

## ORIGIN AND MOTIVATION

The Gonihedric Ising model arises from bosonic string theory as a possible discretisation of the area swept out by a string worldsheet moving through spacetime (1). The name comprises the greek words gonika (angle) and hedra (face) as a reminder of the origin.  
 The lattice sph model allows the application of sophisticated generalised ensemble Monte Carlo methods that are tailored to tackle many open questions on the characteristics of phases occurring.

## THE GONIHEDRIC ISING MODEL

- spins  $\sigma_i \in \{-1, +1\}$  on each vertex of a 3D cubic lattice, linear lattice size  $L$ .
- plaquettes in the dual lattice separate contiguous spins with opposite signs
- bulk of plaquettes defines a surface

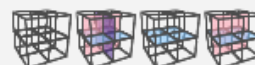
Hamiltonian:

$$H = -2\kappa \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \frac{\kappa}{2} \sum_{\langle i,j,k \rangle} \sigma_i \sigma_j - \frac{1-\kappa}{2} \sum_{\langle i,j,k,l \rangle} \sigma_i \sigma_j \sigma_k \sigma_l$$

= only the *area* instead of *surface area* contributes to the partition function (2)  
 =  $\kappa$  ... surface self-avoidance control parameter, complete self-avoidance for  $\kappa \rightarrow \infty$   
 • Special Case  $\kappa = 0$ :

$$H = -\frac{1}{2} \sum_{\langle i,j,k \rangle} \sigma_i \sigma_j \sigma_k \quad (1)$$

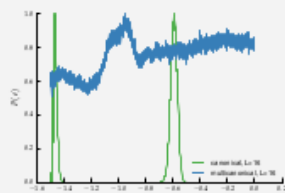
- interaction without energetic penalty
- strong first order phase transition
- Zero-temperature ground state: elementary cubes with same energy



- composing the bulk ground state for periodic boundary conditions, even lattice sizes
- flp of whole planes parallel to either one of the  $x, y, z$  -planes allowed
- bulk ground state degeneracy  $g = 2^{3L}$ , with  $L$  being the linear lattice size

## METHOD

- canonical simulations suffer from *superficial slowing* down near the phase transition, rare states are suppressed by a factor  $\propto \exp(-2\beta\sigma L^2)$  with interface tension  $\sigma$ , inverse temperature  $\beta$
- multicanonical simulation: promote rare states  $\mu$  with energy  $E_\mu = H(\mu) = \epsilon(\mu)L^2$  using



## FINITE-SIZE SCALING

- expected first-order finite-size scaling (3,4):

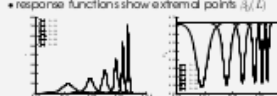
$$\beta_f(L) = \beta_f^c + aL^{-3} + O(L^{-6}) \quad (2)$$

- valid for extremal points  $\beta_f$  of response functions like heat capacity  $C$  or Binder's cumulant  $U_4$

$$C = \beta^2 ((E^2) - (E)^2), \quad U_4 = 1 - \frac{(E^4)}{\beta^2 (E^2)^2}$$

## RESULTS FOR THE ORIGINAL MODEL $\mathcal{H}(\kappa = 0)$

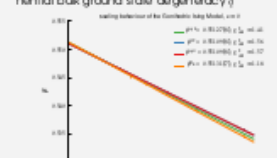
- response functions show extremal points  $\beta_f(L)$



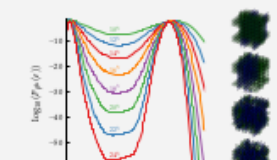
- best agreement with scaling ansatz

$$\beta_f(L) = \beta_f^c + aL^{-2} + O(L^{-6}) \quad (3)$$

- amplitude in Eq. (2) is  $a \propto \ln(g) \propto \ln(2^{3L}) \propto L$
- finite-size scaling corrections in *increased* due to exponential bulk ground state degeneracy  $g$



- energy probability density at  $\beta^*$



- reduced interface tension  $\tilde{\sigma} = \beta\sigma$  can be extracted for several lattice sizes,

$$\tilde{\sigma}(L) = \frac{1}{2L^2} \ln \left( \frac{\max(P_{\text{flat}}(L))}{\max(P_{\text{int}}(L))} \right)$$

and linear regression leads to

$$\tilde{\sigma}(L \rightarrow \infty) = 0.118(3)$$

## THE DUAL MODEL $\mathcal{H}^d$

$$H^d = -\frac{1}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \frac{1}{2} \sum_{\langle i,j,k \rangle} \tau_i \tau_j - \frac{1}{2} \sum_{\langle i,j,k,l \rangle} \sigma_i \sigma_j \tau_k \tau_l \quad (4)$$

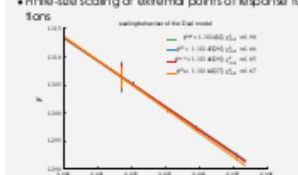
- two spins  $\sigma, \tau$  on each vertex of 3D lattice
- non-trivial interaction of two Ising spin chains due to the last sum
- canonical simulations contradicted each other w.r.t. transition temperatures in the original model

$$\beta_f^c = 0.510(2) \quad \dots \text{from dual model (5)}$$

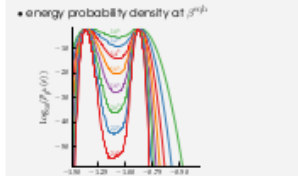
$$\beta_f^c = 0.547(7)(6) \quad \dots \text{from original model (6)}$$

## RESULTS FOR THE DUAL MODEL $\mathcal{H}^d$

- finite-size scaling of extremal points of response functions



- energy probability density at  $\beta^*$



- reduced interface tension

$$\tilde{\sigma}(L \rightarrow \infty) = 0.122(6)$$

## AGREEMENT OF THE MODELS

- inverse transition temperatures associated by

$$\beta^* = -\ln \left( \frac{\beta_f^c}{2} \right)$$

- resulting in

$$\beta_f^c = 0.512(15) \quad \dots \text{from dual model}$$

$$\beta_f^c = 0.5131(7) \quad \dots \text{from original model}$$

⇒ our multicanonical data is consistent

## CONCLUSIONS & OUTLOOK

Multicanonical simulation of the Gonihedric Ising model with vanishing energy penalty for intersecting surfaces and a dual model have been conducted. The response functions of the energy show pronounced peaks and their locations have been analysed by finite-size scaling. It was found that the dual model and the original model coincide when one assumes an unusual finite-size scaling ansatz for both models. The ansatz was supported by taking the ground state degeneracy under consideration. Recently, order parameters for the model have been determined that could significantly improve the estimations on the transition temperature (7).

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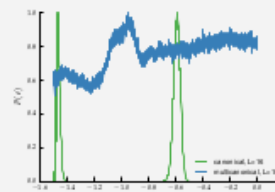
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- Spin case  $\kappa = 1$ :
 
$$\mathcal{H} = -\frac{1}{2} \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l \quad (1)$$
- interface energy and entropic penalty
- strong first order phase transition
- Zero-temperature ground state:
  - elementary cubes with same energy
- composing the bulk ground state for periodic boundary conditions, even lattice sizes
- flip of whole planes parallel to either one of the  $xy, yz, xz$  planes allowed
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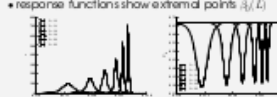
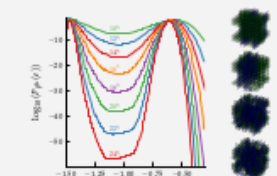


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## RESULTS FOR THE ORIGINAL MODEL $\mathcal{H}(\kappa = 1)$

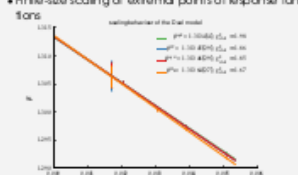
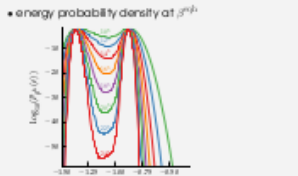
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- energy probability density at  $\beta_f^L$ 

- reduced interface tension  $\tilde{\sigma} = \beta\epsilon_f$  can be extracted for several lattice sizes,
 
$$\tilde{\sigma}(L) = \frac{1}{2L^2} \ln \left( \frac{\max P_\mu^{\text{MC}}(L)}{\max P_\mu^{\text{MC}}(L)} \right)$$
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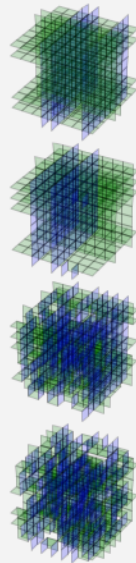
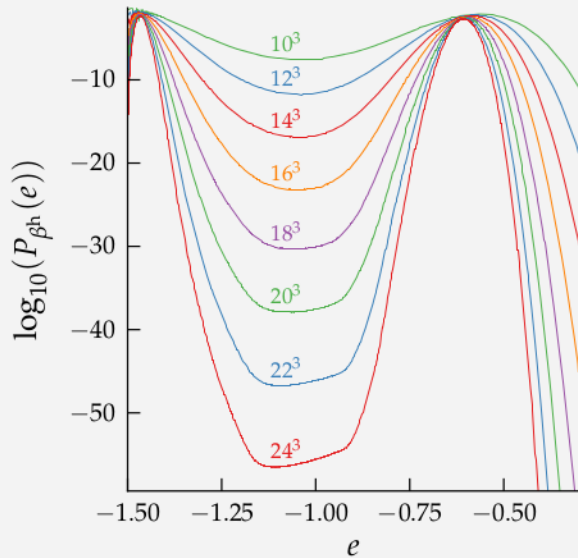
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$$H = -\frac{1}{2} \sum_{[i,j,k,l]} s_i s_j s_k s_l$$



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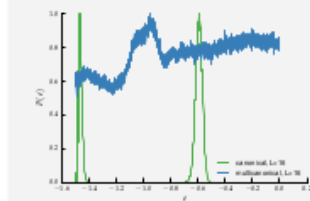
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- Zero-temperature ground state:
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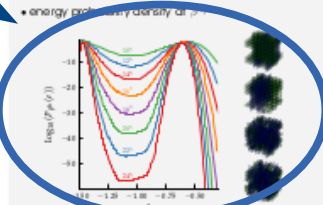
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$$C = \beta^2 \left( \langle E^2 \rangle - \langle E \rangle^2 \right), \quad U_4 = 1 - \frac{\langle E^4 \rangle}{3 \langle E^2 \rangle^2}$$

## RESULTS FOR THE ORIGINAL MODEL $\mathcal{H}(\kappa = 0)$

- response functions show extremal points  $\beta_c(L)$
- best agreement with scaling ansatz
 
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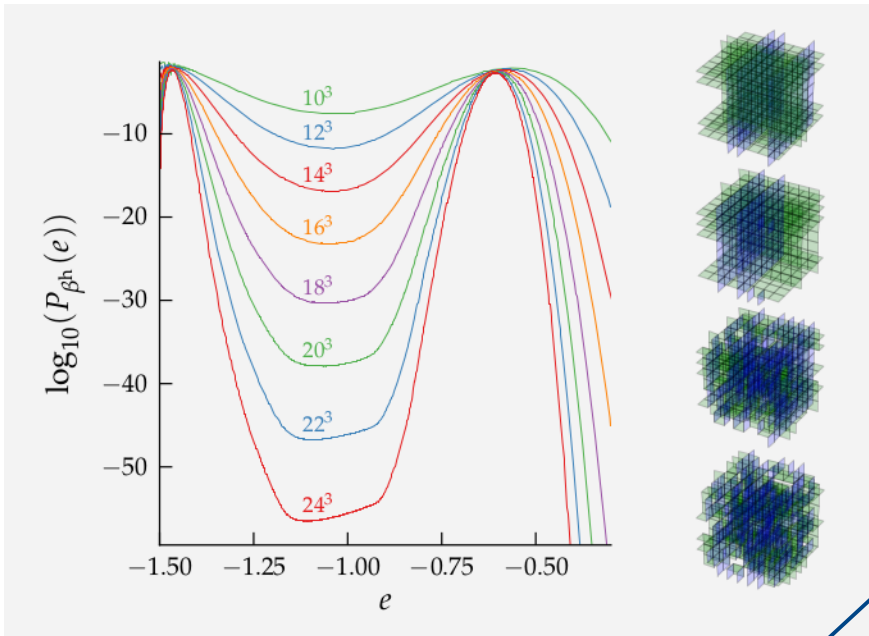
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$$\beta^{C_v}(L) = \beta^\infty - \frac{\log(q)}{\Delta e L^3} + O\left(\frac{1}{L^6}\right)$$

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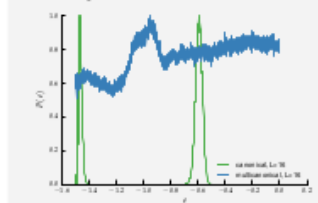
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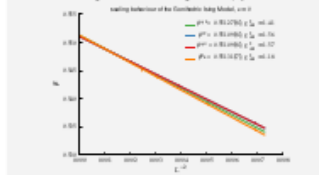
- expectation values of response functions (3,4):
 
$$\langle \beta(L) \rangle = \beta_0^\infty + aL^{-3} + O(L^{-6})$$
- valid for observables  $\beta(L)$  of response functions like first capacity  $C$  or Binder's cumulant  $U_3$ 

$$C = \beta^2 \left( \langle E^2 \rangle - \langle E \rangle^2 \right), \quad U_3 = 1 - \frac{\langle E^3 \rangle}{3 \langle E \rangle^3}$$

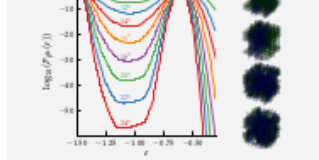
### RESULTS FOR THE ORIGINAL MODEL $\mathcal{H}(\kappa=0)$

- response functions show extremal points  $\beta(L)$
- best agreement with scaling ansatz
 
$$\beta(L) = \beta_0^\infty + aL^{-2} + O(L^{-6})$$
- amplitude in Eq. (2) is  $a \propto \ln(q) \propto \ln(2^{2L}) \propto L$

$\Rightarrow$  finite-size scaling corrections in increased due to exponential bulk ground state degeneracy  $g$



- energy probability density at  $\beta^*$



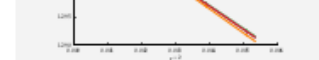
- reduced interface tension  $\tilde{\sigma} = \beta^* \epsilon$  can be extracted for several lattice sizes,
 
$$\tilde{\sigma}(L) = \frac{1}{2L^2} \ln \left( \frac{\max(P_{\text{real}}(L))}{\max(P_{\text{MC}}(L))} \right)$$
- and linear regression leads to
 
$$\tilde{\sigma}(L \rightarrow \infty) = 0.118(3)$$

### THE DUAL MODEL $\mathcal{H}^d$

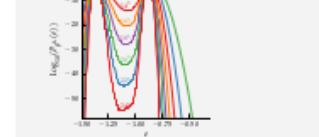
- two spins  $s_i \in \{-1, +1\}$  on each vertex of 3D lattice
- non-trivial interaction of two Ising spin chains due to the last sum
- canonical simulations contradicted each other w.r.t. transition temperatures in the original model
  - $\beta_0^\infty = 0.510(2)$  ...from dual model (9)
  - $\beta_0^\infty = 0.547(7)(6)$  ...from original model (8)

### RESULTS FOR THE DUAL MODEL $\mathcal{H}^d$

- finite-size scaling of extremal points of response functions



- energy probability density at  $\beta^*$



- reduced interface tension
 
$$\tilde{\sigma}(L \rightarrow \infty) = 0.122(6)$$

### AGREEMENT OF THE MODELS

- inverse transition temperatures associated by
 
$$\beta^* = -\ln \left( \tanh \left( \frac{\beta^*}{2} \right) \right)$$
- resulting in
  - $\beta_0^\infty = 0.51123(15)$  ...from dual model
  - $\beta_0^\infty = 0.51131(7)$  ...from original model
- $\Rightarrow$  our multicanonical data is consistent

### CONCLUSIONS & OUTLOOK

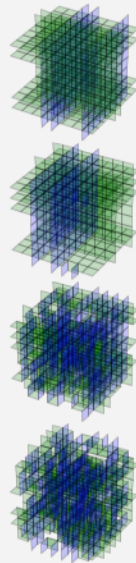
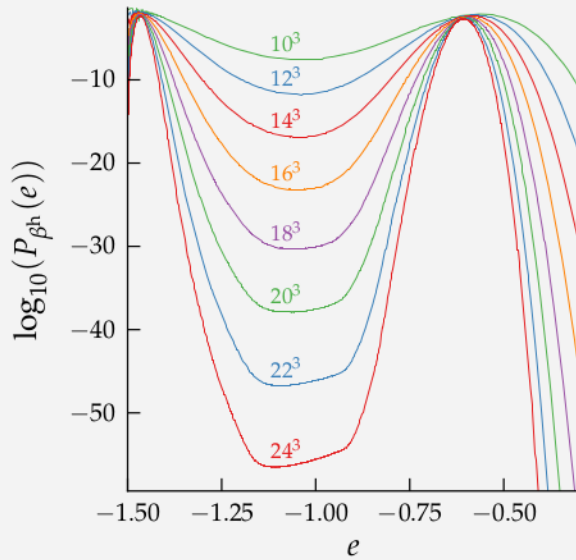
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$$H = -\frac{1}{2} \sum_{[i,j,k,l]} s_i s_j s_k s_l$$



$$\beta^{C_v}(L) = \beta^\infty - \frac{\log(q)}{\Delta e L^3} + O\left(\frac{1}{L^6}\right); \quad \log(q) \propto L$$

Marco Müller<sup>1</sup>, Desmond A. Johnston<sup>2</sup>, Wolfhard Janke<sup>1</sup>  
<sup>1</sup>Institut für Theoretische Physik, Universität Leipzig, Germany  
<sup>2</sup>Math. Dept., Heriot-Watt University, Edinburgh, United Kingdom

## ORIGIN AND MOTIVATION

The Gonihedric Ising model arises from bosonic string theory as a possible discretisation of the area swept out by a string worldsheet moving through spacetime (1). The name comprises the greek words gonik (angle) and hedra (face) as a reminder of the origin.

The lattice spin model allows the application of sophisticated generalised ensemble Monte Carlo methods that are tailored to tackle many open questions on the characteristics of phases occurring.

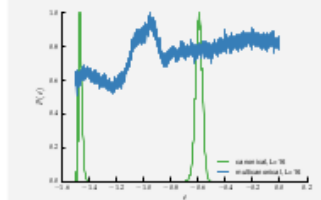
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- spins  $s_i \in \{-1, +1\}$  on each vertex of a 3D cubic lattice, linear lattice size  $L$ .
- plaquettes in the dual lattice separate contiguous spins with opposite signs
- bulk of plaquettes defines a surface
- Hamiltonian:
 
$$H = -2\kappa \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \frac{\kappa}{2} \sum_{\langle i,j,k \rangle} \sigma_i \sigma_j - \frac{1-\kappa}{2} \sum_{\langle i,j,k,l \rangle} \sigma_i \sigma_j \sigma_k \sigma_l$$
- only the area instead of surface area contributes to the partition function (2)
- $\kappa$ ... surface self-avoidance control parameter, complete self-avoidance for  $\kappa \rightarrow \infty$
- Special Case  $\kappa = 0$ :
 
$$H = -\frac{1}{2} \sum_{\langle i,j,k,l \rangle} \sigma_i \sigma_j \sigma_k \sigma_l \quad (1)$$

- interaction without energetic penalty
  - strong first order phase transition
  - Zero-temperature ground state:
    - elementary cubes with same energy
- 
- composing the bulk ground state for periodic boundary conditions, even lattice sizes
  - flip of whole planes parallel to either one of the  $xy, yz, xz$  planes allowed
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- canonical simulations suffer from *supercritical slowing down* near the phase transition, rare states are suppressed by a factor  $\propto \exp(-2\beta e L^3)$  with interface tension  $\sigma$ , inverse temperature  $\beta$
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- two spins  $s_i \in \{-1, +1\}$  on each vertex of 3D lattice
- non-trivial interaction of two Ising spin chains due to the last sum
- canonical simulations contradicted each other w.r.t. transition temperatures in the original model
 
$$\beta_f^\infty = 0.510(2) \quad \dots \text{from dual model (5)}$$

$$\beta_f^\infty = 0.547(7)(6) \quad \dots \text{from original model (6)}$$

## RESULTS FOR THE DUAL MODEL $\mathcal{H}^d$

- finite-size scaling of extremal points of response functions
- energy probability density at  $\beta_f^*$
- reduced interface tension
 
$$\tilde{\sigma}(L \rightarrow \infty) = 0.122(6)$$

## AGREEMENT OF THE MODELS

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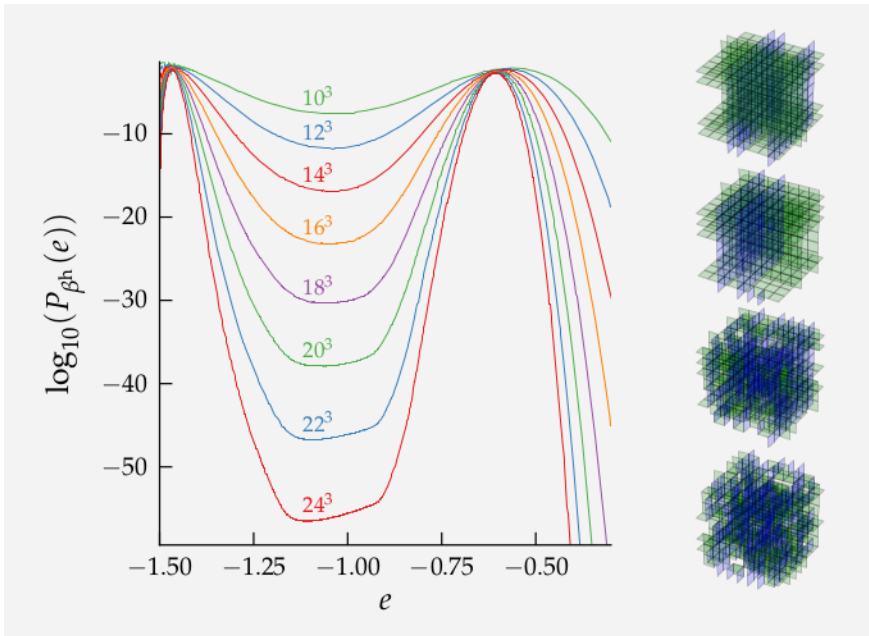
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Marco Müller<sup>1</sup>, Desmond A. Johnston<sup>2</sup>, Wolfhard Janke<sup>1</sup>  
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- only the area instead of surface area contributes to the partition function (2)
- $\kappa$ ... surface self-avoidance control parameter, complete self-avoidance for  $\kappa \rightarrow \infty$
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$$C = \beta^2((E^2) - (E)^2), \quad U_4 = 1 - \frac{(E^4)}{\beta(E^2)^2}$$

### RESULTS FOR THE ORIGINAL MODEL $\mathcal{H}(\kappa = 0)$

- response functions show extremal points  $\beta_f(L)$
- best fit ansatz (log ansatz)
 
$$\beta_f(L) = \beta_f^\infty + aL^{-2} + O(L^{-4})$$
- amplitude in eq. (3) is  $\ln(g) \pm \ln(2^{2L}) \propto L$
- finite-size scaling corrections in increased due to exponential bulk ground state degeneracy  $g$
- energy probability density at  $\beta_f^*$

- reduced interface tension  $\tilde{\tau} = \beta\epsilon$  can be extracted for several lattice sizes,
 
$$\tilde{\tau}(L) = \frac{1}{2L^2} \ln \left( \frac{\max P_\mu^{\text{MC}}(L)}{\max P_\mu^{\text{can}}(L)} \right)$$
 and linear regression leads to
 
$$\tilde{\tau}(L \rightarrow \infty) = 0.118(3)$$

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$$\beta_f^\infty = 0.510(2) \quad \dots \text{from dual model (5)}$$

$$\beta_f^\infty = 0.547(7)(6) \quad \dots \text{from original model (6)}$$

### RESULTS FOR THE DUAL MODEL $\mathcal{H}^d$

- finite-size scaling of extremal points of response functions
- energy probability density at  $\beta_f^*$
- reduced interface tension
 
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### AGREEMENT OF THE MODELS

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### CONCLUSIONS & OUTLOOK

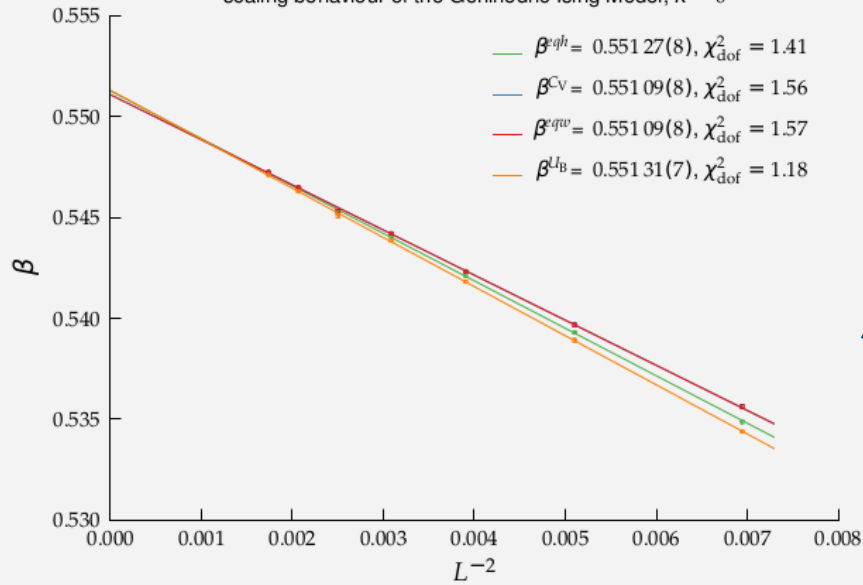
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scaling behaviour of the Gonihedric Ising Model,  $\kappa = 0$



$$\beta^{C_v}(L) = \beta^\infty - \frac{\log(q)}{\Delta e L^3} + O\left(\frac{1}{L^6}\right); \quad \log(q) \propto L$$

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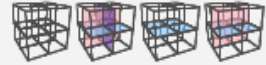
Hamiltonian:

$$H = -\frac{1}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \frac{\kappa}{2} \sum_{(i,j,k)} \sigma_i \sigma_j - \frac{1-\kappa}{2} \sum_{(i,j,k,l)} \sigma_i \sigma_j \sigma_k \sigma_l$$

- only the arc size instead of surface area contributes to the partition function (2)
- $\kappa$ : surface self-avoidance control parameter, complete self-avoidance for  $\kappa \rightarrow \infty$
- Special Case  $\kappa = 0$ :

$$H = -\frac{1}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j \quad (1)$$

- interaction without energetic penalty
- strong first order phase transition
- Zero-temperature ground state: elementary cubes with same energy



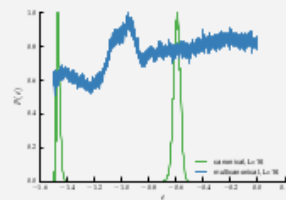
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- canonical simulations suffer from supercritical slowing down near the phase transition, rare states are suppressed by a factor  $\propto \exp(-2\beta\epsilon L^2)$  with interface tension  $\epsilon$ , inverse temperature  $\beta$
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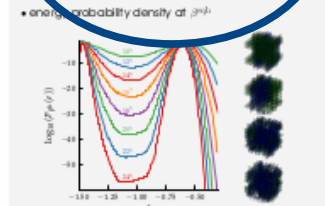
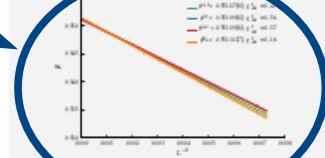


## FINITE-SIZE SCALING

- expected first-order finite-size scaling (3,4):
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- valid for extremal points  $\beta_f$  of response functions like heat capacity  $C$  or Binder's cumulant  $U_4$
- $$C = \beta^2 \left( \langle E^2 \rangle - \langle E \rangle^2 \right), \quad U_4 = 1 - \frac{\langle E^4 \rangle}{3 \langle E^2 \rangle^2}$$

## RESULTS FOR THE ORIGINAL MODEL $\mathcal{H}(\kappa = 0)$

- response functions show extremal points  $\beta_f(L)$
  - best agreement with scaling ansatz
- $$\beta_f(L) = \beta_f^\infty + aL^{-2} + O(L^{-6}) \quad (3)$$
- amplitude in Eq. (2) is  $a \propto b(q) \propto \ln(2^{3L}) \propto L$
  - finite-size scaling corrections increased due to exponential bulk ground state degeneracy



- reduced interface tension  $\tilde{\epsilon} = \beta\epsilon$  can be extracted for several lattice sizes,

$$\tilde{\epsilon}(L) = \frac{1}{2L^2} \ln \left( \frac{\max P_{\text{real}}(L)}{\max P_{\text{MC}}(L)} \right)$$

and linear regression leads to

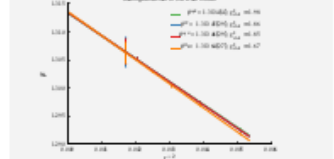
$$\tilde{\epsilon}(L \rightarrow \infty) = 0.118(3)$$

## THE DUAL MODEL $\mathcal{H}^d$

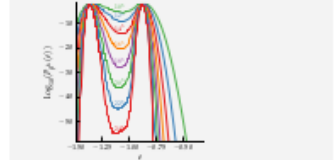
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  - non-trivial interaction of two Ising spin chains due to the last sum
  - canonical simulations contradicted each other w.r.t. transition temperatures in the original model
- $$H^d = -\frac{1}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \frac{1}{2} \sum_{\langle i,j \rangle} \tau_i \tau_j - \frac{1}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tau_i \tau_j \quad (4)$$
- $\beta_f^{\text{Cv}} = 0.512(15)$  ... from dual model (5)
  - $\beta_f^{\text{Cv}} = 0.551(7)$  ... from original model (6)

## RESULTS FOR THE DUAL MODEL $\mathcal{H}^d$

- finite-size scaling of extremal points of response functions



- energy probability density at  $\beta^qtw$



- reduced interface tension
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- $$\beta^* = -\ln \left( \frac{\beta^{\text{Cv}}}{\beta^{\text{Cv}}} \right)$$
- resulting in
- $$\beta_f^{\text{Cv}} = 0.512(15) \quad \dots \text{from dual model}$$
- $$\beta_f^{\text{Cv}} = 0.551(7) \quad \dots \text{from original model}$$
- $\Rightarrow$  our multicanonical data is consistent

## CONCLUSIONS & OUTLOOK

Multicanonical simulation of the Gonihedric Ising model with vanishing energy penalty for intersecting surfaces and a dual model have been conducted. The response functions of the energy show pronounced peaks and their locations have been analysed by finite-size scaling. It was found that the dual model and the original model coincide when one assumes an unusual finite-size scaling ansatz for both models. The ansatz was supported by taking the ground state degeneracy under consideration. Recently, order parameters for the model have been determined that could significantly improve the estimations on the transition temperature (7).

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