Random quantum magnets in $d \geq 2$ dimensions: Critical behavior and entanglement entropy

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Introduction

- Quantum phase transitions
  - takes place at $T = 0$
  - due to quantum flactuation
  - by varying a quantum control parameter
- Examples
  - rare-earth magnetic insulators
  - heavy-fermion compounds
  - high-temperature superconductors
  - two-dimensional electron gases
  - quantum magnets: $\mathrm{LiHoF_4}$
- Disorder often play an important rǒle
  - Anderson localization
  - many-body localization
  - quantum spin glasses: $\mathrm{LiHo_xY_{1-x}F_4}$
- Paradigmatic model: random transverse Ising model (RTIM)
  - in 1d exact results, also through strong disorder RG method
    - infinite disorder scaling at the critical point
    - dynamical (Griffiths-McCoy) singularities outside the critical point
  - in 2d numerical implementation of the SDRG method
    - infinite disorder scaling
    - but contradictionary results through the quantum cavity approach

Aim of the present talk

- Improve the numerical algorithm of the SDRG method
- Study the critical behavior of the RTIM for $d > 2$
- Study Erdős-Rényi random graphs ($d \to \infty$)
- Study the boundary critical behavior at surfaces, corners, edges
- Study the entanglement entropy and its singularity at the critical point

Random transverse Ising model

$$\boxed{\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i^z\sigma_j^z - \sum_i h_i\sigma_i^x}$$

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- J_{ij} couplings
- independent random numbers from the distribution $p(J)$
- h_i transverse fields
- independent random numbers from the distribution $q(h)$

.

- box-h disorder
  - $p(J) = \Theta(J)\Theta(1-J)$ ($\Theta(x)$: Heaviside step-function)
  - $q(h) = \frac{1}{h_b}\Theta(h)\Theta(h_b - h)$
- fix-h disorder
  - $p(J) = \Theta(J)\Theta(1-J)$
  - $q(h) = \delta(h - h_f)$

Quantum control parameter: $\theta = \log(h_b)$ or $\theta = \log(h_f)$.

Strong disorder RG approach

(Ma, Dasgupta, Hu 1979, D.S. Fisher 1992, F.I. & Monthus 2005)

- sort the couplings and transverse fields, $\Omega = \max(J_i, h_i)$
- eliminate the largest parameter - reduce the number of spins by one
- generate new effective parameters between the remaining spins
  - $\boxed{\Omega = J_{ij}}$ i and j form a ferromagnetic cluster - aggregation
    in an effective field: $\tilde{h}_{ij} = \frac{h_i h_j}{J_{ij}}$
    having a moment: $\tilde{\mu}_{ij} = \mu_i + \mu_j$
  - $\boxed{\Omega = h_i}$ site i is decimated out - annihilation
    new effective couplings between sites j and k: $\tilde{J}_{jk} = \frac{J_{ij} J_{ik}}{h_i}$
- repeate the transformation
- at the fixed point Ω is reduced to $\Omega^* = 0$.
- final result: set of connected clusters with different masses, μ, decimated at different energies, Ω.

Exact results in 1D

Infinite disorder fixed point (IDFP)

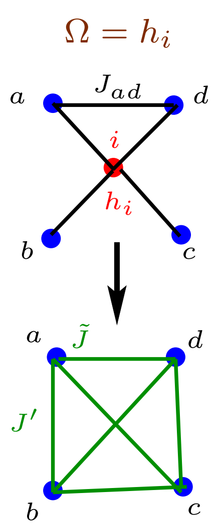
- Critical point: $\overline{\log(J)} = \overline{\log(h)}$
  - distribution of the effective parameters is logarithmically broad
  - asymptotically exact decimation steps
  - strongly anisotropic scaling
    $\ln \Omega_L \sim L^{\psi}$, $\boxed{\psi = 1/2}$
  - large effective spin clusters
    size: $\xi \sim |\theta - \theta_c|^{-\nu}$ $\boxed{\nu = 2}$

    moment: $\mu \sim [\ln(\Omega_0/\Omega)]^{\phi} \sim L^{d_f}$
    $d_f = \phi \psi = d - x$, $\boxed{\phi = \frac{\sqrt{5}+1}{2}}$, $\boxed{x = \frac{3-\sqrt{5}}{4}}$

- In the Griffiths phase: $\theta - \theta_c > 0$
  - non-singular static behaviour: $\xi < \infty$
  - singular dynamical behaviour:
    $\Omega \sim L^{-z}$ dynamical exponent: $z = z(\theta)$
    $\chi(T) \sim T^{-1+d/z}$, $\quad C_v(T) \sim T^{d/z}$

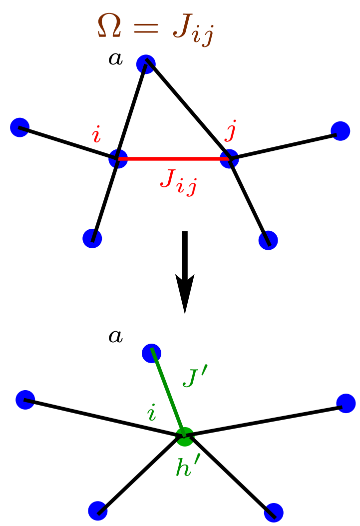
Numerical implementation of the RG procedure for $D > 1$

- differences with the 1D procedure
  - change in the topology
  - application of the maximum rule (valid at IDFP)



$J' = J_{ai}J_{bi}/h_i$

$\tilde{J} = \max(J_{ad}, J_{ai}J_{di}/h_i)$



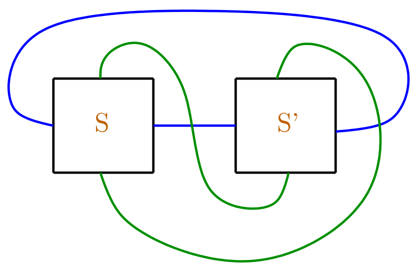
$J' = \max(J_{ai}, J_{aj})$

$h' = h_i h_j / J_{ij}$

- problems with the naïve implementation
  - h-decimations induce several new bonds
  - the lattice transforms to a fully connected cluster
  - slow algorithm: for N sites works in $\mathcal{O}(N^3)$ time
- improved algorithm
  - concept of local maxima - which can be decimated independently
  - concept of optimal RG trajectory - along which the time is minimal
  - filtering out irrelevant bonds - getting rid of latent couplings
  - improved algorithm works in $\mathcal{O}(N \log N)$ time

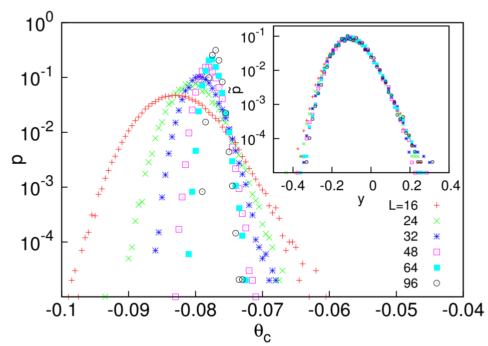
Bulk critical behavior

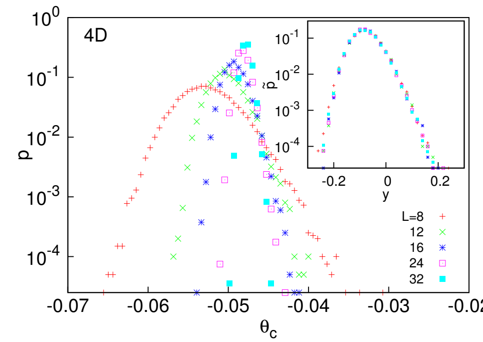
- Finite-size critical points - $\theta_c(S,L)$
  - two-copies of the same sample (S and S') are coupled together



  - continuously increase θ and monitor the clusters, which are built of identical sites in the copies
  - at $\theta_c(S,L)$ the last correlated cluster disappears, thus for $\theta > \theta_c(S,L)$ we are in the paramagnetic phase

- **Distribution of pseudocritical points**





- Finite-size scaling
  - shift of the mean:

$$\left|\theta_c - \overline{\theta_c}(L)\right| \sim L^{-1/\nu_s}$$

  - width of the distribution:

$$\Delta\theta_c(L) \sim L^{-1/\nu_w}$$

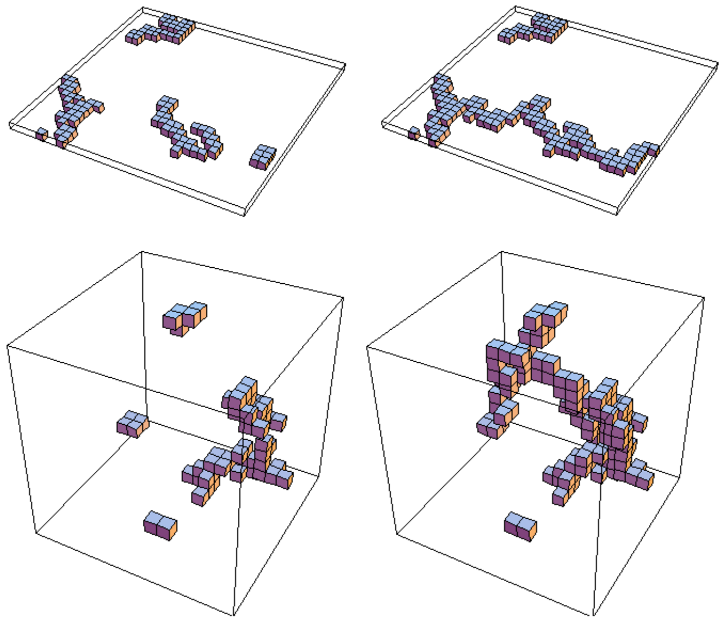
  - numerical estimates:

$$\boxed{\nu_s = \nu_w}$$

like in a conventional random fixed point

Scaling at the critical point

- Cluster structure



correlation (left) and energy (right) clusters

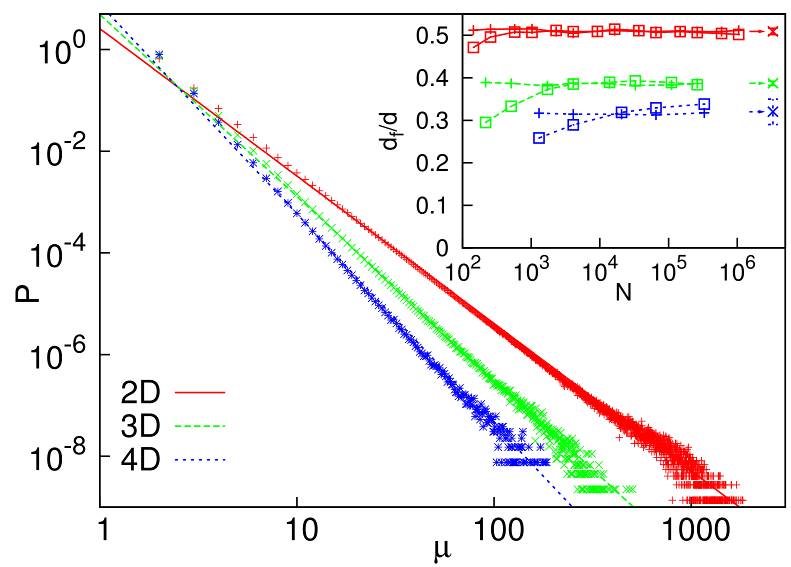
- Correlation clusters → magnetization
  - mass: $\mu = N^{\#}$ of connected sites
  - typical mass: $\mu \sim L^{d_f}$

- distribution function:

$$P_L(\mu) = L^{d_f} \tilde{P}(\mu L^{-d_f})$$

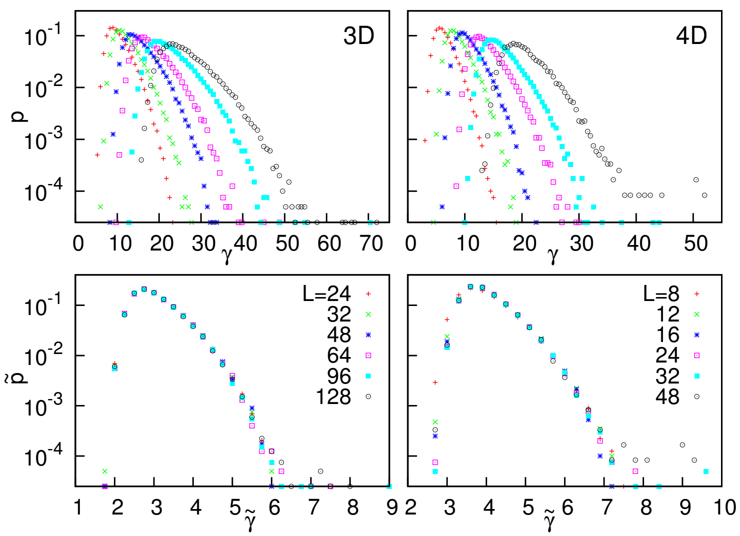
- power-law tail for large $\mu L^{-d_f} = u$

$$\tilde{P}(u) \sim u^{-\tau}, \text{ with } \tau = 1 + \frac{d}{d_f}$$

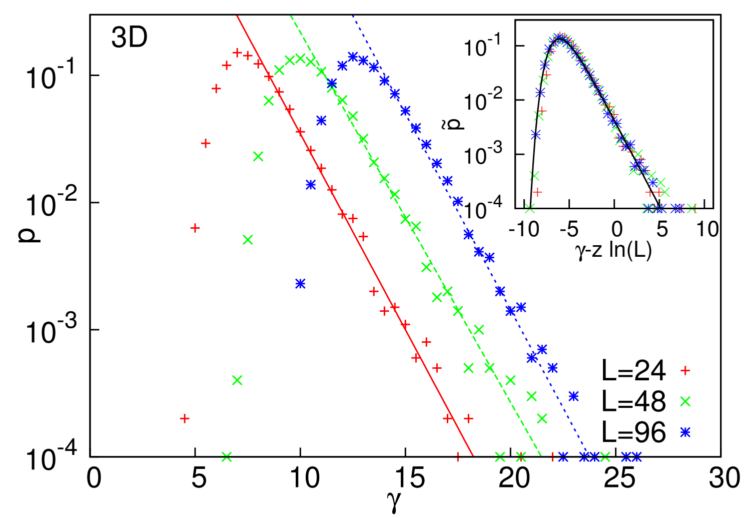


Energy clusters → dynamical scaling

- **energy scale:** ε_L smallest gap associated with the energy cluster
- Critical point
  - typical value: $\gamma_L \sim L^{\psi}$
  - scaling combination $\tilde{\gamma} = (\gamma_L - \gamma_0)L^{-\psi}$
  - **Infinite disorder scaling**



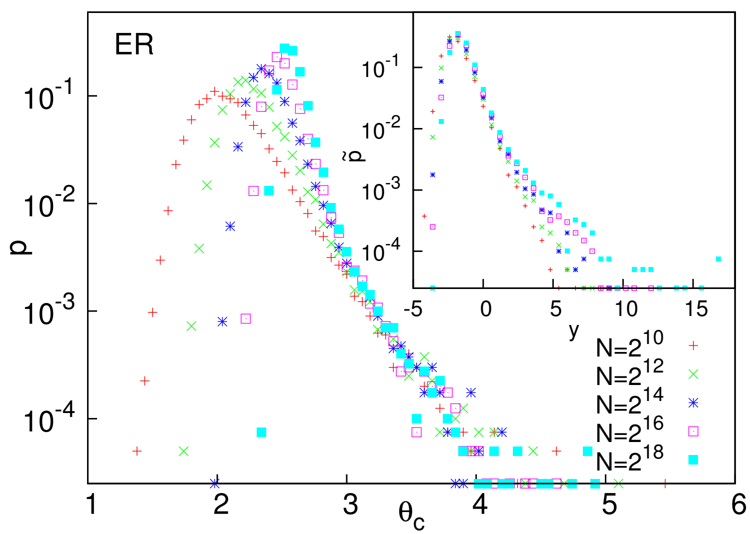
- We use: $\gamma_L = \log \varepsilon_L$
- (disordered) Griffiths phase
  - typical value: $\gamma_L \sim z \log(L)$ $[\varepsilon_L \sim L^{-z}]$
  - distr.: $\log p(\gamma) \approx -(d/z)\gamma, \quad (\gamma \gg 1)$
  - scaling comb.: $\tilde{\gamma} = (\gamma_L - z\ln(L) - \gamma_0$



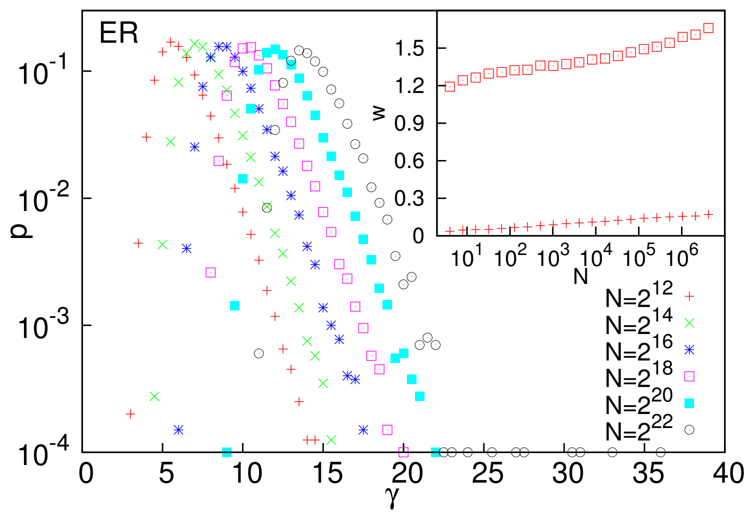
Erdős-Rényi (ER) random graphs $D \to \infty$

- Construction
  - N sites
  - $kN/2$ edges in random positions
  - $k > 1$ random graph is percolating

- Distribution of the pseudocritical points



- log-energy scaling



inset: fixed-h +, box-h ⊡

- logarithmically infinite disorder scaling

- width of the distribution:
  $W \approx W_0 + W_1 \log^{\varepsilon} N$
  $\varepsilon = 1.3(2)$

Bulk critical parameters

	1D	2D	3D	4D	ER
L_{max} N_{max}		2048 4.2×10^6	128 2.1×10^6	48 5.3×10^6	 4.2×10^6
$\theta_c^{(b)}$ $\theta_c^{(f)}$	0 $-1.$	1.6784(1) $-0.17034(2)$	2.5305(10) $-0.07627(2)$	3.110(5) $-0.04698(10)$	2.775(2) $-0.093(1)$
$d\nu_w$ $d\nu_s$	2.	2.48(6) 2.50(6)	2.90(15) 2.96(5)	3.3(1) 2.96(10)	7.(2) 5.(1)
x/d	$\frac{3-\sqrt{5}}{4}$	0.491(8)	0.613(3)	0.68(3)	0.81(2)
ψ/d	1/2	0.24(1)	0.15(2)	0.11(2)	0. (*log*)

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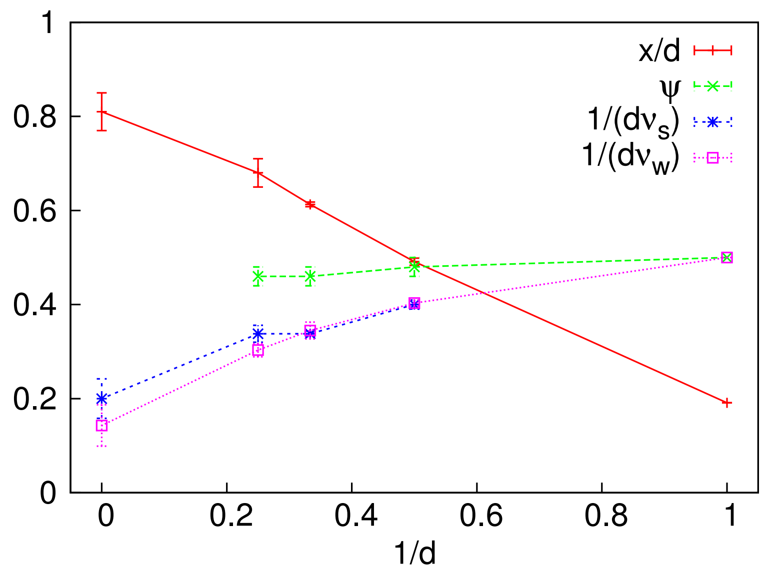


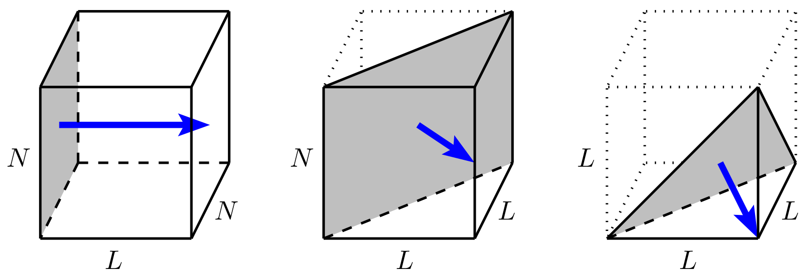
Conclusions at this point

- Infinite disorder fixed point at any dimensions
- Strong disorder RG approach is asymptotically exact
- Spin glass and random ferromagnet are in the same universality class

Boundary critical behavior

- Different finite geometries

  . slab wedge pyramid



- Different local exponents

  . surface edge corner

- magnetization profiles
  - from the fixed boundary: (Fisher-de Gennes)

    $m_l \sim l^{-x_b}, \quad x = x_b, \quad l \ll L$
  - from the free part

    $m_{l'} \sim (l')^{x_{\alpha b}}, \; l' = L_\alpha - l + 1 \ll L_\alpha$

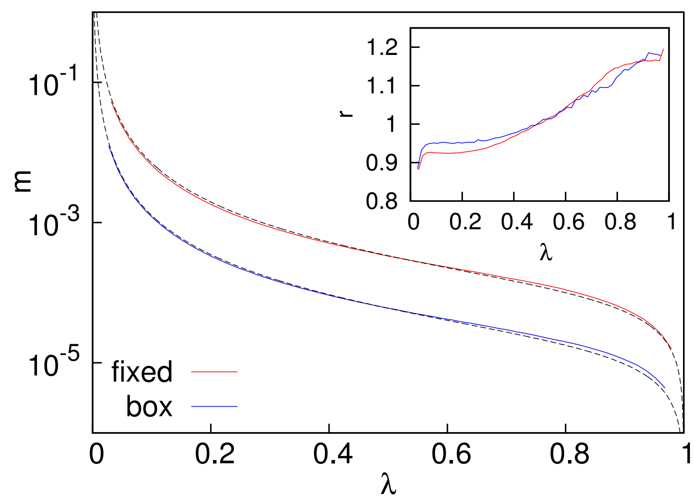
    $x_{\alpha b} = x_\alpha - x, \; \alpha : s, \; c, \; e$

  - interpolation formula

    $m_l = \frac{A}{L^x}[\sin(\pi\lambda)]^x [\cos(\pi\lambda/2)]^{x_\alpha}$

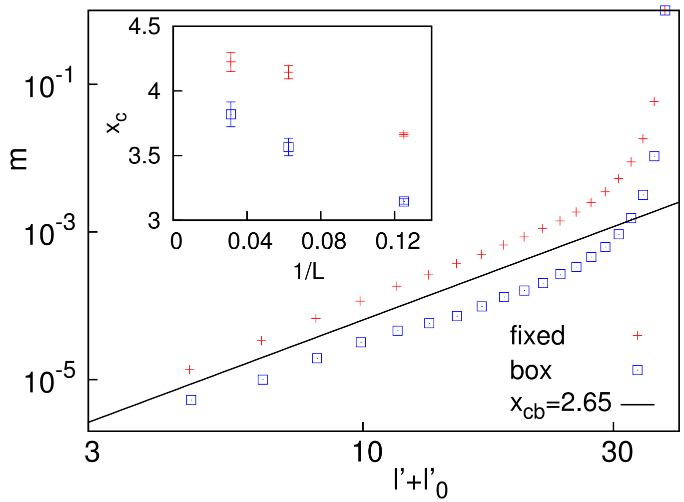
    $\lambda = l/L_\alpha$

- results in 3D



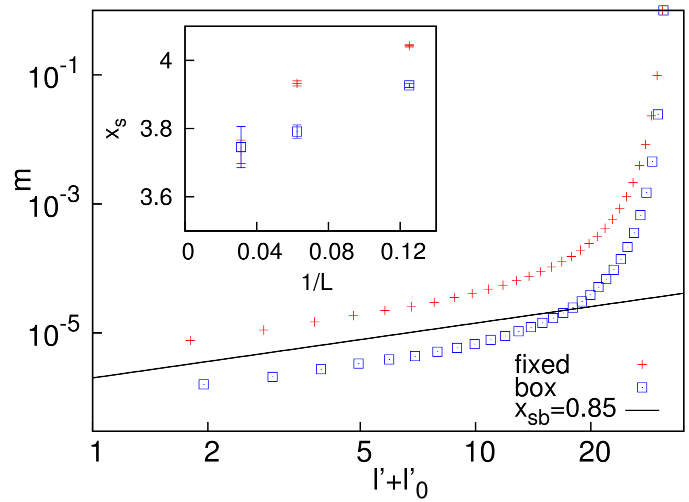
slab geometry, inset: ratio with the interpolation formula

- results in 3D



pyramid geometry, inset: corner exp.

- results in 4D



slab geometry, inset:surface exp.

Boundary critical exponents - summary

	bulk		surface		corner		edge	
	x	x_b	x_s	x_{sb}	x_c	x_{cb}	x_e	x_{eb}
1D	$(3-\sqrt{5})/4$		0.5					
2D	0.982(15)	0.98(1)	1.60(2)	0.65(2)	2.3(1)	1.35(10)		
3D	1.840(15)	1.855(20)	2.65(15)	0.84(7)	4.2(2)	2.65(25)	3.50(15)	1.75(15)
4D	2.72(12)	2.72(10)	3.7(1)	0.85(15)				

Entanglement entropy

- entanglement entropy between a subsystem: A and the enviroment: B:

$$\mathscr{S}_A = -\mathrm{Tr}(\rho_A \log_2 \rho_A)$$

- $\rho_A = \mathrm{Tr}_B|0><0|$: reduced density matrix with $|0>$ the ground state of the complete system: which is a set of independent clusters

- each connected cluster with c number of spins is in a GHZ-state:

$$\frac{1}{\sqrt{2}}(|\underbrace{\uparrow\uparrow\cdots\uparrow}_{c\text{ times}}\rangle + |\underbrace{\downarrow\downarrow\cdots\downarrow}_{c\text{ times}}\rangle)$$

- for the GHZ-state
  - $\rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$
  - $\mathscr{S}_{GHZ} = 1$

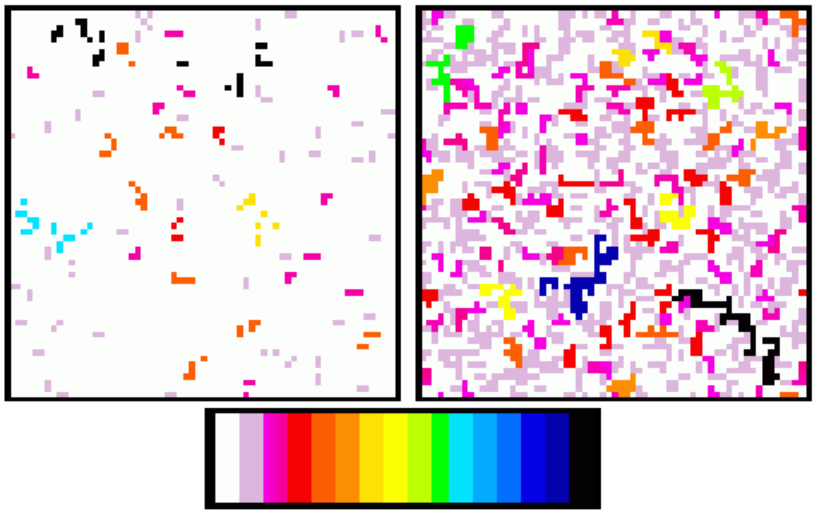
- each cluster contained both in A and B gives 1 contribution to $\mathscr{S}_A$

- $\boxed{\mathscr{S}_A \sim L^{d-1}}$: area law

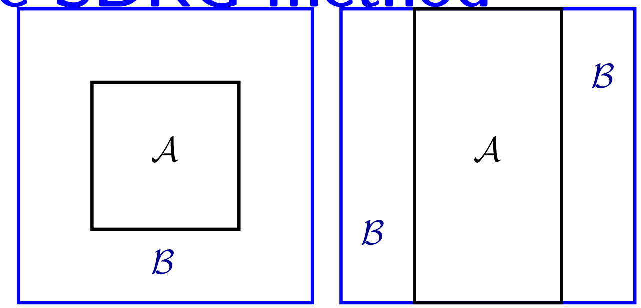
- corrections at the critical point?
  - are they singular?
  - form: $\sim L^{d-1} \ln\ln L$ or $\sim L^{d-1} + b \ln L$?
  - origin: corner and/or bulk?
  - are they universal?
  - related to a diverging ξ?

Numerical calculation by the SDRG method

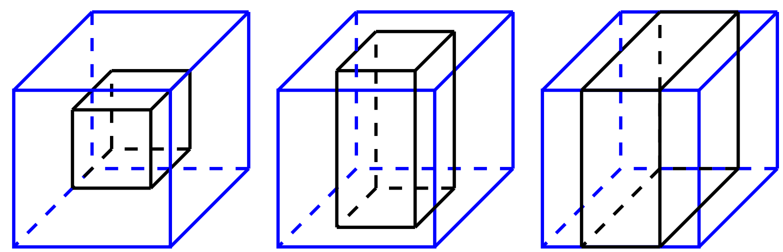
- for a sample obtain the clusters through renormalization



$L = 64$ box-h (left) fixed-h (right)



. square (cube) strip (slab)



$\mathcal{S}_0^3$ $\mathcal{S}_1^3$ $\mathcal{S}_2^3$

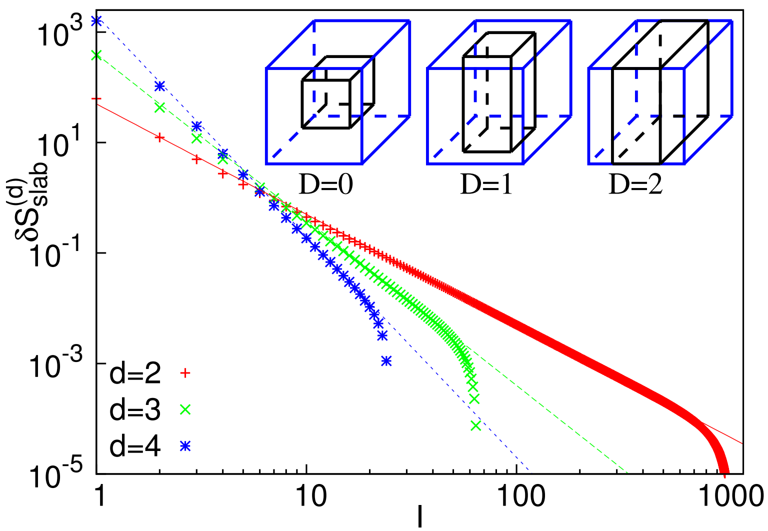
- Calculation in different geometries

cube column slab

Slab geometry

- the same area (term) for different ℓ: $\mathscr{S}^{(d)}_{\rm slab}(L,\ell) = a_{d-1} f_{d-1} + corr(\ell),\ f_{d-1} = L^{d-1}$
- the finite difference is related to the correction

$$\delta\mathscr{S}^{(d)}_{\rm slab}(L,\ell) = \mathscr{S}^{(d)}_{\rm slab}(L,\ell+1) - \mathscr{S}^{(d)}_{\rm slab}(L,\ell)$$



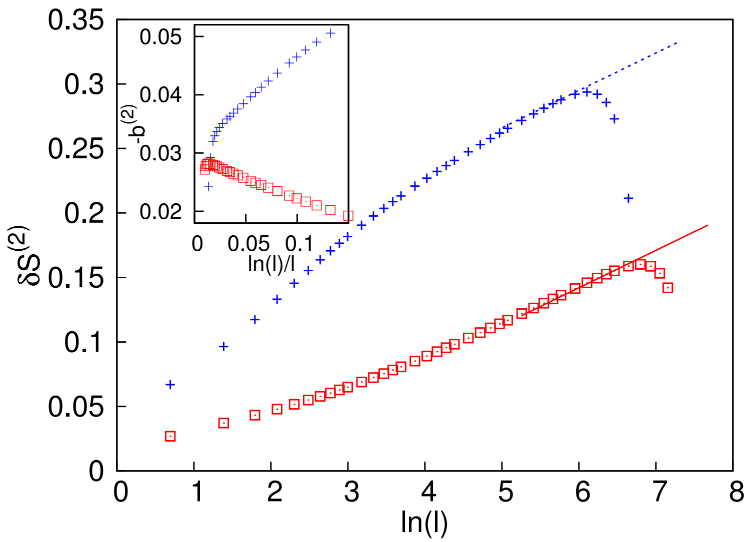
- numerical data at the critical point:

$$\boxed{\delta\mathscr{S}^{(d)}_{\rm slab}(L,\ell) \sim \ell^{-d}}$$

- no singular corrections!
- phenomenological explanation
  - domains which contribute to the entropy are of size $\xi \leq \ell$
  - finite-size corrections are for $\xi \approx \ell$
  - number of these blobs are $n_{bl} \sim (L/\ell)^{d-1}$
  - each blob has the same $\mathscr{O}(1)$ correction
  - total correction: $\mathscr{S}^{(d)}_{\rm slab}(L,\ell) - a_{d-1} f_{d-1} \sim n_{bl} \sim \ell^{-d+1}$
- singular contributions are due to corners!

Cube (square) geometry in 2d

- corner correction to the area-law:
  $\mathscr{S}^{(2)}_{\rm cube}(\ell) = a_1 f_1 + \mathscr{S}^{(2)}_{\rm cr}(\ell)$
- calculate the difference:
  $\delta\mathscr{S}^{(2)}(\ell) \equiv \mathscr{S}^{(2)}(\ell) - 2\mathscr{S}^{(2)}(\ell/2)$
  $\approx \mathscr{S}^{(2)}_{\rm cr}(\ell) - 2\mathscr{S}^{(2)}_{\rm cr}(\ell/2)$

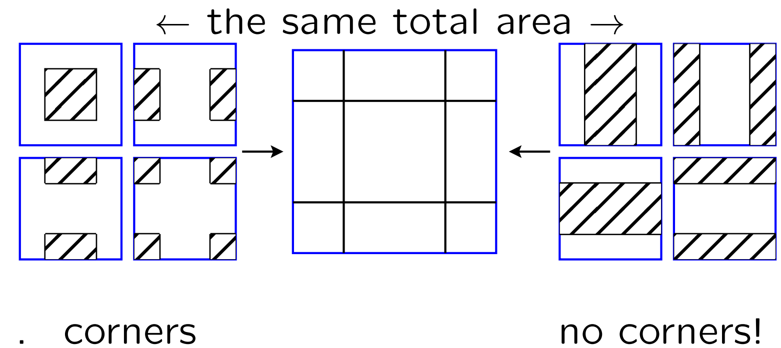


fixed-h ($+$) and box-h ($\square$)

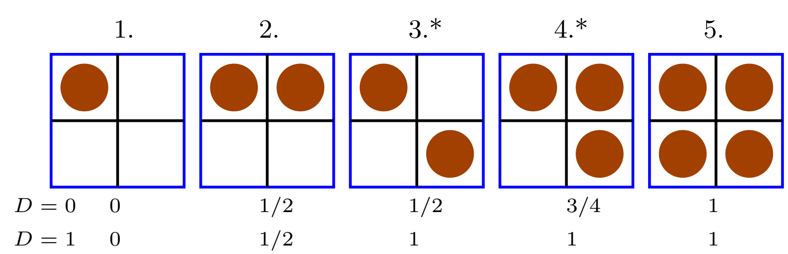
- numerical data:

$$\delta\mathscr{S}^{(2)}(\ell) \simeq \mathscr{S}^{(2)}_{\rm cr}(\ell) + cst \simeq -b^{(2)} \ln \ell + cst$$

- universal logarithmic correction:
  $b^{(2)} = -0.029(1)$
- **direct calculation of the corner contribution for $\ell = L/2$**



- relation with the cluster geometry



Cube geometry in $d > 2$

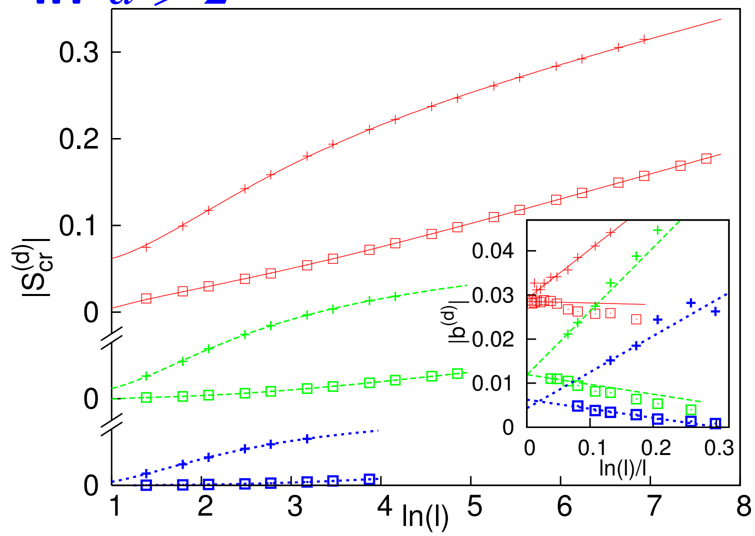
- edge and corner corrections to the area-law:
  $\mathscr{S}^{(d)}_{\text{cube}}(\ell) = a_{d-1} f_{d-1} + \sum_{E=1}^{d-2} a_E f_E + \mathscr{S}^{(d)}_{\text{cr}}(\ell)$

- $1 \le E < d-1$: dimension of the edge, $f_E \sim L^E$

- $a_E - a_E(\ell) \sim \ell^{-E}$, as for the surface term

- **direct calculation of the corner contribution for $\ell = L/2$**

$$\mathscr{S}^{(d)}_{cr} = \sum_{D=0}^{d-1} \left(-\frac{1}{2}\right)^D \binom{d}{D} \mathscr{S}^{(d)}_D$$



fixed-h (+) and box-h (□)
2d,3d and 4d

- numerical data:
  $$\boxed{\mathscr{S}^{(d)}_{\text{cr}}(\ell) \simeq -b^{(d)} \ln \ell + cst}$$

- universal logarithmic correction:
  $b^{(2)} = -0.029(1)$, $b^{(3)} = 0.012(2)$ and $b^{(4)} = -0.006(2)$

Phenomenological explanation

- consider 2-site clusters (at the end of the RG)

- "corner clusters" have points at hyper-cubes connected by the main diagonal

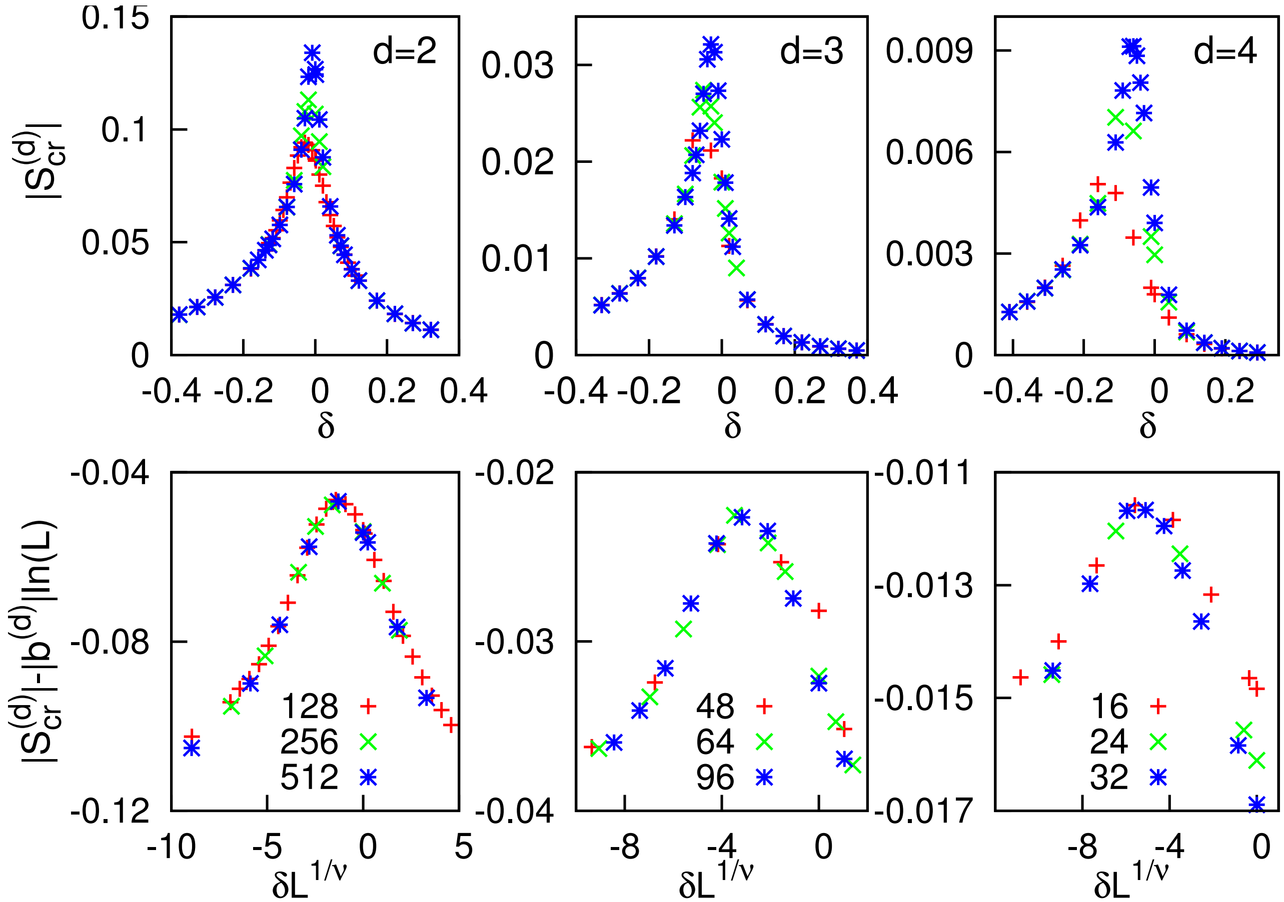
- relative coordinates of the 2-site clusters:
  $0 \leq x_j \leq L/2, \; j = 1, 2, \ldots, d)$ (periodic b.c.)

- the corner entropy (averaging over all positions):
  $-2\prod_{j=1}^{d}(-x_j/L)$

- probability of a 2-site cluster of length r is:
  the average pair-correlation function: $C_{av}(r) \approx n_r^2$
  $n_r \sim r^{-d}$: the density of non-decimated sites

- average contribution:
  $\mathscr{S}_{\rm cr}^{(d)}(\ell) \sim -\int_1^{\ell} {\rm d}x_1 \ldots \int_1^{\ell} {\rm d}x_d \prod_{j=1}^{d}(-x_j/r^2)$
  $\sim (-1)^{d+1} \int_1^{\ell} (r^{d-1} r^d)/r^{2d} {\rm d}r$
  $\sim (-1)^{d+1} \ln \ell$

Corner correction outside the critical point



The singularity is related to a diverging correlation length!

Conclusions

- Infinite disorder fixed point at any dimensions
- Strong disorder RG approach is asymptotically exact
- Universal (disorder independent) bulk and local exponents
- Entanglement entropy: logarithmic correction to the area law due to corners at the critical point
- Disordered model is (at least) so well understood as the pure model