Numerical study of the branching tree of states in spin glasses

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The Sherrington-Kirkpatrick model

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Edwards-Anderson model

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij} s_i s_j, \quad s_i = \pm 1$$

- $J_{ij} = \pm 1$ with 50% probability.
- Disorder and frustration
- Order parameter from the overlap

$$q = \lim_{t\to\infty} \frac{1}{N} \sum_x \langle s_x(0) s_x(t)\rangle_t \quad \rightsquigarrow \quad q = \frac{1}{N}\sum_x \langle s_x\rangle^2 .$$

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- The EA model is too difficult to handle analytically.
- We consider its mean-field version:

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- $p(q)$ is smooth in $[0, q_\mathrm{M})$, but has a δ function at q_M
- The order parameter is not a number, but a function $q(x)$.
- $x(q)$ is the cumulative probability of q: $x(q) = \int_0^q \mathrm{d}q' p(q')$.

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(ferromagnet: consider $\langle A \rangle = \frac{1}{2} \langle A \rangle_+ + \frac{1}{2} \langle A \rangle_-$).

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- Example: the overlap:

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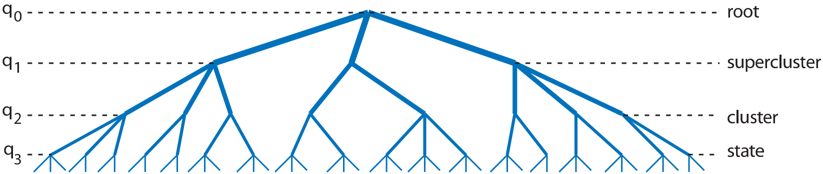
- It turns out that the states live in an ultrametric space, using $q_{\alpha\beta}$ to define a distance
- We can classify the states in a taxonomic tree, which branches out as we break the replica symmetry.

The tree of states (I)



- Consider a simplified case where q can only take four discrete values (equivalently $K = 3$ steps of RSB).
- We can group the configurations (organisms) $\rightarrow$ states (species) $\rightarrow$ clusters (geni) $\rightarrow$ superclusters (families).

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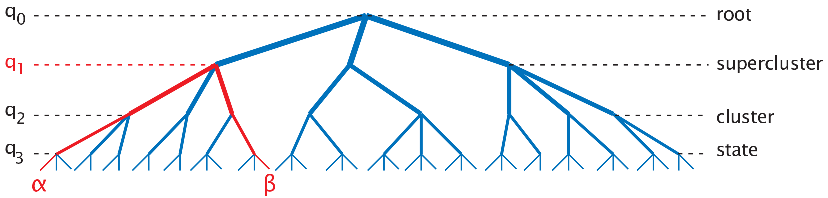

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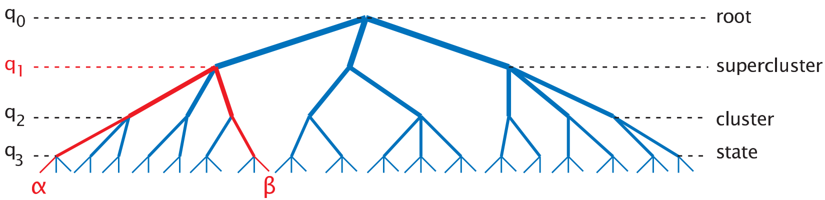

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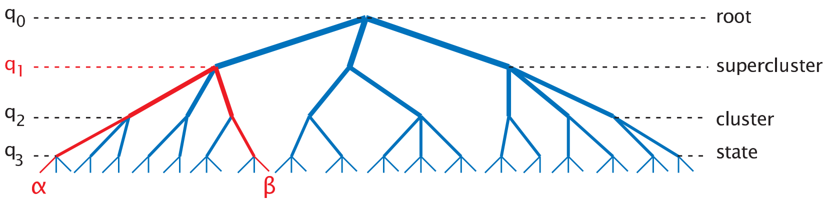

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- We can consider a decomposition in clusters, instead of states:

$$\langle A \rangle = \sum_I W_I \langle A \rangle_I , \qquad W_I = \sum_{\alpha\in I} w_\alpha .$$

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- We want to generate explicit realisations of this tree.

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- Universality: everything is encoded in $x(q)$.

Generating the tree from the ground up (I)

- The formulation of the f_l considering a single q level in isolation is convenient for analytical computations.
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- We eliminate at each level all the clusters with $W_l < \epsilon$ (equivalent to neglecting all the states with $w_\alpha < \epsilon$).
- We are losing a total probability of $\sim \epsilon^{1-x(q_\mathrm{M})}$.

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- Notice that the first step, the root, is going from $q = 0 \to q_0$. Since $x(q = 0) = 0$, in the first step we have independent f_i.

The cavity method

- We have seen how to generate the tree, we now want to compute physical quantities from it. We consider a cavity approach.
- We add an $N+1$ spin s_0 to a system with N spins s_i.
- The average properties of s_0 are the same as those of the s_i.
- Consider the N-spin system in equilibrium and compute the cavity field

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- The cavity field h_α for each state is the sum of $K+1$ Gaussian random variables:

$$h_\alpha = h_\alpha^0 + h_\alpha^1 + \ldots + h_\alpha^K,$$

  - h_α^0 has variance βq_0 and is common to the whole tree.
  - h_α^i has variance $\beta(q_i - q_{i-1})$ and is common to all the states down the same branch.

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- The cavity step shifts the free energies:

$$w'_\alpha \sim w_\alpha \cosh(\beta h_\alpha)$$

$\Longrightarrow$ we have an iterative method to refine the tree

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- Let us study the solution at x_M fixed to the known correct value
$x_\mathrm{M} \approx 0.233122$.

$q(x)$ at $T = 0.85$

- We start with $q_{\mathrm{M}} = 0.23 \approx x_{\mathrm{M}}$ and iterate.
- We consider $\epsilon = 10^{-5} \Longrightarrow \sim 1 - \epsilon^{1-x_{\mathrm{M}}} = 99.99\%$ of the probability.
- $K = 20$ is enough, since $q(x)$ is so simple.
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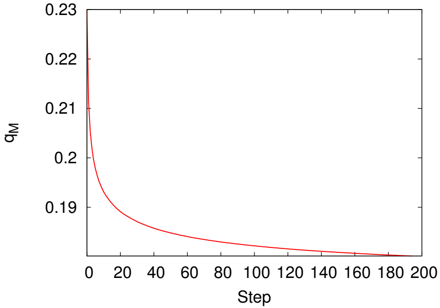

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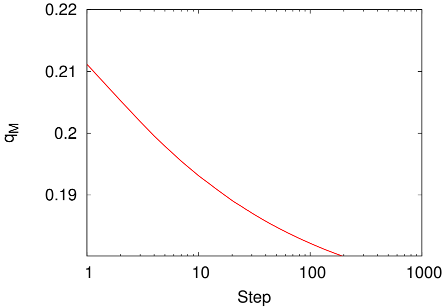

The convergence is very slow (logarithmic).

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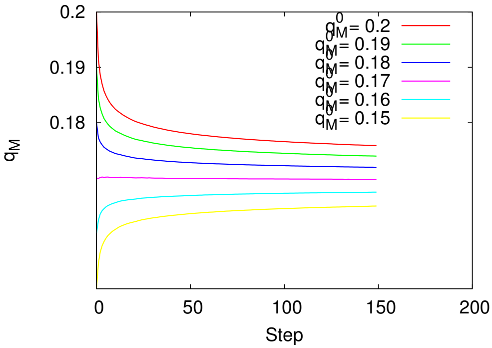

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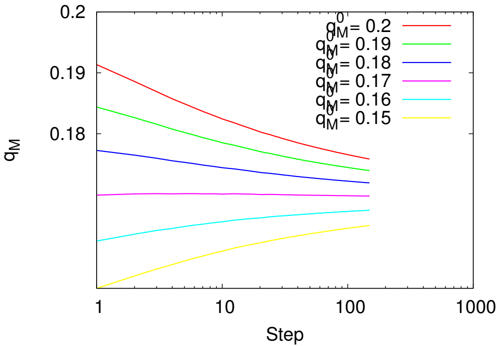

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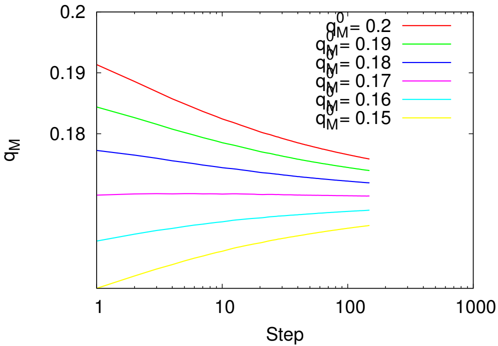

- The evolution is monotonic $\Longrightarrow$ it is easy to find the stable solution.
- After 100 steps, starting with $q_{\mathrm{M}}^{(0)} = 0.17$, we find $q_M^{(100)} = 0.169687(3)$ (expected value $q_{\mathrm{M}} \approx 0.169691$).

Replicon

- The spin-glass susceptibility is $\chi_{\mathrm{SG}} = \overline{\left((1 - m_0^2)^2\right)^2} / \left[1 - \beta^2 \overline{(1 - m_0^2)^2}\right]$.
- It diverges for $T < T_{\mathrm{c}}$ so

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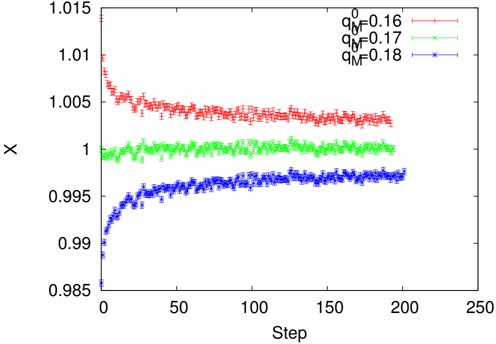

- We obtain $X = 0.99972(33)$.

Conclusions

- We have reviewed the main properties of the tree of states in mean-field spin glasses.
- We have shown how to generate explicit realisations of this tree, in a self-consistent way.
- The states computed from the tree can be used to study physical quantities.
- Applications
    - Evaluate all the correlation functions of the model for fixed q.
    - Study other mean-field models (such as the full-RSB solution for the spin glass in a Bethe lattice).
    - Finite-size effects: generate the $P_{\mathcal{J}}(q)$ for single trees (single samples) and study the smoothing of the individual peaks.

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THANK YOU!