Potts Models with Invisible States

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Plan of talk

Invisible states

Getting a first order transition where symmetry suggests second

Thin graphs

Mean field theory by using r-regular (“thin”) random graphs

Order of transition

- Order of transition - dimension, symmetry

$$\mathcal{H}_q = -\sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j}$$

Two dimensions ferromagnetic, $q = 2, 3, 4$ continuous

- Mean field, $q = 2$ continuous

- R. Tamura, S. Tanaka and N. Kawashima - some observations didn't fit paradigm

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$$\mathcal{H} = \lambda J_1 \sum_{\langle i,j \rangle_{\text{axis } 1}} \mathbf{s}_i \cdot \mathbf{s}_j + J_1 \sum_{\langle i,j \rangle_{\text{axis } 2,3}} \mathbf{s}_i \cdot \mathbf{s}_j + J_3 \sum_{\langle\langle i,j \rangle\rangle} \mathbf{s}_i \cdot \mathbf{s}_j,$$

Tinkering...

Tinkering usually *softens* phase transitions

Fisher renormalization

$$\alpha \to -\frac{\alpha}{1-\alpha}$$

Effects of disorder, Harris criterion etc

Spin models on planar random graphs - KPZ
Ising:

$$\alpha = 0 \to \alpha = -1$$

Invisible States

Introduce states which contribute to entropy but *not* (internal) energy

Still contribute to free energy

Do this for Potts model, see what happens to symmetry $\leftrightarrow$ order

Invisible States

Entropy, but no energy

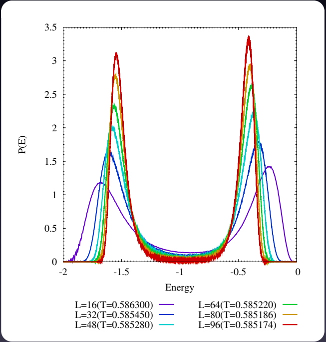
$$\mathcal{H}_{(q,r)} = -\sum_{\langle i,j \rangle} \delta_{s_i,s_j} \sum_{\alpha=1}^{q} \delta_{s_i,\alpha}\delta_{s_j,\alpha}, \qquad s_i = 1, \cdots, q, q+1, \cdots, q+r,$$

Rewrite

$$\mathcal{H}'_{(q,r)} = -\sum_{\langle i,j \rangle} \delta_{\sigma_i,\sigma_j} \sum_{\alpha=1}^{q} \delta_{\sigma_i,\alpha}\delta_{\sigma_j,\alpha} - T \ln r \sum_{i} \delta_{\sigma_i,0}, \qquad \sigma_i = 0, 1, \cdots, q,$$

Resulting Phase Diagram

Two dimensions (Monte-Carlo) $(3, 27)$ near PT



Analytic Observations

$(2, r)$ Potts is equivalent to BEG model

$$\begin{aligned} \mathcal{H}_{\mathrm{BEG}} &= -\frac{1}{2}\sum_{\langle i,j\rangle} t_i t_j - \frac{1}{2}\sum_{\langle i,j\rangle} t_i^2 t_j^2 - \mu \sum_i \left(1 - t_i^2\right), \\ t_i &= +1,\ 0,\ -1 \end{aligned}$$

if we take

$$\mu = T \ln r$$

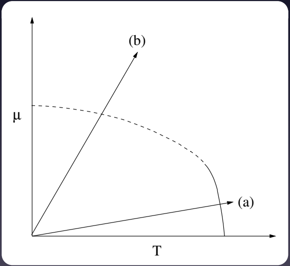
$$\mathcal{H}'_{(q,r)} = -\sum_{\langle i,j\rangle} \delta_{\sigma_i,\sigma_j} \sum_{\alpha=1}^{q} \delta_{\sigma_i,\alpha}\delta_{\sigma_j,\alpha} - T \ln r \sum_i \delta_{\sigma_i,0}, \qquad \sigma_i = 0, 1, \cdots, q,$$

Phase Diagram

$$\mathcal{H}_{\mathrm{BEG}} \;\; = \;\; -\frac{1}{2}\sum_{\langle i,j \rangle} t_i t_j - \frac{1}{2}\sum_{\langle i,j \rangle} t_i^2 t_j^2 - \mu \sum_i \left(1 - t_i^2\right) \;\; .$$

invisibility relation....

$$\mu = T \ln r$$



2nd order for $(2,1)$, $(2,2)$, and $(2,3)$-state Potts model

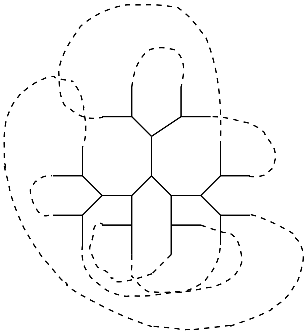
Universality

Critical r value depends on an external field

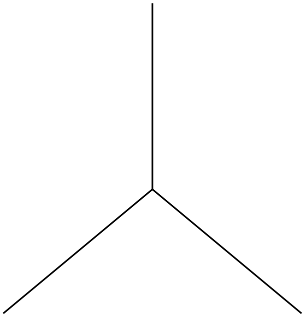
Suggests this is not universal

Investigate using mean field

Mean field theory using “Thin” Graphs



A not very big 3-regular graph

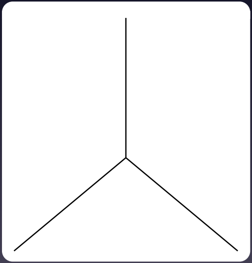


Individual cubic vertex

Counting r-regular random graphs

$$N_n = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda^{2n+1}} \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} \; \exp\left(-\frac{1}{2}\phi^2 + \frac{\lambda}{6}\phi^3 \right)$$

$$N_n = \left(\frac{1}{6}\right)^{2n} \frac{(6n-1)!!}{(2n)!}.$$



Decorated Thin Graphs - spin models

$$Z_n = \sum_{G^n} \sum_{\{\sigma\}} \exp\left(\beta \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j} \right)$$

$$Z_n \times N_n = \oint \frac{d\lambda}{\lambda^{2n+1}} \int d\phi_i \exp\left(-S_q\right)$$

$$S_q = \frac{1}{2} \sum_{i=1}^{q} \phi_i^2 - c \sum_{i<j} \phi_i \phi_j - \frac{\lambda}{3} \sum_{i=1}^{q} \phi_i^3 \ , \quad c = \frac{1}{\exp(2\beta) + q - 2}$$

Scale out λ: $\phi_i \to \phi_i / \lambda$, saddle point in ϕ_i

$$\frac{\partial S}{\partial \phi_i} = 0$$

BEG on Thin Graphs

Hamiltonian

$$\mathcal{H}_{\rm BEG} = -\frac{1}{2}\sum_{\langle i,j\rangle} t_i t_j - \frac{1}{2}\sum_{\langle i,j\rangle} t_i^2 t_j^2 - \mu \sum_i \left(1 - t_i^2\right) \ .$$

Action

$$\begin{aligned} S_{\rm BEG} &= \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2) - a(\phi_1\phi_3 + \phi_2\phi_3) \\ &- \frac{1}{3}(\phi_1^3 + \phi_2^3 + \Delta\,\phi_3^3) \end{aligned}$$

BEG on Thin Graphs

Dictionary

$$\begin{aligned} \exp(-\beta) &= \frac{a^2}{1-a^2} \\ a^3 \exp(\beta\mu) &= \Delta \,. \end{aligned}$$

$$a = \sqrt{\frac{1}{e^{\beta}+1}}$$

$$0 < a < 1/\sqrt{2}$$

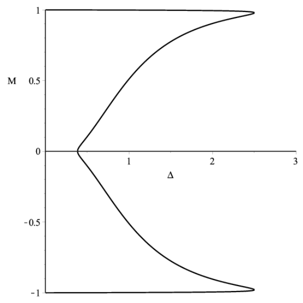
Now Solve...

$$\begin{aligned} \frac{\partial S_{\rm BEG}}{\partial \phi_1} &= \phi_1 - a\phi_3 - \phi_1^2 = 0 \\ \frac{\partial S_{\rm BEG}}{\partial \phi_2} &= \phi_2 - a\phi_3 - \phi_2^2 = 0 \\ \frac{\partial S_{\rm BEG}}{\partial \phi_3} &= \phi_3 - a(\phi_1 + \phi_2) - \Delta\phi_3^2 = 0 \end{aligned}$$

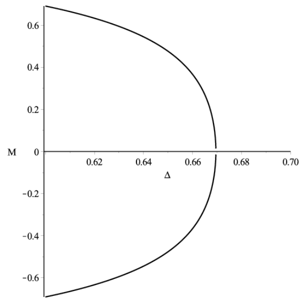
$$M = \frac{\phi_1^3 - \phi_2^3}{\phi_1^3 + \phi_2^3 + \Delta\,\phi_3^3}$$

Magnetization

$$M_{sp} = \pm \frac{\sqrt{\Delta(\Delta - 2a \pm 2a\sqrt{1-4\Delta a})}(2\Delta - a \pm a\sqrt{1-4\Delta a})}{2\Delta^2 - 6\Delta a \pm 4\Delta a\sqrt{1-4\Delta a} + 1 \mp \sqrt{1-4\Delta a}}$$



$a = 0.1$ (Low T)



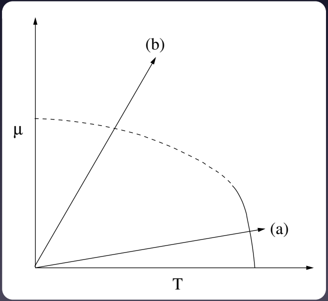
$a = 0.38$ (Higher T)

Phase Diagram

$$\mathcal{H}_{\mathrm{BEG}} \;=\; -\frac{1}{2}\sum_{\langle i,j\rangle} t_i t_j - \frac{1}{2}\sum_{\langle i,j\rangle} t_i^2 t_j^2 - \mu \sum_i \left(1 - t_i^2\right)\ .$$

invisibility relation....

$$\mu = T \ln r\ , \qquad a^3 r = \Delta$$



Translation

$0 < a < a_t = 1/\sqrt{8}$ First order

$1/\sqrt{8} < a < 1/2$ Second order

$1/2 < a < 1/\sqrt{2}$ No transition

$\Delta = a^3 r$, so $r = 16$

In fact (Izmailyan) with z neighbours

$$r_c(z) = \frac{4z}{3(z-1)} \left(\frac{z-1}{z-2} \right)^z$$

Conclusions

It is straightforward to obtain the mean-field BEG phase diagram on "thin" (i.e. r-regular) random graphs and the Bethe lattice

On 3-regular random graphs you need $r = 17$ invisible states to make the $(2, r)$ state Potts model transition first order

On z-regular random graphs you need

$$r_c(z) = \frac{4z}{3(z-1)} \left(\frac{z-1}{z-2} \right)^z$$

On *planar* 4-regular random graphs you need 223 (!)

References

R. Tamura, S. Tanaka and N. Kawashima, Prog. Theor. Phys. **124** (2010) 381; J. Phys: Conf. Ser. **297** (2011) 012022.

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