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CQT
Computational Quantum Field Theory

Multicanonical Analysis of the Gonihedric Ising Model

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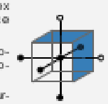
[2] Math. Dept., Heriot-Watt University, Edinburgh, United Kingdom

Origin and Motivation

The Gonihedric Ising model arises from bosonic string theory as a possible discretisation of the area swept out by a string worldsheet moving through spacetime (1). The name comprises the greek words gonia (angle) and hedra (face) as a reminder of the origin.

The lattice spin model allows the application of sophisticated generalised ensemble Monte Carlo methods that are tailored to tackle many open questions on the characteristics of phases occurring.

The Gonihedric Ising model



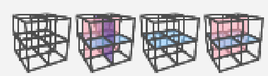
- spins $\sigma \in \{-1,+1\}$ on each vertex of a 3D cubic lattice, linear lattice size L
- plaquettes in the dual lattice separate contiguous spins with opposite signs
- bulk of plaquettes defines a surface
- Hamiltonian:

$$\mathcal{H} = -2\kappa \sum_{\langle i,j\rangle} \sigma_i\sigma_j + \frac{\kappa}{2} \sum_{\langle\langle i,j\rangle\rangle} \sigma_i\sigma_j - \frac{1-\kappa}{2} \sum_{[i,j,k,l]} \sigma_i\sigma_j\sigma_k\sigma_l$$

  - only *linear size* instead of *surface area* contributes to the partition function (2)
  - κ ... surface self-avoidance control parameter, complete self-avoidance for $\kappa \to \infty$
- Special Case $\kappa = 0$:

$$\mathcal{H} = -\frac{1}{2} \sum_{[i,j,k,l]} \sigma_i\sigma_j\sigma_k\sigma_l \qquad (1)$$

  - intersection without energetic penalty
  - strong first order phase transition
- Zero-temperature ground state:
  - elementary cubes with same energy



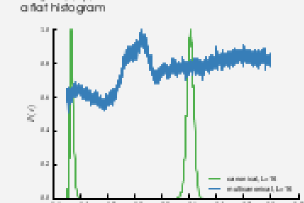
  - composing the bulk ground state for periodic boundary conditions, even lattice sizes
  - flip of whole planes parallel to either one of the xy, yz, zx-planes allowed
  - bulk ground state degeneracy $g = 2^{3L}$, with L being the linear lattice size

Method

- canonical simulations suffer from *supercritical slowing down* near the phase transition, rare states are suppressed by a factor $\propto \exp(-2\beta\sigma L^2)$ with interface tension σ, inverse temperature β
- multicanonical simulation: promote rare states μ with energy $E_\mu = \mathcal{H}\{\mu\} =: e(\mu)L^3$ using

$$p_\mu^{\mathrm{muca}}(\beta) = \exp(-\beta E_\mu - f(E_\mu))$$

- weights $f(E_\mu)$ are determined beforehand to achieve a flat histogram



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Finite-Size Scaling

- expected first-order finite-size scaling (3,4):

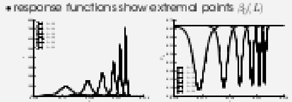
$$\beta_i(L) = \beta_0^\infty + aL^{-3} + \mathcal{O}\left(L^{-6}\right) \qquad (2)$$

- valid for extremal points β_i of response functions like heat capacity C or Binder's cumulant U_B

$$C = \beta^2\left(\left\langle E^2\right\rangle - \langle E\rangle^2\right), \qquad U_\mathrm{B} = 1 - \frac{\langle E^4\rangle}{3\langle E^2\rangle^2}$$

Results for the original model $\mathcal{H}(\kappa = 0)$

- response functions show extremal points $\beta_i(L)$

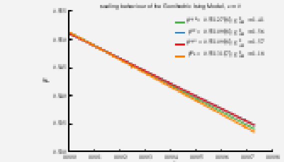


- best agreement with scaling ansatz

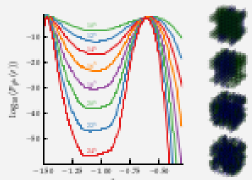
$$\beta_i(L) = \beta_0^\infty + \hat{a}L^{-2} + \mathcal{O}\left(L^{-6}\right). \qquad (3)$$

- amplitude in Eq. (2) is $a \propto \ln(g) \propto \ln(2^{3L}) \propto L$

⇒ finite-size scaling corrections increased due to exponential bulk ground state degeneracy g



- energy probability density at β^{eqh}



- reduced interface tension $\bar{\sigma} = \beta\sigma$ can be extracted for several lattice sizes,

$$\bar{\sigma}(L) = \frac{1}{2L^2}\ln\left(\frac{\max[P_{\beta^{eqh}}(L)]}{\min[P_{\beta^{eqh}}(L)]}\right),$$

and linear regression leads to

$$\bar{\sigma}(L \to \infty) = 0.1183(6)$$

The dual model $\mathcal{H}^\mathrm{d}$

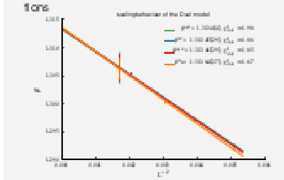
$$\mathcal{H}^\mathrm{d} = -\frac{1}{2}\sum_{\langle i,j\rangle_x} \sigma_i\sigma_j - \frac{1}{2}\sum_{\langle i,j\rangle_y} \tau_i\tau_j - \frac{1}{2}\sum_{\langle i,j\rangle_z} \sigma_i\sigma_j\tau_i\tau_j. \qquad (4)$$

- two spins σ, τ on each vertex of 3D lattice
- non-trivial interaction of two Ising spin chains due to the last sum
- canonical simulations contradicted each other w.r.t. transition temperatures in the original model:

$\beta_0^\infty = 0.510(2)$ ... from dual model (5)
$\beta_0^\infty = 0.54757(63)$ ... from original model (6)

Results for the dual model $\mathcal{H}^\mathrm{d}$

- Finite-size scaling of extremal points of response functions



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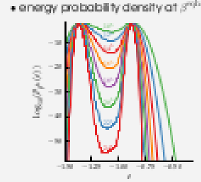

- reduced interface tension

$$\bar{\sigma}(L \to \infty) = 0.1223(6).$$

Agreement of the models

- inverse transition temperatures associated by

$$\beta^* = -\ln\left(\tanh\left(\frac{\beta^\infty}{2}\right)\right).$$

- resulting in

$\beta_0^\infty = 0.55123(15)$ ... from dual model
$\beta_0^\infty = 0.55131(7)$ ... from original model

⇒ our multicanonical data is consistent

Conclusions & Outlook

Multicanonical simulations of the Gonihedric Ising model with vanishing energy penalty for intersecting surfaces and a dual model have been conducted. The response functions of the energy show pronounced peaks and their locations have been analysed by finite-size scaling. It was found that the dual model and the original model coincide when one assumes an unusual finite-size scaling ansatz for both models. The ansatz was supported by taking the ground state degeneracy under consideration.
Recently, order parameters for the model have been determined that could significantly improve the estimations on the transition temperature (7).

References

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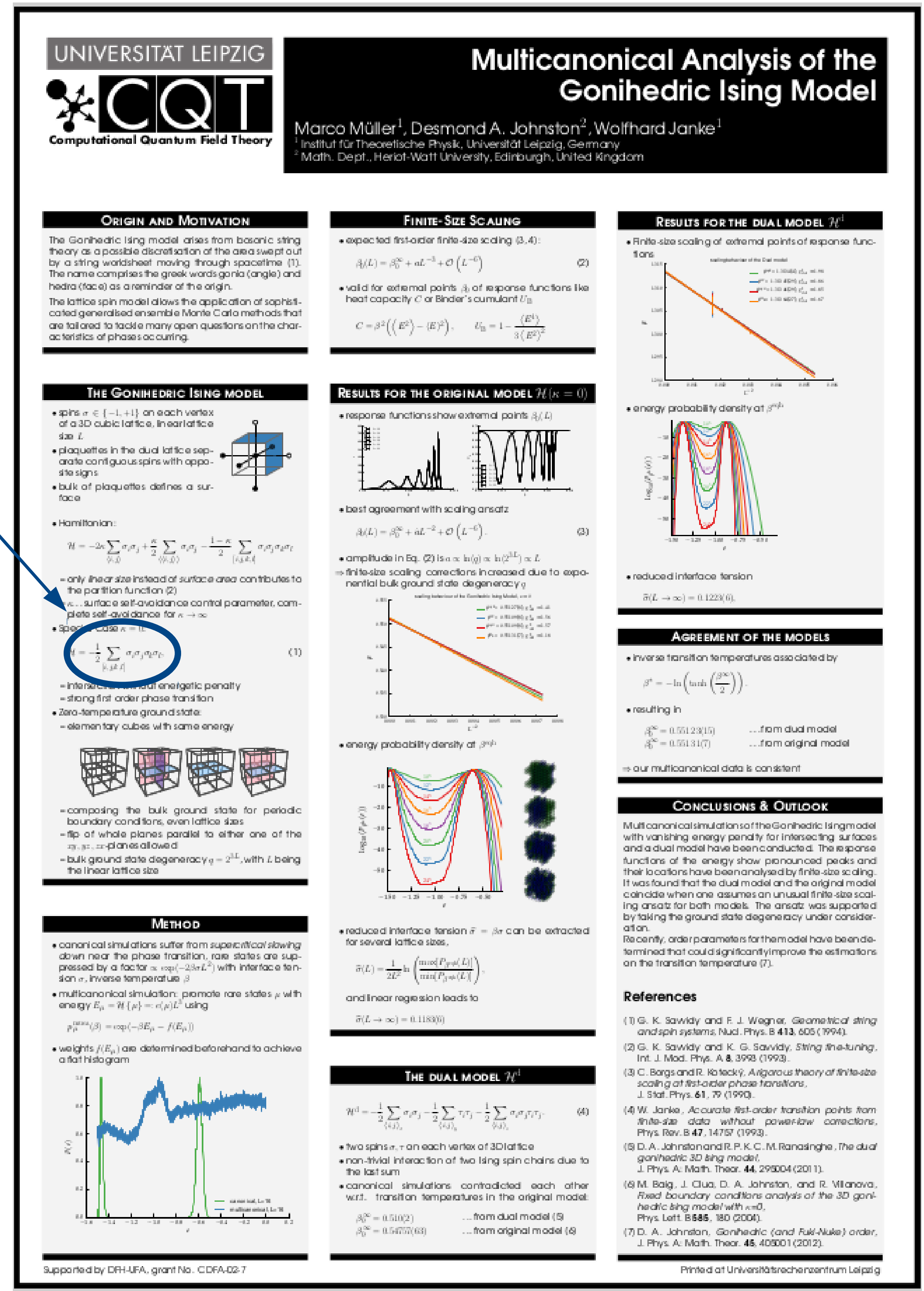

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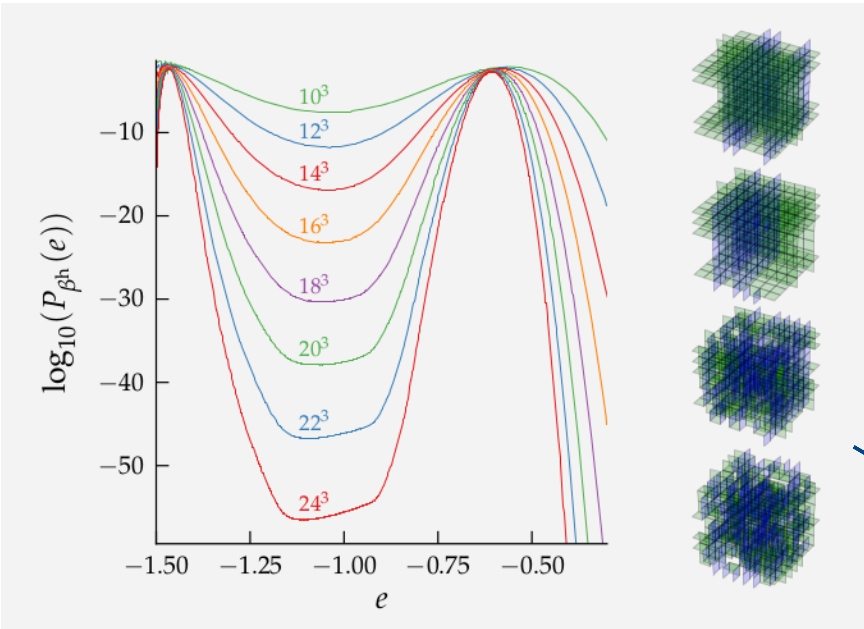

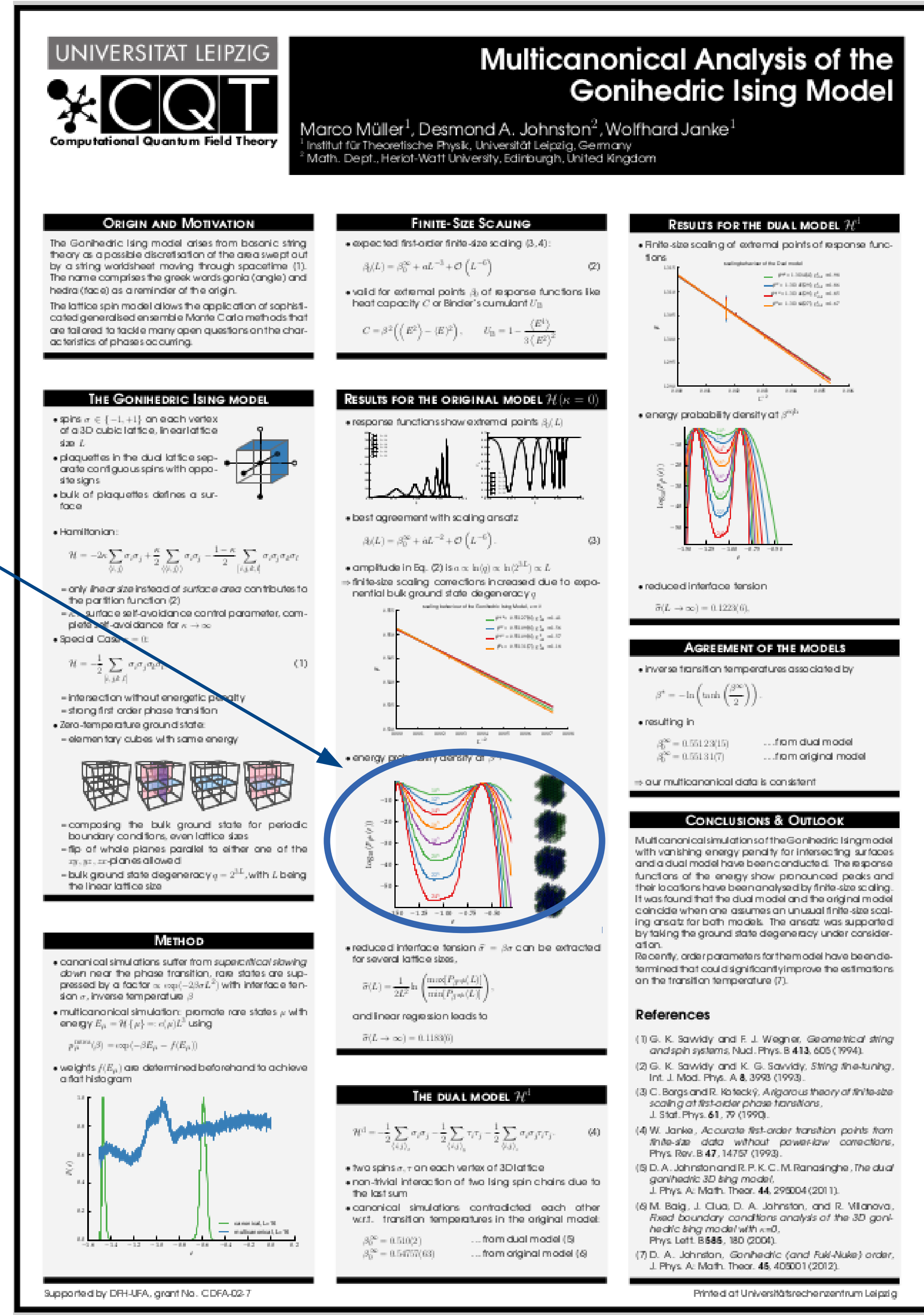

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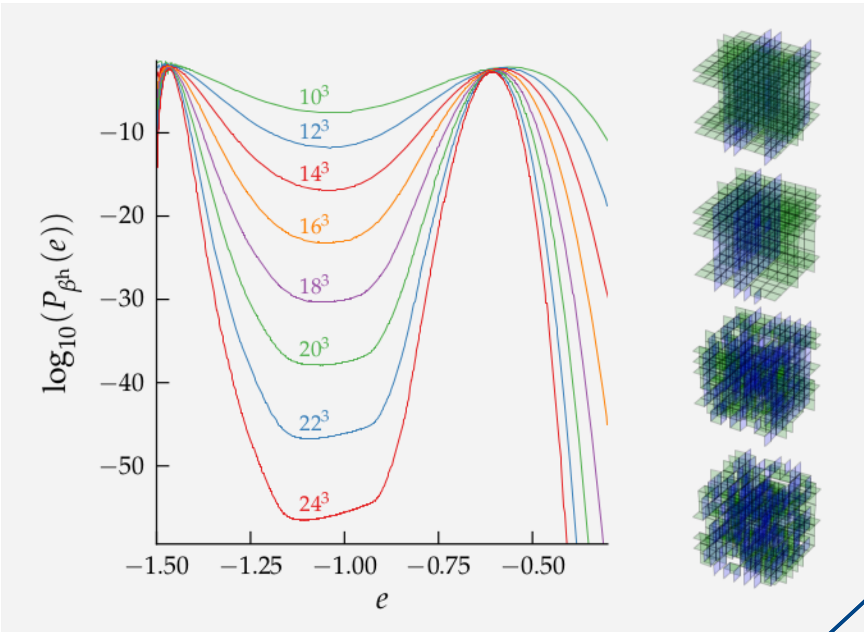
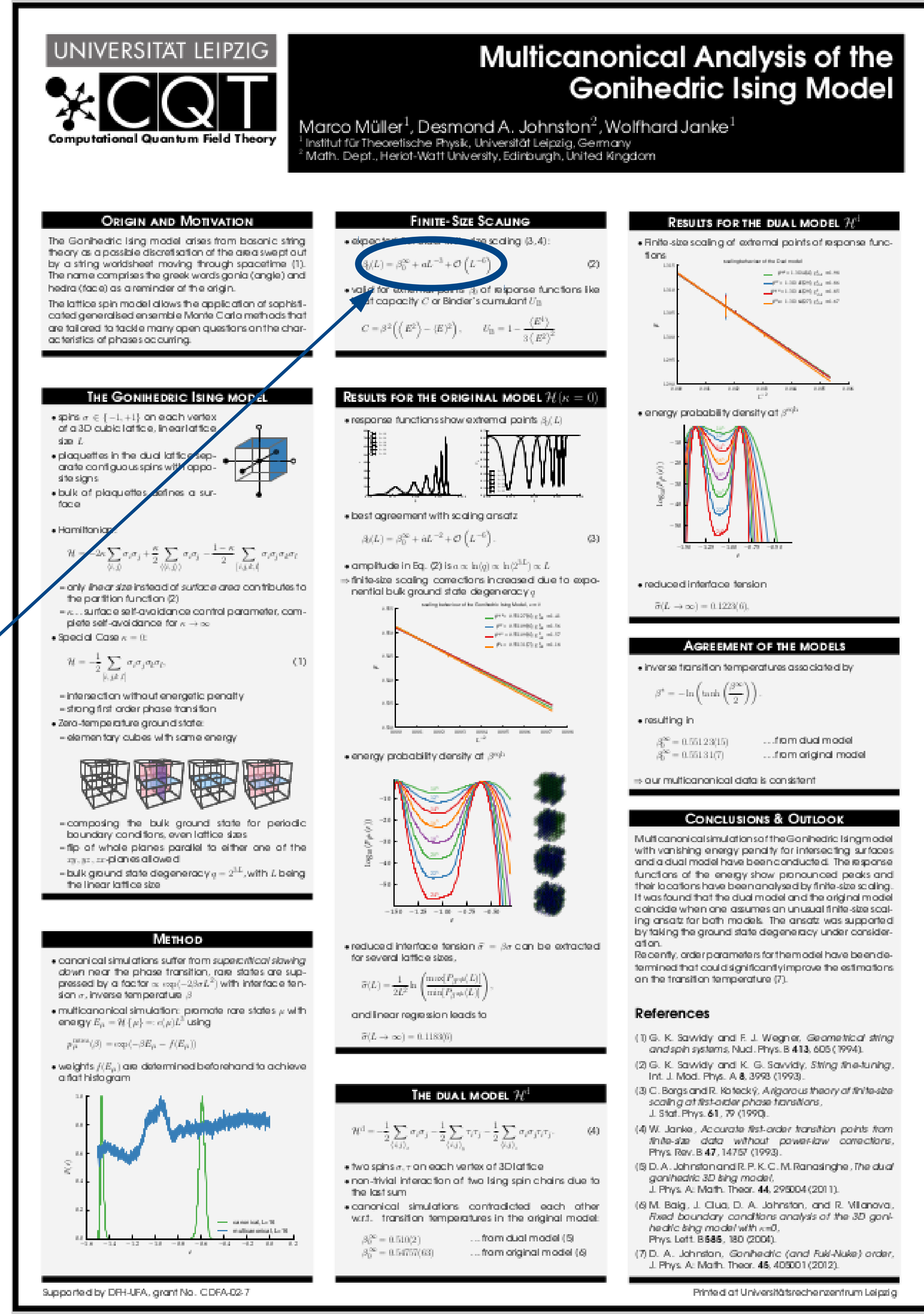


$$\beta^{C_V}(L) = \beta^{\infty} - \frac{\log(q)}{\Delta e\, L^3} + O\left(\frac{1}{L^6}\right)$$



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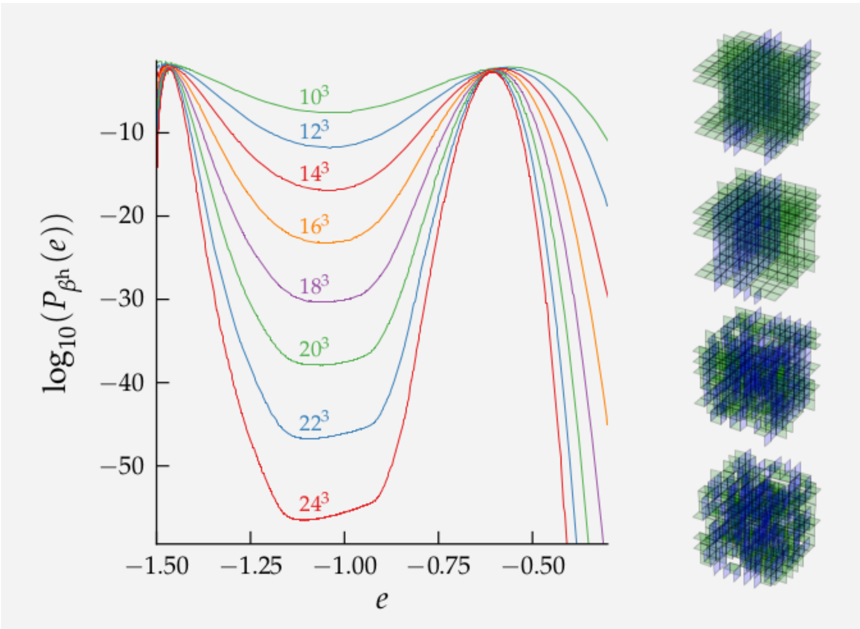

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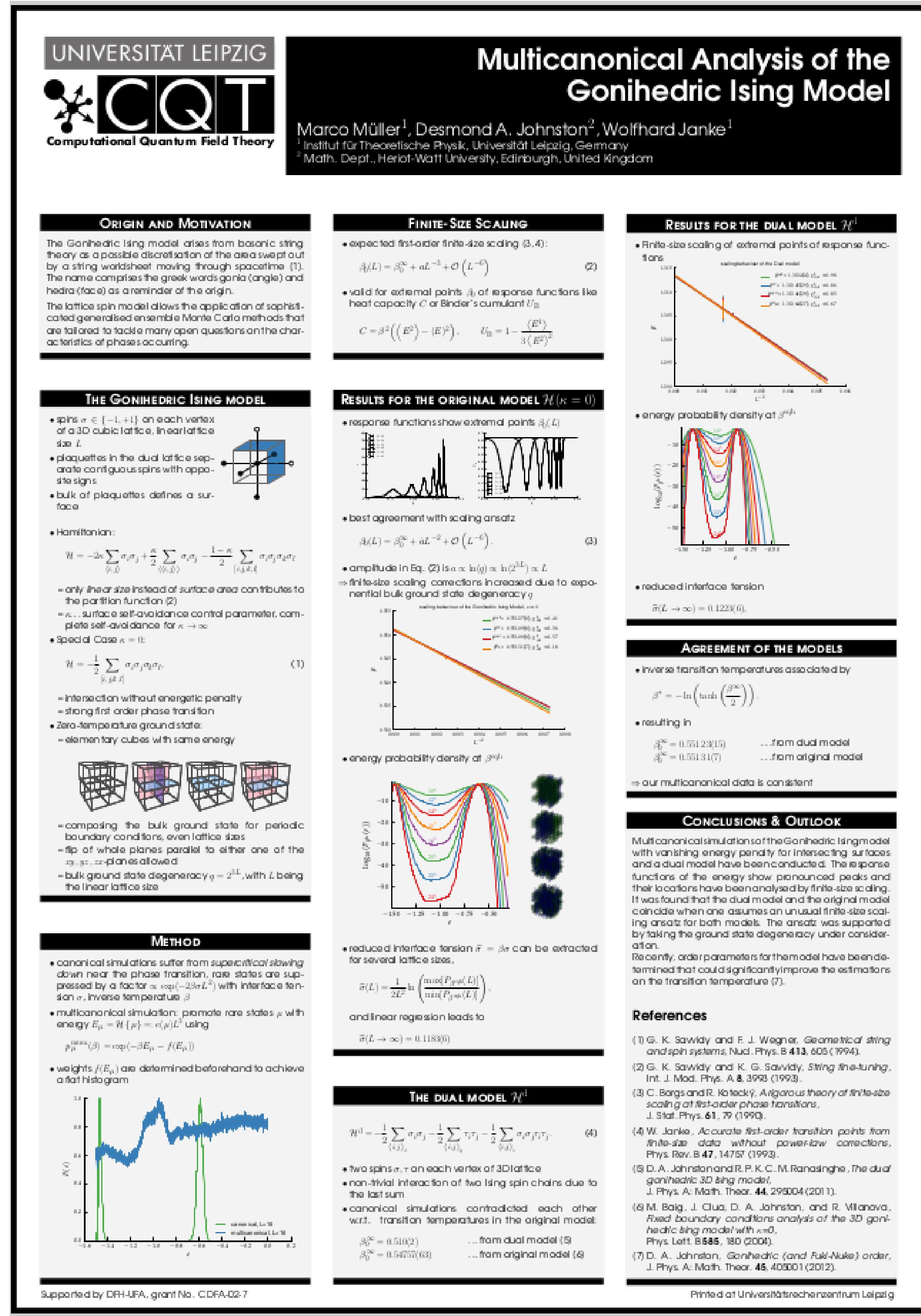

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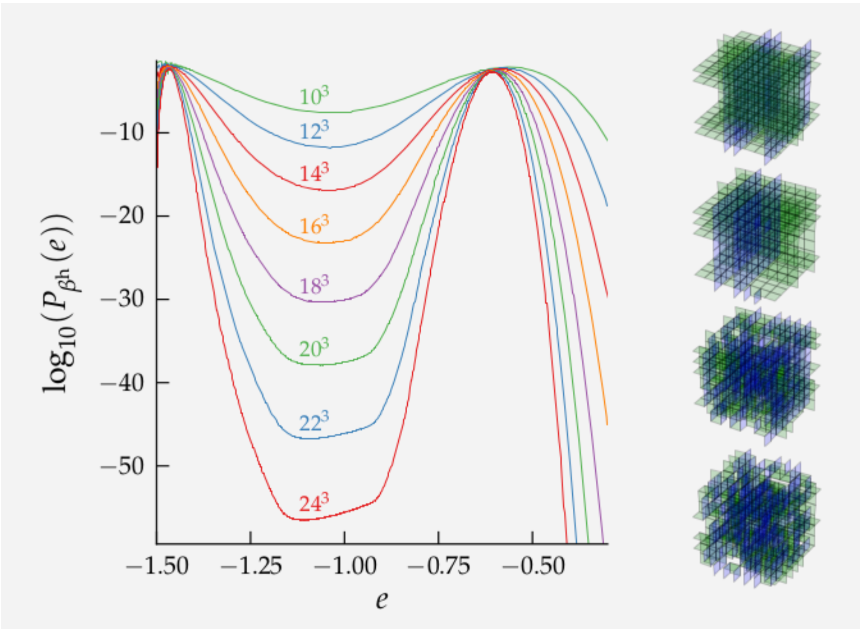

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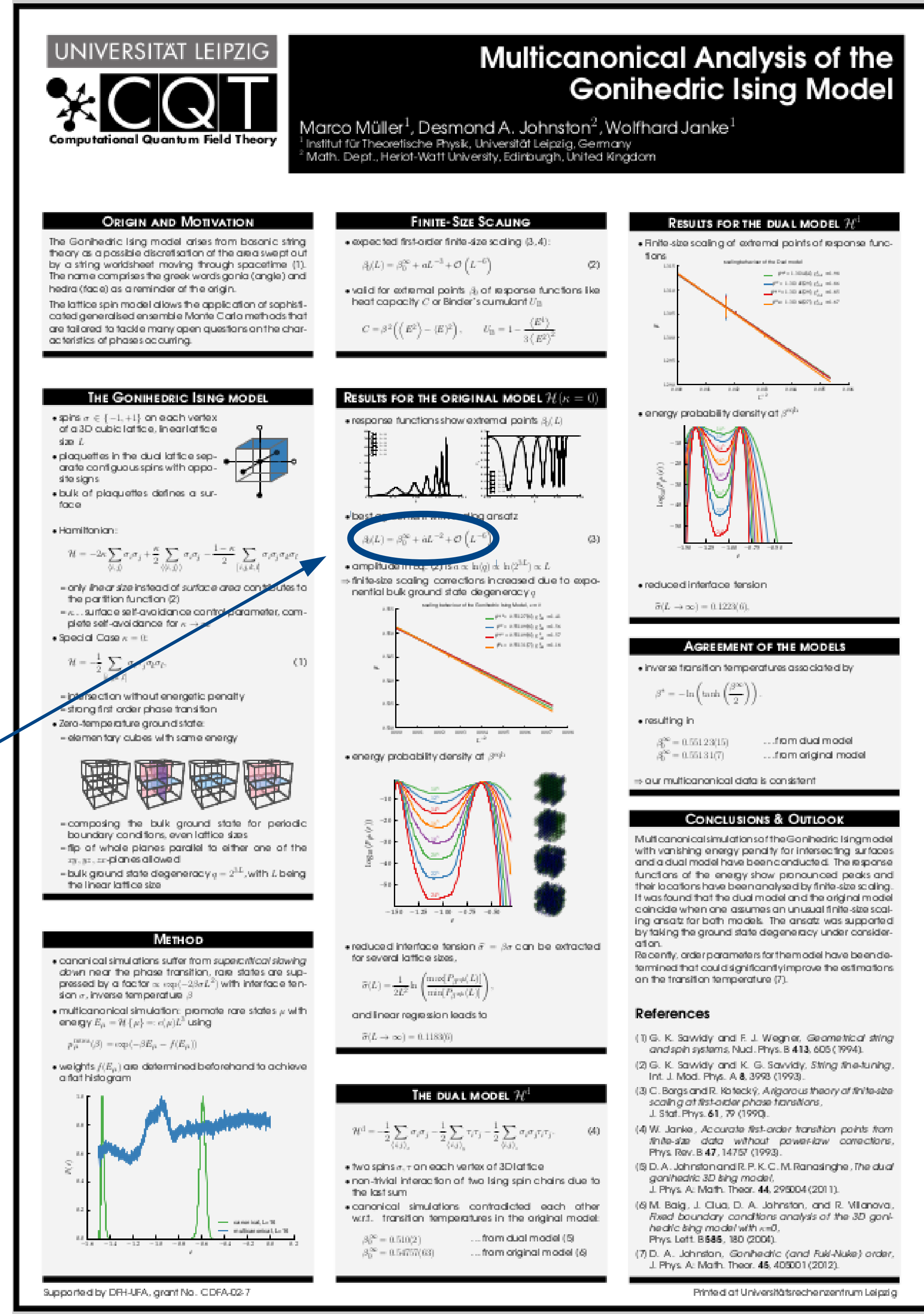

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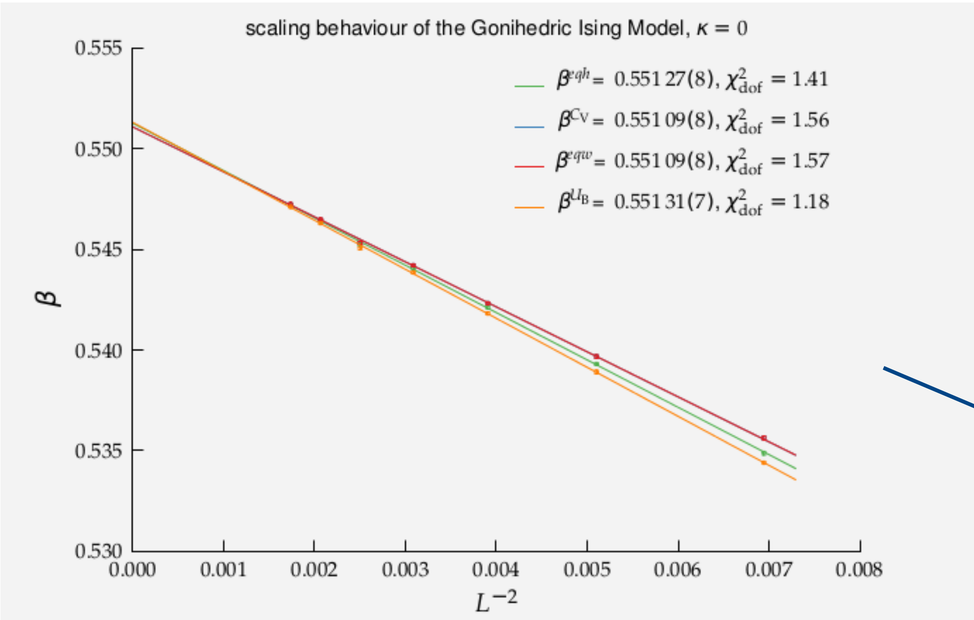

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