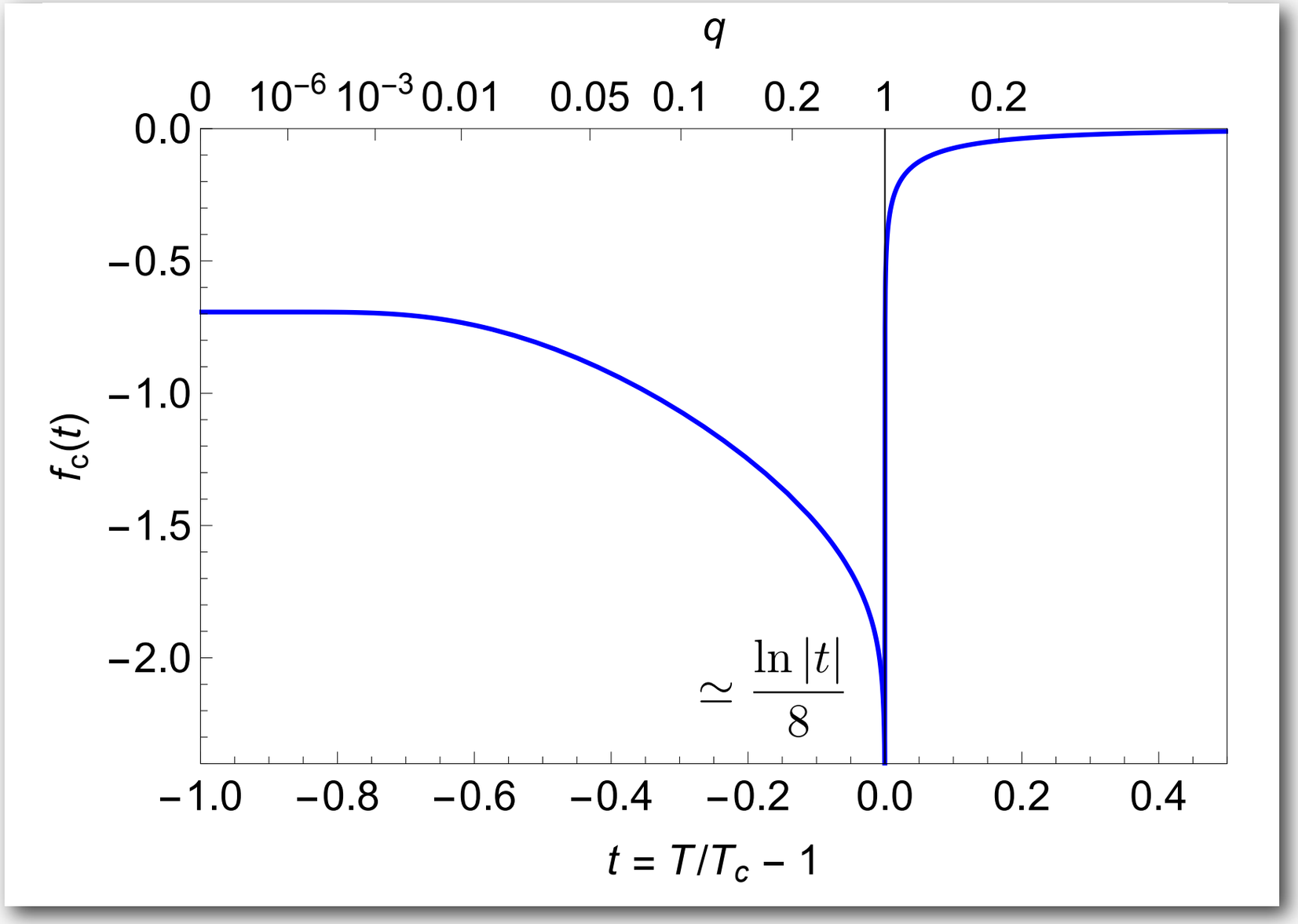
$$f^{>}_{\mathrm{c,sing}} \simeq \frac{\pi^2}{16\ln q} - \frac{2C}{\pi} + \frac{3\ln 2}{2} \simeq \frac{\ln t}{8} + const$$

- $f_{\mathrm{c}}$ universal at criticality, related to central charge $c$ [5]

[5] J.L. Cardy and I. Peschel, Nucl. Phys. B 300, 377 (1988)

# Exact corner free energy

$$e^{-f_c^<} = 2\Pi\left(\begin{matrix}-2 & 3 & -2 & -1 & -2 & 3 & -2 & 0\\ -2 & \frac{1}{2} & 2 & 0 & -2 & -\frac{1}{2} & 2 & 0\end{matrix}\middle| q\right) \qquad e^{-f_c^>} = \Pi\left(\begin{matrix}0 & -3 & 0 & 0\\ -1 & 0 & 1 & 0\end{matrix}\middle| q^2\right)$$



# Logarithmic contributions

- both $F$ and $F_\infty$ have logarithmic contributions near $T_\mathrm{c}$

$$F(t{=}0, L_\perp, L_\parallel) \simeq f_\mathrm{b}(0)\, L_\parallel L_\perp + f_\mathrm{s}(0)\,[L_\perp + L_\parallel] - \frac{1}{16}\ln(L_\perp L_\parallel) + \mathcal{O}(1)$$

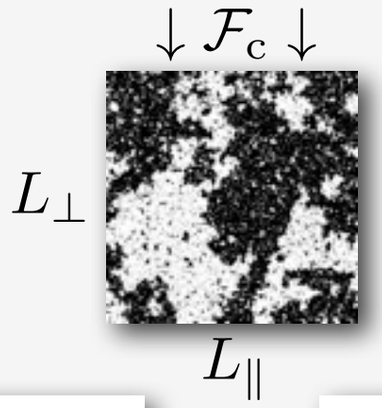
$$F_\infty(t, L_\perp, L_\parallel) \simeq f_\mathrm{b}(t)\, L_\parallel L_\perp + f_\mathrm{s}(t)\,[L_\perp + L_\parallel] + \frac{1}{8}\ln t + \mathcal{O}(1)$$

- contributions combine to scaling variable $x$

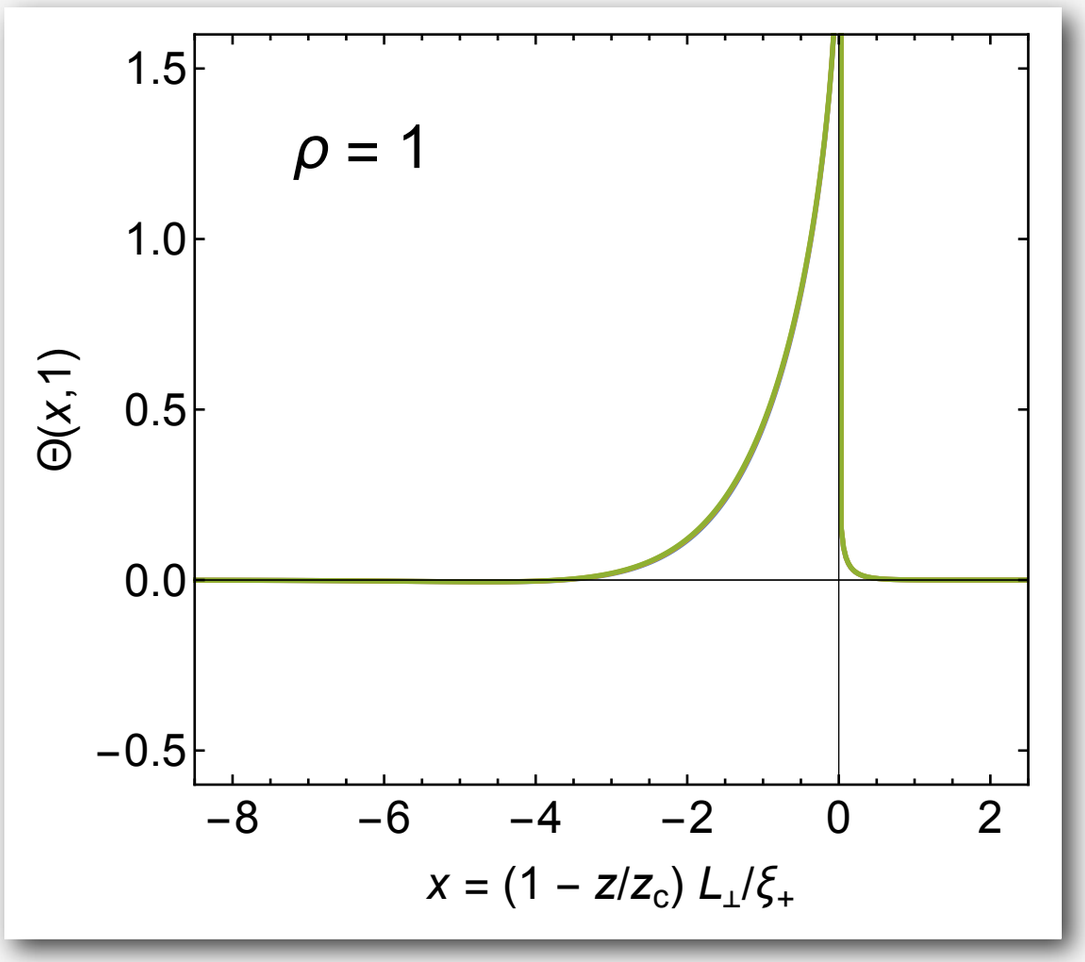
$$\delta F\left(t, L_\perp, L_\parallel = \tfrac{L_\perp}{\rho}\right) \simeq -\frac{1}{8}\ln(tL_\perp) + \mathcal{O}(1)$$

$$\Theta(x, \rho) \simeq -\frac{1}{8}\ln(x) + \mathcal{O}(1)$$

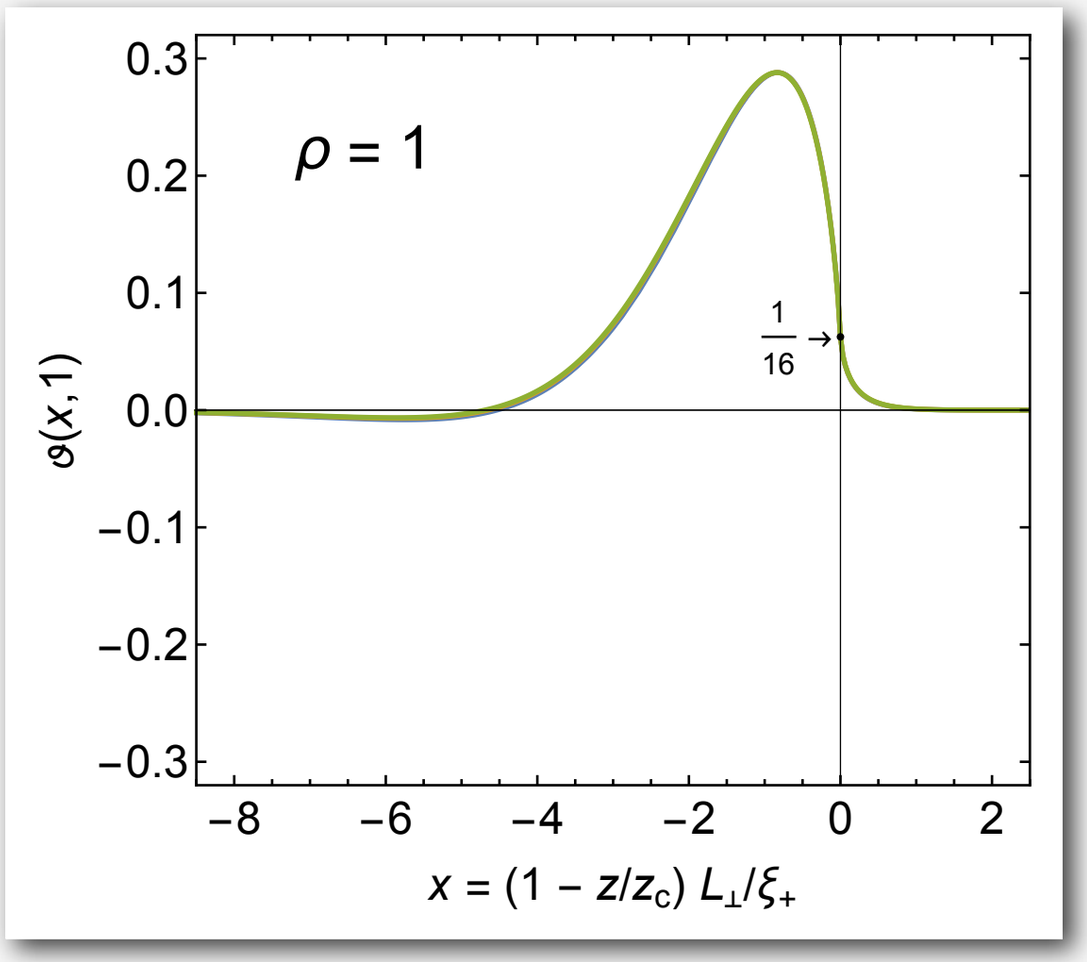
# The square ($\rho = 1$)




$$\vartheta(x,1) = -\frac{x}{d\nu}\frac{\partial}{\partial x}\Theta(x,1)$$

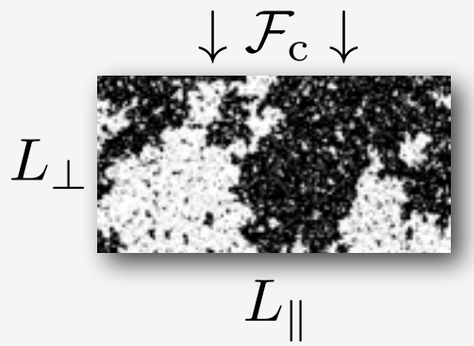


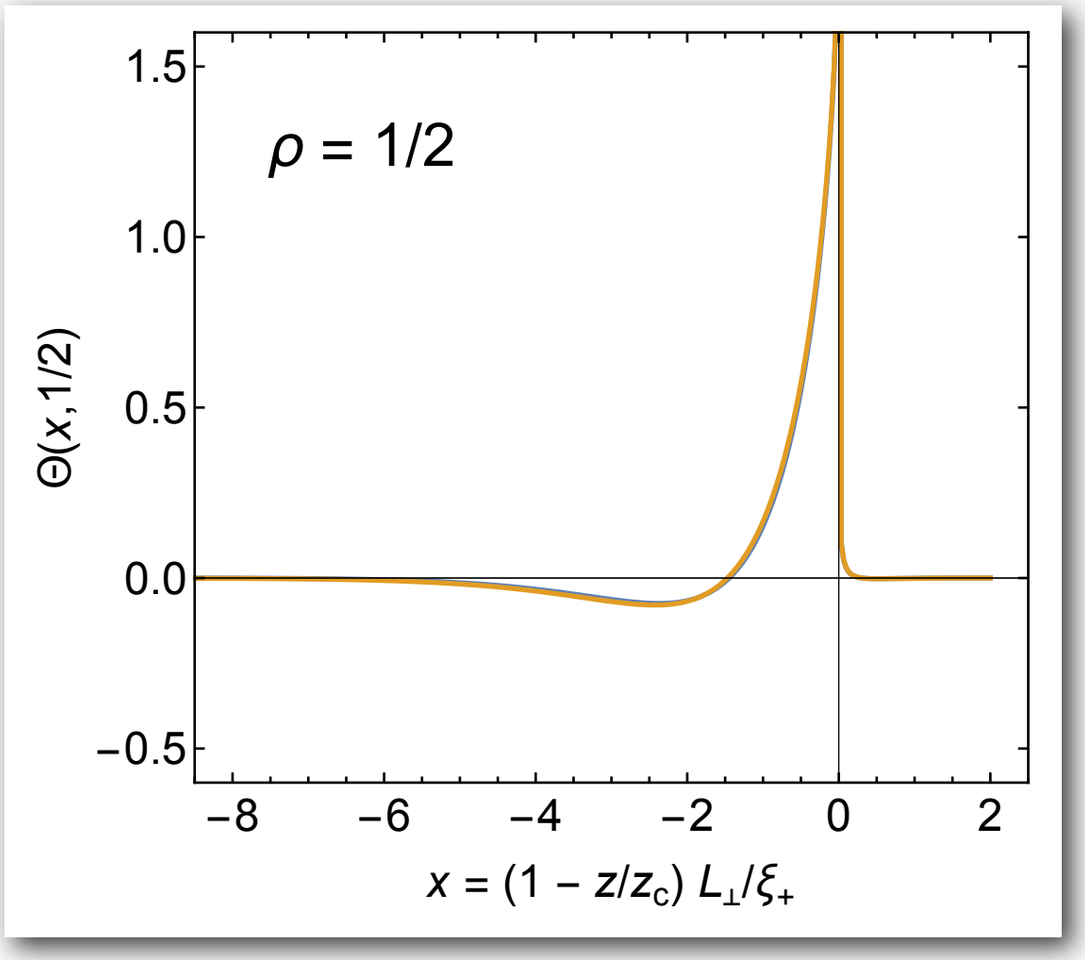

residual free energy



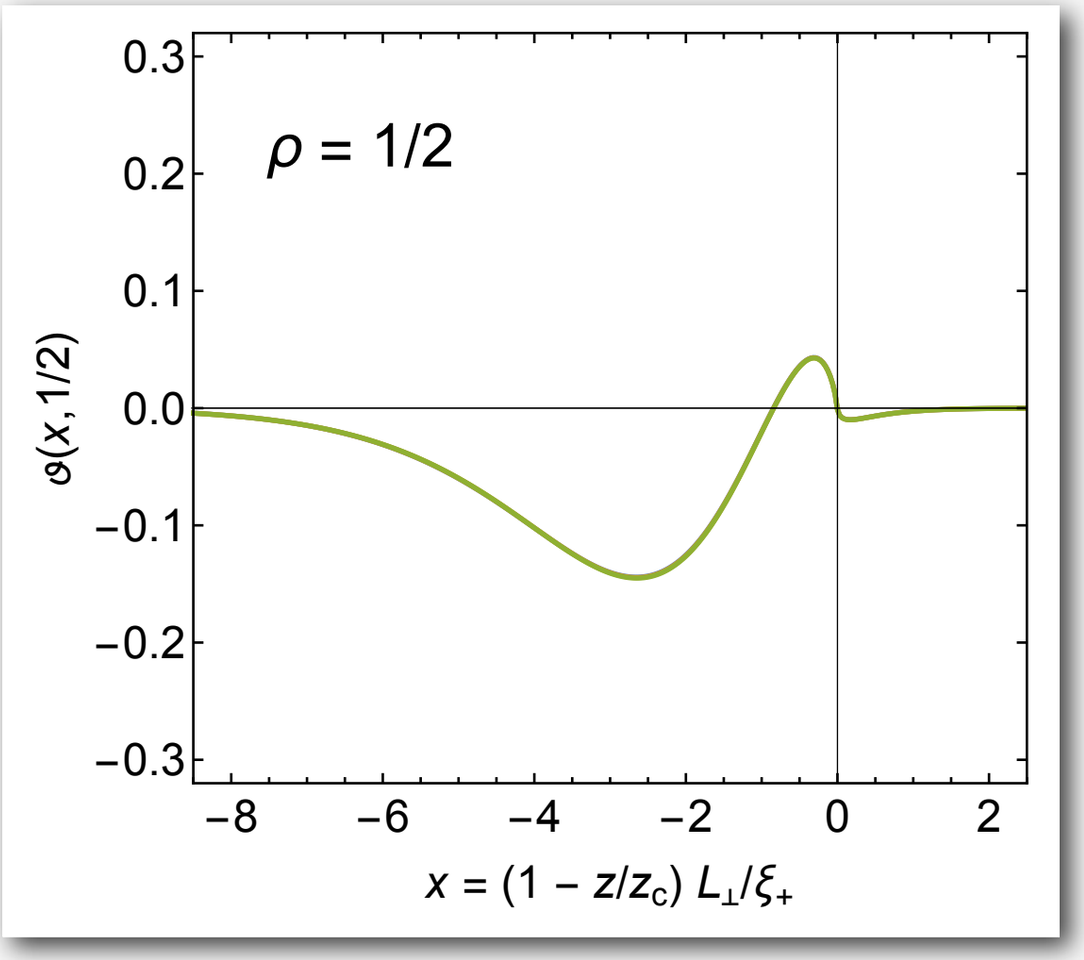

Casimir force

# $\rho = 1/2$



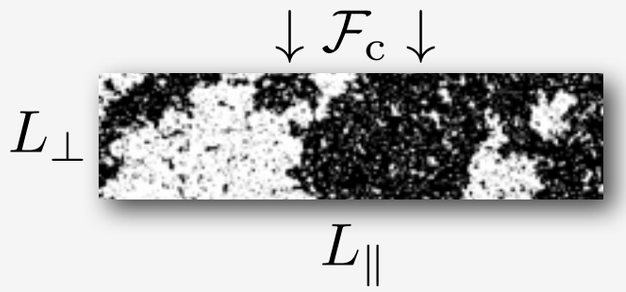



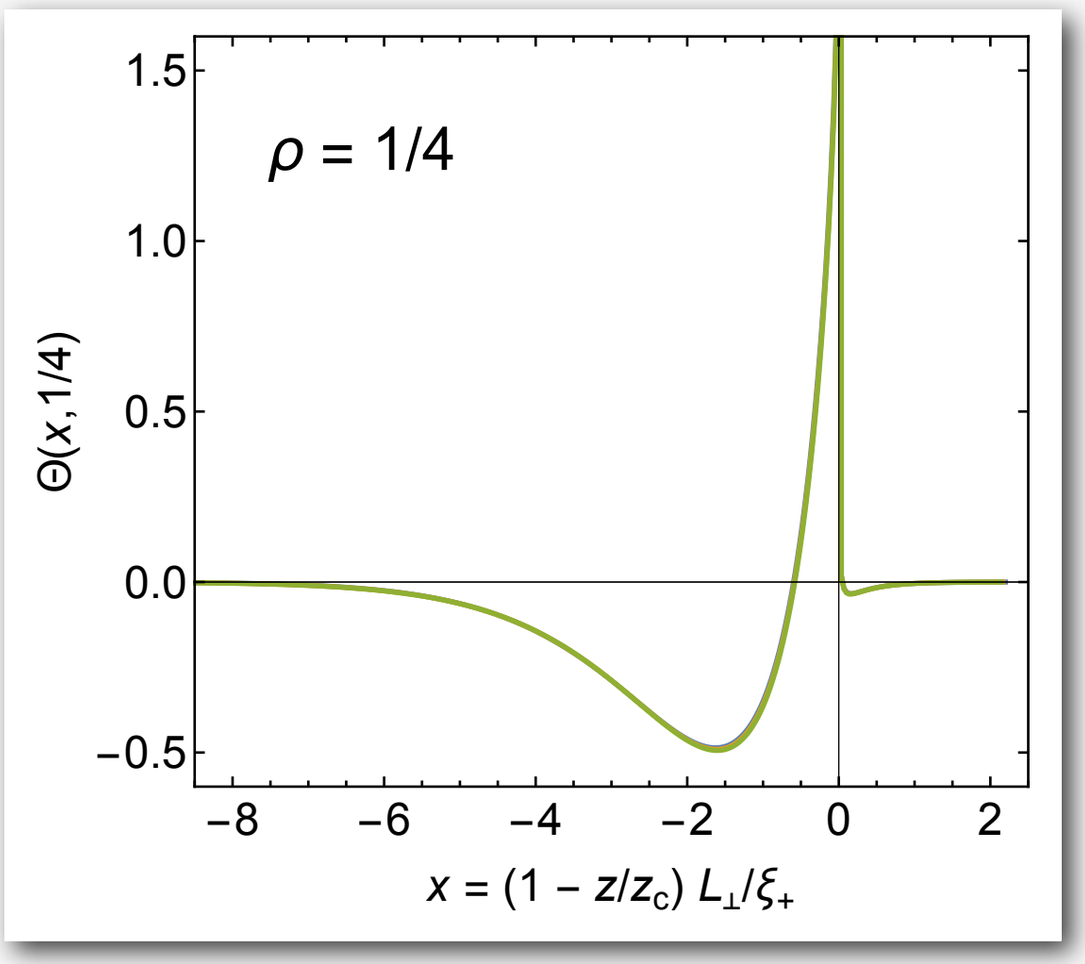

residual free energy



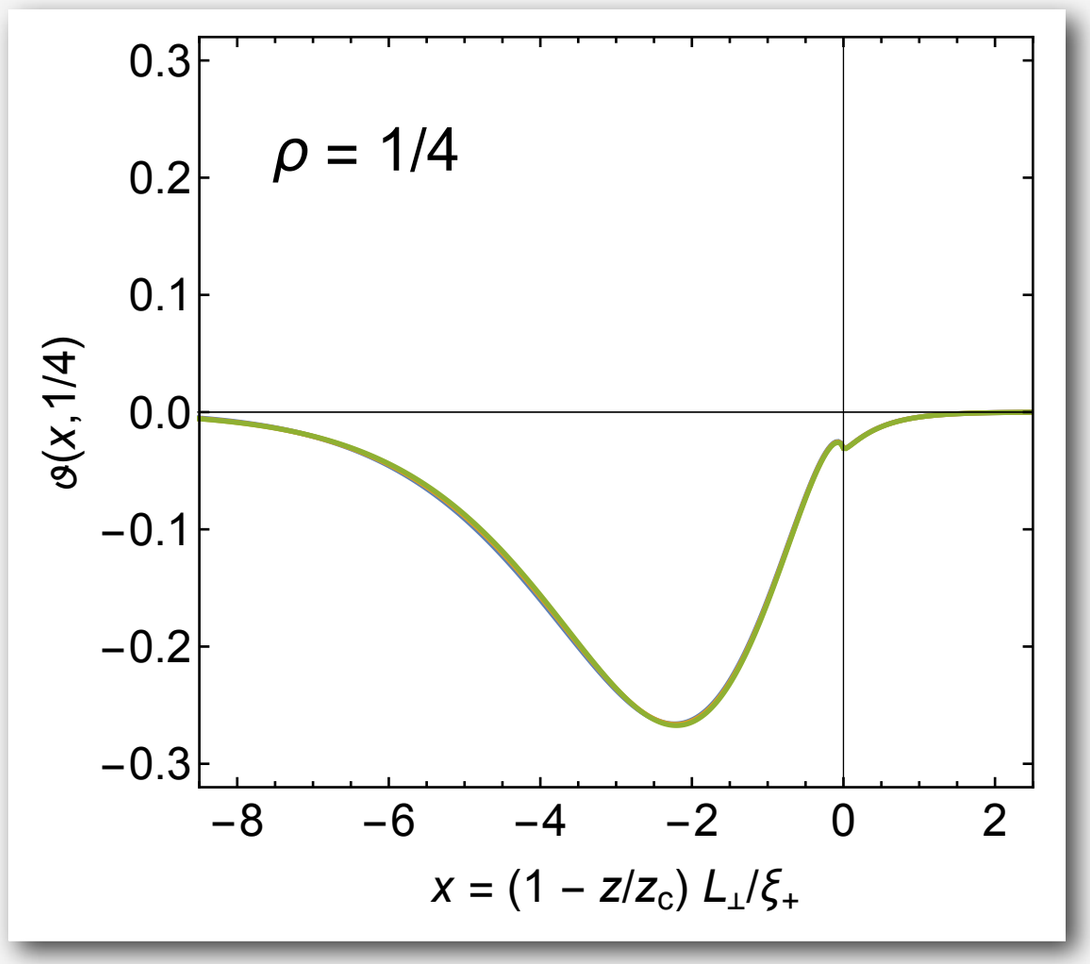

Casimir force

# $\rho = 1/4$



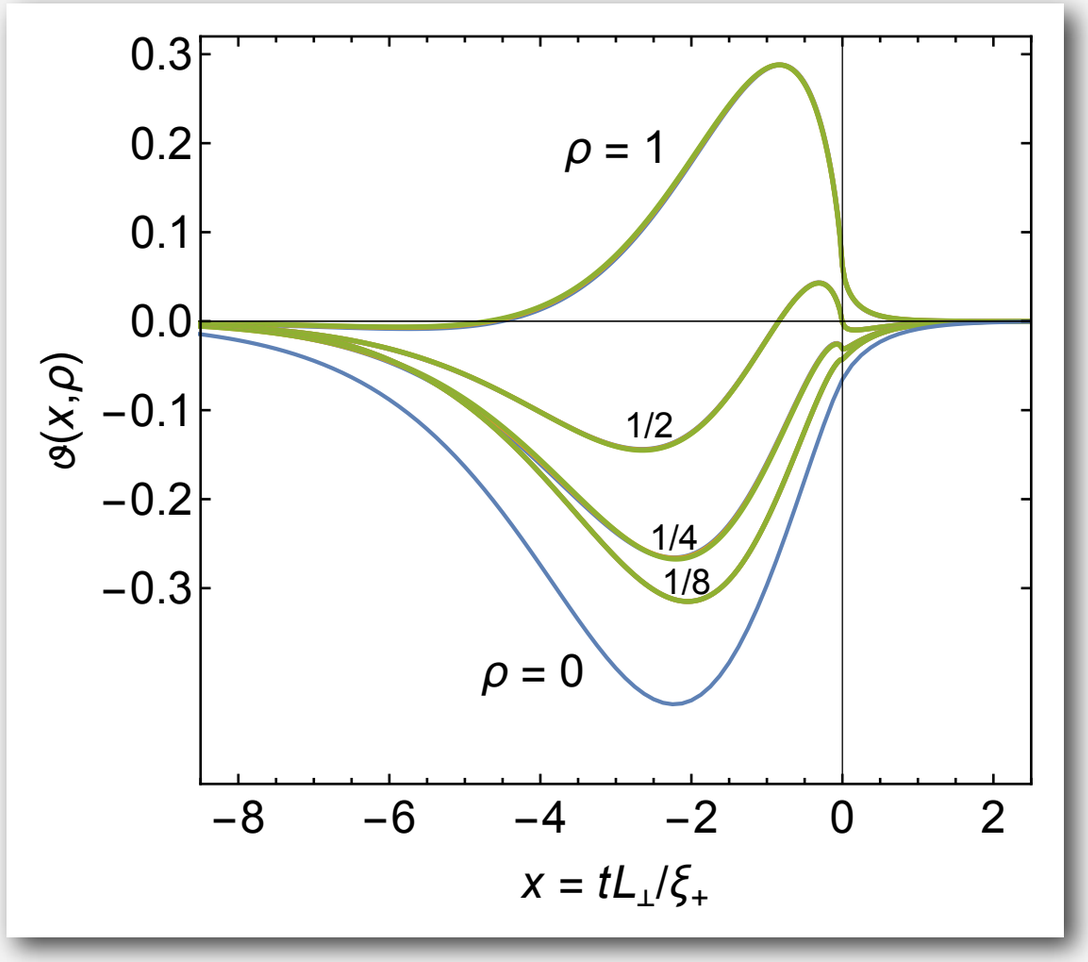

residual free energy




Casimir force

# Conclusions

- exact infinite product representations for b/s/c free energies
- finite size scaling works for systems with corners
- residual free energy shows unusual log divergence at $T_c$




- corner contributions in 3d?
- $q$-products not fully understood
- integral formula for $f_c$?

Thank You!