Spherical models of interface growth

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Overview :

1. Magnets and growing interfaces : analogies
2. Interface growth & KPZ universality class
3. Interface growth and 1^{st} Arcetri model
4. Conclusions

1. Magnets and growing interfaces : analogies

Common properties of critical and ageing phenomena :

* **collective** behaviour,
  very **large** number of interacting degrees of freedom
* **algebraic** large-distance and/or large-time behaviour
* described in terms of **universal** critical **exponents**
* very **few** relevant scaling operators
* justifies use of extremely **simplified** **mathematical models**
  with a remarkably rich and complex behaviour
* yet of **experimental significance**

Magnets

thermodynamic equilibrium state

order parameter $\phi(t,\mathbf{r})$

phase transition, at critical temperature T_c :

magnetisation :

$m(t) = \langle\phi(t,\mathbf{r})\rangle|_{T=T_c} \sim t^{-\beta/(\nu z)}$

relaxation, after quench to $T \leq T_c$

autocorrelator

$C(t,s) = \langle\phi(t,\mathbf{r})\phi(s,\mathbf{r})\rangle_c$

Interfaces

growth continues forever

height profile $h(t,\mathbf{r})$

same generic behaviour throughout

roughness :

$w(t)^2 = \left\langle\left(h(t,\mathbf{r}) - \overline{h}(t)\right)^2\right\rangle \sim t^{2\beta}$

relaxation, from initial substrate :

autocorrelator $C(t,s) =$

$\left\langle\left(h(t,\mathbf{r}) - \overline{h}(t)\right)\left(h(s,\mathbf{r}) - \overline{h}(s)\right)\right\rangle$

ageing scaling behaviour :

when $t,s\to\infty$, and $y := t/s > 1$ fixed, expect

$$C(t,s) = s^{-b} f_C(t/s) \quad \text{and} \quad f_C(y) \stackrel{y\to\infty}{\sim} y^{-\lambda_C/z}$$

b, β, ν and dynamical exponent z : **universal** & related to stationary state

autocorrelation exponent λ_C : **universal** & independent of stationary exponents

Magnets

exponent value $b = \begin{cases} 0 & ; \ T < T_c \\ 2\beta/\nu z & ; \ T = T_c \end{cases}$

Interfaces

exponent value $b = -2\beta$

models :

Magnets

(a) gaussian field

$$\mathcal{H}[\phi] = -\frac{1}{2}\int \mathrm{d}\mathbf{r}\,(\nabla\phi)^2$$

(b) **Ising model**

$$\mathcal{H}[\phi] = -\frac{1}{2}\int \mathrm{d}\mathbf{r}\,\left[(\nabla\phi)^2 + \tau\phi^2 + \frac{g}{2}\phi^4\right]$$

such that $\tau = 0 \leftrightarrow T = T_c$

dynamical Langevin equation (Ising) :

$$\begin{aligned} \partial_t \phi &= -D\frac{\delta\mathcal{H}[\phi]}{\delta\phi} + \eta \\ &= D\nabla^2\phi + \tau\phi + g\phi^3 + \eta \end{aligned}$$

Interfaces

(a) Edwards-Wilkinson (EW) :

$$\partial_t h = \nu\nabla^2 h + \eta$$

(b) **Kardar-Parisi-Zhang** (KPZ) :

$$\partial_t h = \nu\nabla^2 h + \frac{\mu}{2}(\nabla h)^2 + \eta$$

$\eta(t,\mathbf{r})$ is the usual white noise, $\langle\eta(t,\mathbf{r})\eta(t',\mathbf{r}')\rangle = 2T\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$

phase transition exactly solved $d = 2$

relaxation exactly solved $d = 1$

Onsager '44, Glauber '63, ...

growth exactly solved $d = 1$

Calabrese & Le Doussal '11

Sasamoto & Spohn '10

Question : obtain qualitative understanding by approximate treatment of the non-linearity ?

Ising model : **yes, certainly !** $\boxed{\Rightarrow \text{spherical model !}}$

Berlin & Kac 52
Lewis & Wannier 52

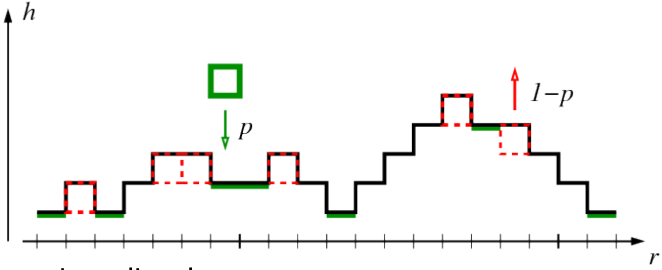
(a) for a lattice model : replace Ising spins $s_i = \pm 1 \mapsto s_i \in \mathbb{R}$, with (mean) spherical constraint $\sum_i s_i^2 = \mathcal{N}$
(b) for continuum field : replace $\phi^3 \mapsto \phi\langle\phi^2\rangle$ and spherical constraint $\int \mathrm{d}\mathbf{r}\langle\phi^2\rangle \sim 1$.

Interest : analytically solvable for any d and in more general contexts than Ising model, all exponents . . . known exactly. Very useful to illustrate general principles in a specific setting. New universality class, distinct from the Ising model ($\mathrm{O}(N)$ model with $N \to \infty$).

Question : $\boxed{\text{can one find a similar treatment for the KPZ equation ?}}$
Are there new universality class(es) for interface growth ?
Behaviour different from the rather trivial EW-equation ?

2. Interface growth & KPZ class

deposition (evaporation) of particles on a substrate $\rightarrow$ height profile $h(t, \mathbf{r})$
generic situation : RSOS (**r**estricted **s**olid-**o**n-**s**olid) model — KIM & KOSTERLITZ 89



p = deposition prob.
$1 - p$ = evap. prob.

here $p = 0.98$

some universality classes :

(a) KPZ $\partial_t h = \nu \nabla^2 h + \frac{\mu}{2} (\nabla h)^2 + \eta$ — KARDAR, PARISI, ZHANG 86

(b) EW $\partial_t h = \nu \nabla^2 h + \eta$ — EDWARDS, WILKINSON 82

η is a gaussian white noise with $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2\nu T \delta(t - t') \delta(\mathbf{r} - \mathbf{r}')$

Family-Viscek scaling on a spatial lattice of extent L^d : $\overline{h}(t) = L^{-d}\sum_j h_j(t)$

FAMILY & VISCEK 85

$$w^2(t;L) = \frac{1}{L^d}\sum_{j=1}^{L^d}\left\langle \left(h_j(t) - \overline{h}(t)\right)^2\right\rangle = L^{2\zeta} f\left(tL^{-z}\right) \sim \begin{cases} L^{2\zeta} & ; \text{if } tL^{-z} \gg 1 \\ t^{2\beta} & ; \text{if } tL^{-z} \ll 1 \end{cases}$$

β : growth exponent, ζ : roughness exponent, $\boxed{\zeta = \beta z}$

two-time correlator :

limit $L \to \infty$

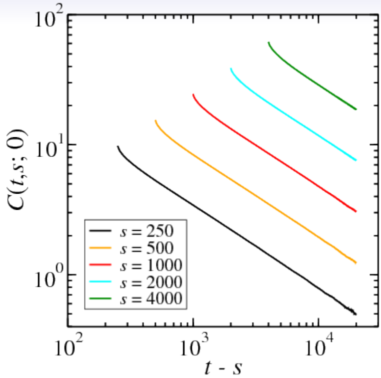
$$C(t,s;\mathbf{r}) = \left\langle \left(h(t,\mathbf{r}) - \langle\overline{h}(t)\rangle\right)\left(h(s,\mathbf{0}) - \langle\overline{h}(s)\rangle\right)\right\rangle = s^{-b} F_C\left(\frac{t}{s}, \frac{\mathbf{r}}{s^{1/z}}\right)$$

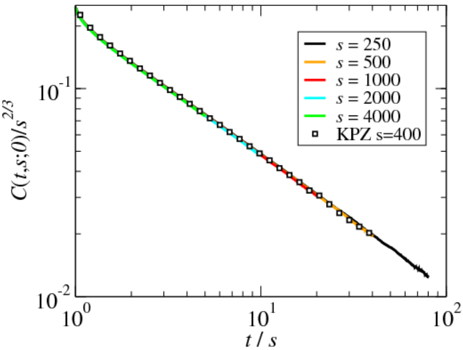
with ageing exponent : $\boxed{b = -2\beta}$

KALLABIS & KRUG 96

expect for $y = t/s \gg 1$: $F_C(y,\mathbf{0}) \sim y^{-\lambda_C/z}$ autocorrelation exponent

$1D$ relaxation dynamics, starting from an initially flat interface





observe all **3** properties of **ageing** : $\left\{ \begin{array}{l} \text{slow dynamics} \\ \text{no TTI} \\ \text{dynamical scaling} \end{array} \right.$

confirm **simple ageing** for the $1D$ KPZ universality class

confirm expected exponents $b = -2/3$, $\lambda_C/z = 2/3$ *pars pro toto*

KALLABIS & KRUG 96 ; KRECH 97 ; BUSTINGORRY *et al.* 07-10 ; CHOU & PLEIMLING 10 ; D'AQUILA & TÄUBER 11/12 ; H.N.P. 12

Values of some growth and ageing exponents in $1D$

model	z	a	b	$\lambda_R = \lambda_C$	β	ζ
KPZ	$3/2$	$-1/3$	$-2/3$	1	$1/3$	$1/2$
exp 1			$\approx -2/3^{\dagger}$	$\approx 1^{\dagger}$	$0.336(11)$	$0.50(5)$
exp 2	$1.5(2)$				$0.32(4)$	$0.50(5)$
EW	2	$-1/2$	$-1/2$	1	$1/4$	$1/2$

liquid crystals

(cancer) cell growth

Takeuchi, Sano, Sasamoto, Spohn 10/11/12

Huergo, Pasquale, Gonzalez, Bolzan, Arvia 12

† scaling holds only for flat interface

Two-time space-time responses and correlators consistent with simple ageing for $1D$ KPZ

Similar results known for EW universality class

ROETHLEIN, BAUMANN, PLEIMLING 06

3. Interface growth & (1st) Arcetri model

? KPZ $\longrightarrow$ $\boxed{\text{intermediate model}}$ $\longrightarrow$ EW **?**

preferentially exactly solvable, and this in $d \geq 1$ dimensions

inspiration : mean **spherical model** of a ferromagnet

Berlin & Kac 52
Lewis & Wannier 52

Ising spins $s_i = \pm 1$ obey $\sum_i s_i^2 = \mathcal{N} = \#$ sites
spherical spins $s_i \in \mathbb{R}$ spherical constraint $\left\langle \sum_i s_i^2 \right\rangle = \mathcal{N}$

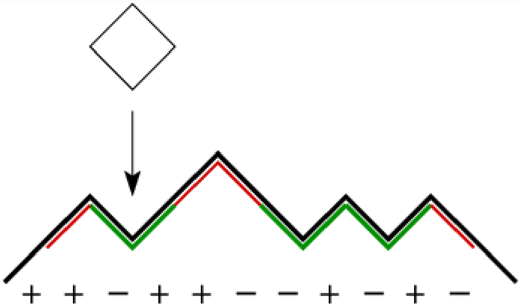
hamiltonian $\mathcal{H} = -J \sum_{(i,j)} s_i s_j - \lambda \sum_i s_i^2$ Lagrange multiplier λ

exponents **non**-mean-field for $2 < d < 4$ and $T_c > 0$ for $d > 2$

kinetics from Langevin equation $\boxed{\partial_t \phi = -D \frac{\delta \mathcal{H}[\phi]}{\delta \phi} + \mathfrak{z}(t)\phi + \eta}$

time-dependent Lagrange multiplier $\mathfrak{z}(t)$ fixed from spherical constraint
all equilibrium and ageing exponents exactly known, for $T < T_c$ and $T = T_c$

Coniglio & Zannetti 89, …Godrèche & Luck '00, Corberi, Lippiello, Fusco, Gonnella & Zannetti 02-14 consider RSOS-adsorption process :



use **not** the heights $h_n(t) \in \mathbb{N}$ on a discrete lattice,

but rather the slopes $u_n(t) = \frac{1}{2}\left(h_{n+1}(t) - h_{n-1}(t)\right)$

? can one let $u_n(t) \in \mathbb{R}$, but impose a spherical constraint ?

? consequences of the 'hardening' of a soft EW-interface by a 'spherical constraint' on the u_n ?

since $u(t,x) = \partial_x h(t,x)$: go from KPZ to Burgers' equation, and

replace its non-linearity by a mean spherical condition

$$\begin{aligned} \partial_t u_n(t) &= \nu\left(u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)\right) + \mathfrak{z}(t) u_n(t) \\ &\quad + \frac{1}{2}\left(\eta_{n+1}(t) - \eta_{n-1}(t)\right) \\ \sum_n \left\langle u_n(t)^2 \right\rangle &= N \end{aligned}$$

$\langle \eta_n(t)\eta_m(s)\rangle = 2T\nu\delta(t-s)\delta_{n,m}$

Extension to $d \geq 1$ dimensions :

define gradient fields $u_a(t,\mathbf{r}) := \nabla_a h(t,\mathbf{r})$, $a = 1, \ldots, d$:

$$\begin{aligned} \partial_t u_a(t,\mathbf{r}) &= \nu \nabla_{\mathbf{r}} \cdot \nabla_{\mathbf{r}} u_a(t,\mathbf{r}) + \mathfrak{z}(t) u_a(t,\mathbf{r}) + \nabla_a \eta(t,\mathbf{r}) \\ \sum_{a=1}^{d} \left\langle u_a(t,\mathbf{r})^2 \right\rangle &= N^d \end{aligned}$$

interface height : $\boxed{\widehat{u}_a(t,\mathbf{q}) = \mathrm{i}\sin q_a\, \widehat{h}(t,\mathbf{q})}$; $\mathbf{q} \neq \mathbf{0}$ in Fourier space exact solution : $\mathbf{q} \neq \mathbf{0}$

$$\widehat{h}(t,\mathbf{q}) = \widehat{h}(0,\mathbf{q})e^{-2t\omega(\mathbf{q})}g(t)^{-1/2} + \int_0^t \mathrm{d}\tau\; \widehat{\eta}(\tau,\mathbf{q})\sqrt{\frac{g(\tau)}{g(t)}}\, e^{-2(t-\tau)\omega(\mathbf{q})}$$

in terms of the auxiliary function $g(t) = \exp\left(-2\int_0^t \mathrm{d}\tau\, \mathfrak{z}(\tau)\right)$,
satisfies Volterra equation

$$g(t) = f(t) + 2T\int_0^t \mathrm{d}\tau g(\tau) f(t-\tau) \;\;,\;\; f(t) := d\,\frac{e^{-4t}I_1(4t)}{4t}\left(e^{-4t}I_0(4t)\right)^{d-1}$$

* for $d=1$, identical to 'spherical spin glass', with $T = 2T_{\mathrm{SG}}$:
hamiltonian $\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij}s_i s_j$; J_{ij} random matrix, its eigenvalues distributed according to Wigner's semi-circle law Cugliandolo & Dean 95

* correspondence spherical spins $s_i \leftrightarrow$ slopes u_n

* kinetics of heights $h_n(t)$ is driven by phase-ordering of the spherical spin glass $=$ 3D kinetic spherical model

phase transition : long-range correlated surface growth for $T \leq T_c$

$$\frac{1}{T_c(d)} = \frac{d}{2}\int_0^\infty \mathrm{d}t\; e^{-dt} t^{-1} I_1(t) I_0(t)^{d-1} \quad ; \quad T_c(1) = 2,\ T_c(2) = \frac{\pi}{\pi - 2}$$

Some results :

upper critical dimension $d^* = 2$

1. $T = T_c,\ d < 2$: sub-diffusive interface motion $\langle h(t) \rangle \sim t^{(2-d)/4}$

   interface width $w(t) = t^{(2-d)/4} \Longrightarrow \beta = \frac{2-d}{4}$

   ageing exponents $a = b = \frac{d}{2} - 1,\ \lambda_R = \lambda_C = \frac{3d}{2} - 1\,;\ z = 2$

2. $T = T_c,\ d > 2$: interface immobile $\langle h(t) \rangle \sim \mathrm{cste.}$

   interface width $w(t) = \mathrm{cste.} \Longrightarrow \beta = 0$

   ageing exponents $a = b = \frac{d}{2} - 1,\ \lambda_R = \lambda_C = d\,;\ z = 2$

3. $T < T_c$:

   interface width $w^2(t) = (1 - T/T_c)t \Longrightarrow \beta = \frac{1}{2}$

   ageing exponents $a = \frac{d}{2} - 1,\ b = -1,\ \lambda_R = \lambda_C = \frac{d-2}{2}\,;\ z = 2$

4. Conclusions

- physical ageing occurs naturally in many **irreversible** systems relaxing towards generic stationary states
  considered here : magnetic systems & surface growth
- quite analogous scaling phenomenologies
- 1$^{\rm st}$ **Arcetri model** captures at least some qualitative properites of KPZ. Specific properties :
  - interface becomes more smooth as $d \to d^* = 2$; for $d > 2$ the EW equation gives the mean-field description
  - at $T = T_c$, the stationary exponents (β, z) are those of EW, but the ageing exponents are different
    confirms explicitly general field-theory renormalisation group
  - for $d = 1$ and $T = T_c$, equivalent to $p = 2$ spherical spin glass
  - new kind of behaviour at $T < T_c$
- in progress : classify all possible 'spherical/Arcetri models' and study their properties