Renormalization Group for Quantum Walks

CompPhys13

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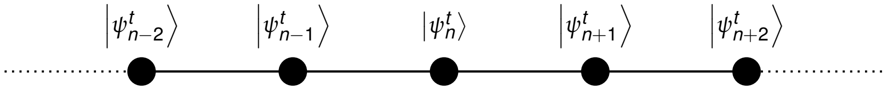
Discrete time walks in one dimension



Discrete Time Evolution

$$\left|\psi^{t+1}\right\rangle = \mathcal{U} \cdot \left|\psi^{t}\right\rangle$$

$$\mathcal{U} = \sum_{n} \mathcal{P} \cdot |n\rangle \langle n-1| + \mathcal{Q} \cdot |n\rangle \langle n+1|$$

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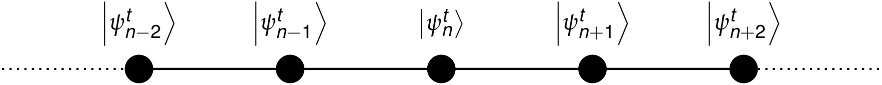

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Classical

$$p^t_n = \left\| \left|\psi^t_n\right\rangle \right\|_1$$

$\mathcal{P} + \mathcal{Q}$ stochastic

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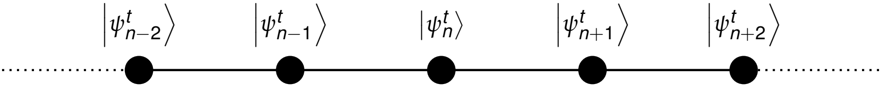

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Classical

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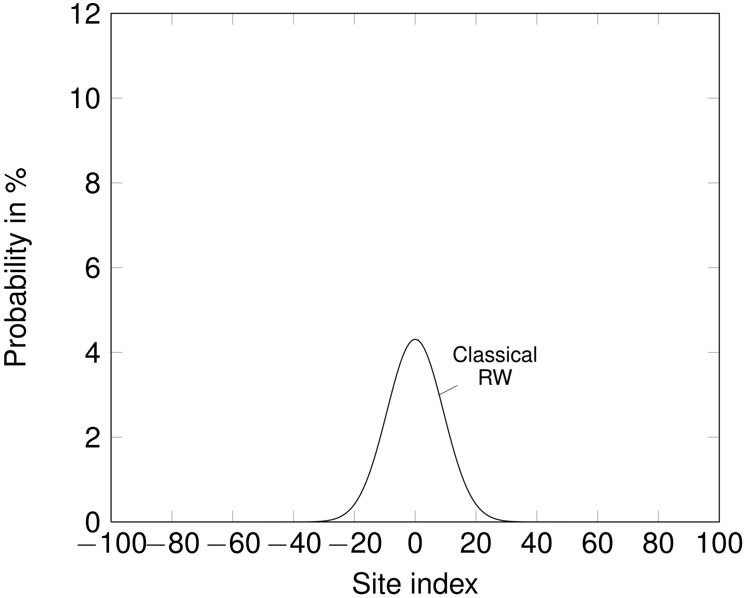
$\mathcal{P} + \mathcal{Q}$ stochastic

Quantum Mechanical

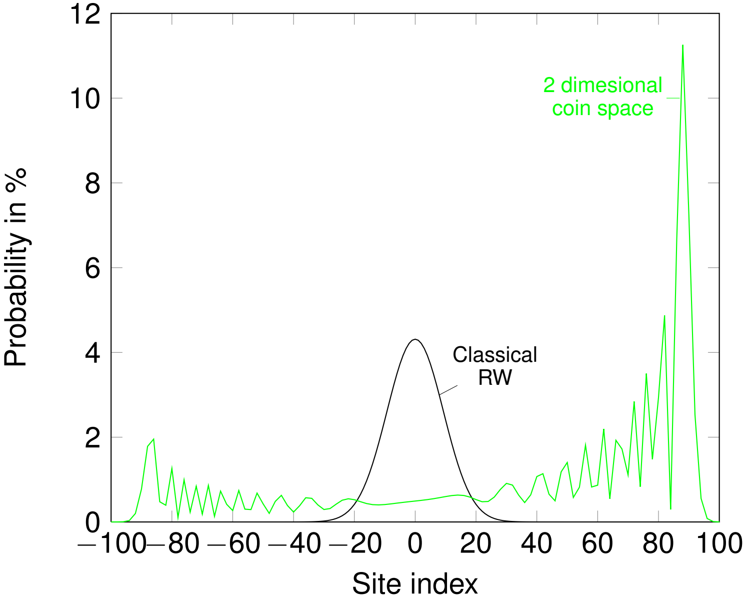
$$p^t_n = \langle \psi^t_n | \psi^t_n \rangle \qquad \mathcal{U} = \mathcal{S} \cdot (\mathbb{1} \otimes \mathcal{C})$$

$$\mathcal{P}\mathcal{P}^\dagger + \mathcal{Q}\mathcal{Q}^\dagger = \mathbb{1} \qquad \mathcal{P}\mathcal{Q}^\dagger = \mathcal{Q}\mathcal{P}^\dagger = 0$$

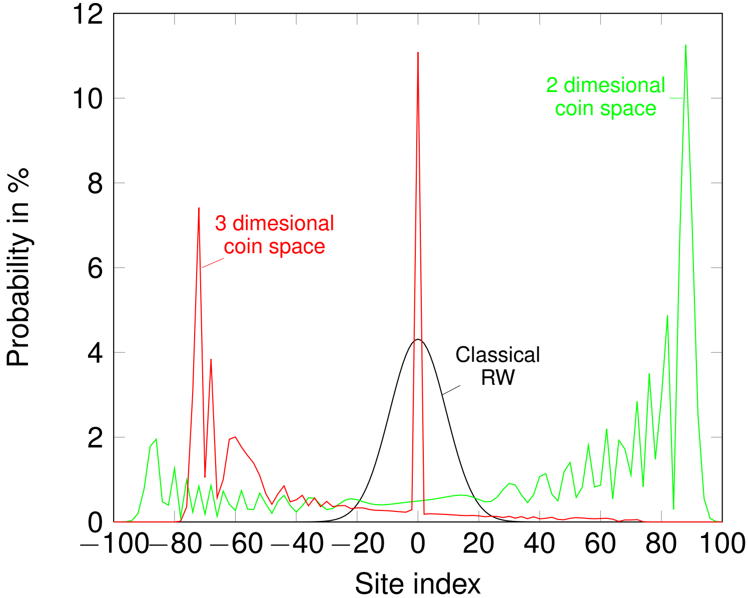
One dimensional walks



One dimensional walks



One dimensional walks



Motivation

- model for quantum transport (Aharonov PRA 1993)
- Grover's search algorithm (Grover PRL 1997)
  find a marked vertex in a graph in $T \sim \mathcal{O}\left(\sqrt{N}\right)$
- more general quantum search algorithm
- general quantum computation (Lovett PRA 2010)

- generally faster spreading compared to the random walk

$$\langle r^2 \rangle = t^{\frac{2}{d_w}} \qquad \text{with } d_w^{QW} \stackrel{?}{=} \frac{1}{2}\, d_w^{RW}$$

- analytic results only on translational invariant lattice

The generating function

The discrete Laplace transform/ The z-Transform

The generating function and its backtransform

$$|\tilde{\psi}(z)\rangle = \sum_{t=0}^{\infty} z^t \cdot |\psi^t\rangle \qquad |\psi^t\rangle = \frac{1}{2\pi\,\mathrm{i}} \oint_{|z|=1} z^{-t-1} \cdot |\tilde{\psi}(z)\rangle \, \mathrm{d}z$$

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$$|\psi^{t+1}\rangle = \mathcal{U} \cdot |\psi^t\rangle \qquad \Longrightarrow \qquad |\tilde{\psi}(z)\rangle = \tilde{\mathcal{U}}(z) \cdot |\tilde{\psi}(z)\rangle + |\psi^0\rangle$$

Recurrence equations become algebraic equations.

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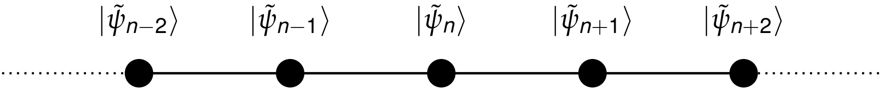
Recurrence equations become algebraic equations.

Generating function for p_n^t in quantum walks

$$\tilde{p}_n(z) = \sum_{t_0}^{\infty} z^t \cdot p_n^t = \frac{1}{2\pi\mathrm{i}} \oint \langle \tilde{\psi}(z/y) | \tilde{\psi}(y) \rangle \, \frac{\mathrm{d}\, y}{y}$$

Renormalization group in one dimension



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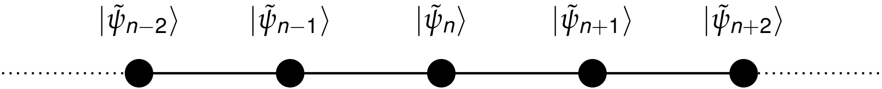

$$\begin{aligned}
&\vdots \\
|\tilde{\psi}_{n-1}\rangle &= \tilde{\mathcal{P}} \cdot |\tilde{\psi}_{n-2}\rangle + \tilde{\mathcal{Q}} \cdot |\tilde{\psi}_{n}\rangle + \tilde{\mathcal{R}} \cdot |\tilde{\psi}_{n-1}\rangle \\
|\tilde{\psi}_{n}\rangle &= \tilde{\mathcal{P}} \cdot |\tilde{\psi}_{n-1}\rangle + \tilde{\mathcal{Q}} \cdot |\tilde{\psi}_{n+1}\rangle + \tilde{\mathcal{R}} \cdot |\tilde{\psi}_{n}\rangle \\
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&\vdots
\end{aligned}$$

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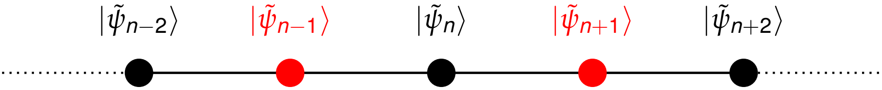

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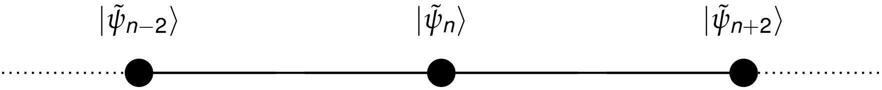
\end{aligned}$$

$$\vdots$$

Renormalization group in one dimension



$$|\tilde{\psi}_n\rangle = \tilde{\mathcal{P}}' \cdot |\tilde{\psi}_{n-2}\rangle + \tilde{\mathcal{Q}}' \cdot |\tilde{\psi}_{n+2}\rangle + \tilde{\mathcal{R}}' \cdot |\tilde{\psi}_n\rangle$$

The one dimensional classical random walk

- the recursion equations:

$$p' = \frac{p^2}{1-r} \qquad q' = \frac{q^2}{1-r} \qquad r' = r + \frac{2pq}{1-r}$$

- three fixed points:

$$(p^*, q^*, r^*) = (1-r^\infty, 0, r^\infty) \qquad t_k \sim L_k^{d_w} \quad \text{with} \quad d_w = 1$$
$$(p^*, q^*, r^*) = (0, 1-r^\infty, r^\infty) \qquad d_w = 1$$
$$(p^*, q^*, r^*) = (0, 0, 1) \qquad d_w = 2$$

The one dimensional quantum walk

- the recursion (matrix) equations:

$$\mathcal{P}' = \mathcal{P} \cdot (\mathbb{1} - \mathcal{R})^{-1} \cdot \mathcal{P} \qquad \mathcal{Q}' = \mathcal{Q} \cdot (\mathbb{1} - \mathcal{R})^{-1} \cdot \mathcal{Q}$$

$$\mathcal{R}' = \mathcal{R} + \mathcal{P} \cdot (\mathbb{1} - \mathcal{R})^{-1} \cdot \mathcal{Q} + \mathcal{Q} \cdot (\mathbb{1} - \mathcal{R})^{-1} \cdot \mathcal{P}$$

- three different regimes:
  - a non-trivial fixed point (despite unitarity)!
  - quasi periodic/chaotic behavior of the parameters
  - subspace of trivial fixed points: $\mathcal{P} = \mathcal{Q} = 0$
- we need a generalization of the Jacobian

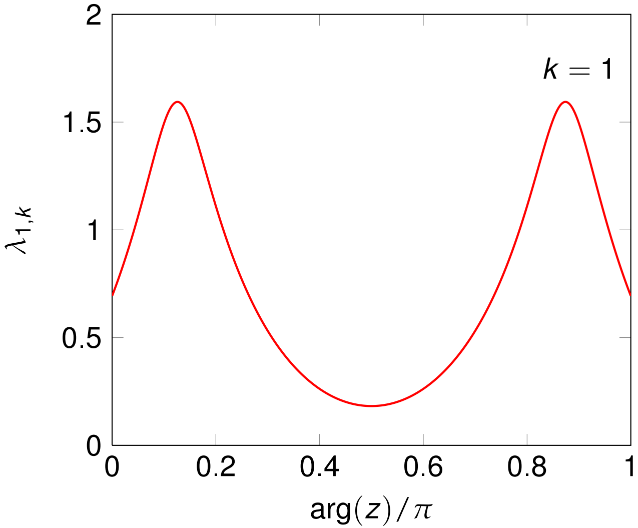
The Lyapunov exponents

- given a dynamical system evolving according to

$$\underline{x}(t+1) = F(\underline{x}(t))$$

- J_t - Jacobian at $x(t)$, $i.e. F\left(x(t)+\delta x\right) \approx x(t+1) + J_t \cdot \delta x$
- $\mathcal{J}_t = \prod_{t'=0}^{t} J_{t'}$ - relates the change in $x(t+1)$ when changing $x(0)$
- $\{\mathrm{e}^{\lambda_{i,t}}\}$ - eigenvalue spectrum of $\Lambda = \left(\mathcal{J}_t^\dagger \cdot \mathcal{J}_t\right)^{\frac{1}{2(t+1)}}$
- $\lambda_{1,\infty}$ - (largest) Lyapunov exponent dominates long term behavior

Lyapunov exponent for the line



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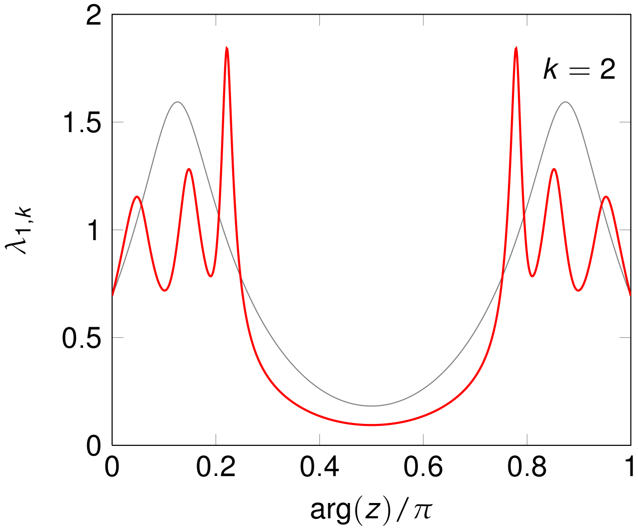

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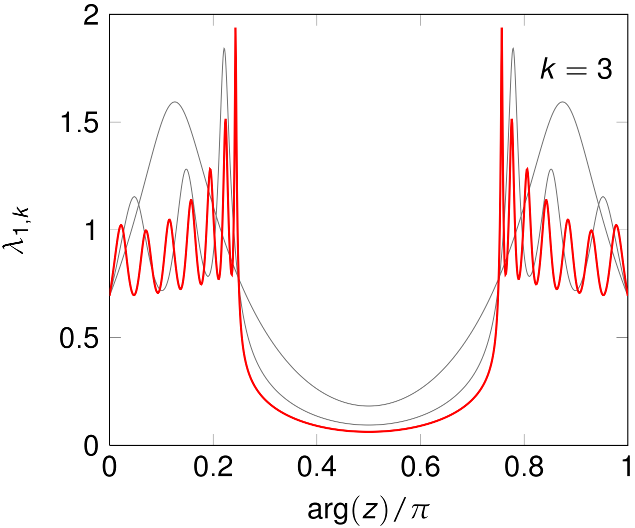

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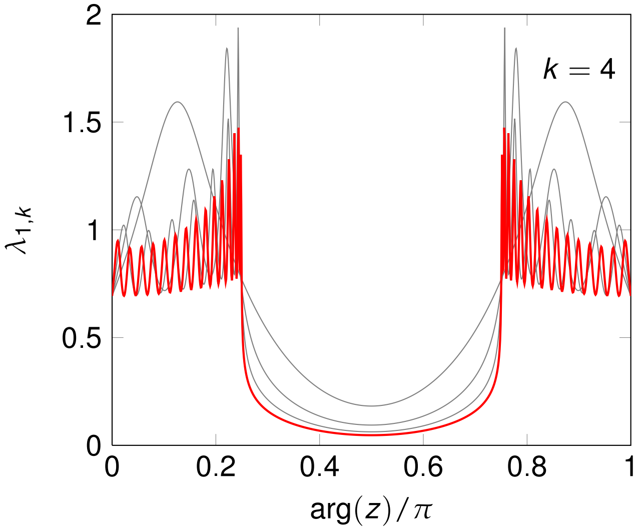

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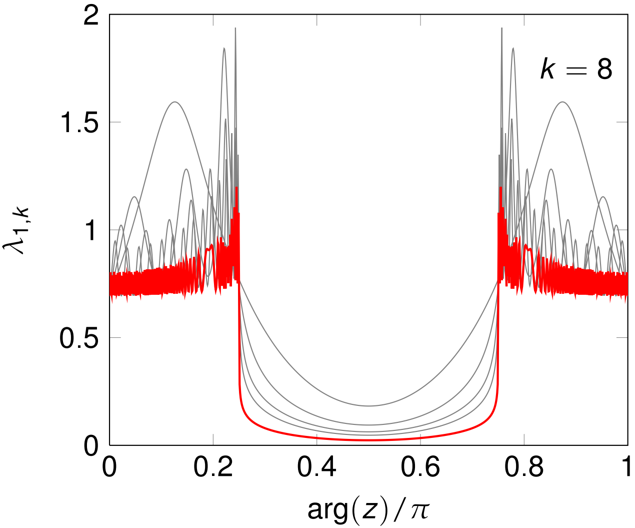

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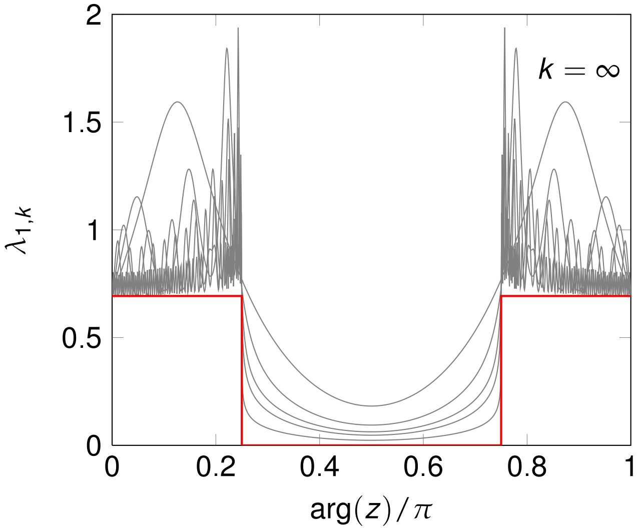
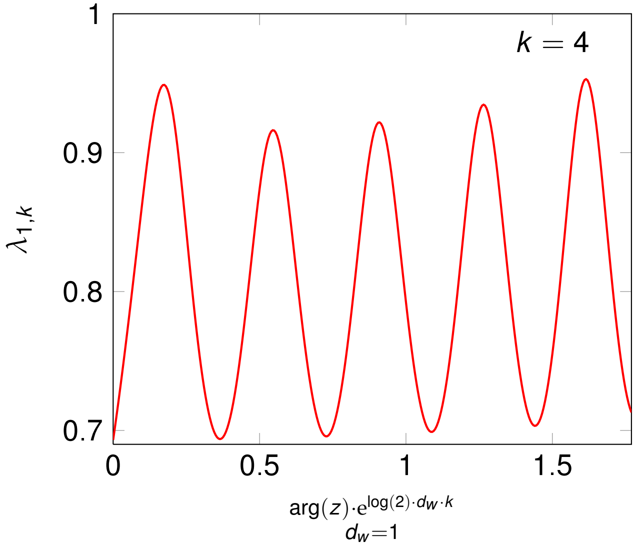


Scaling of the Lyapunov exponent for the line



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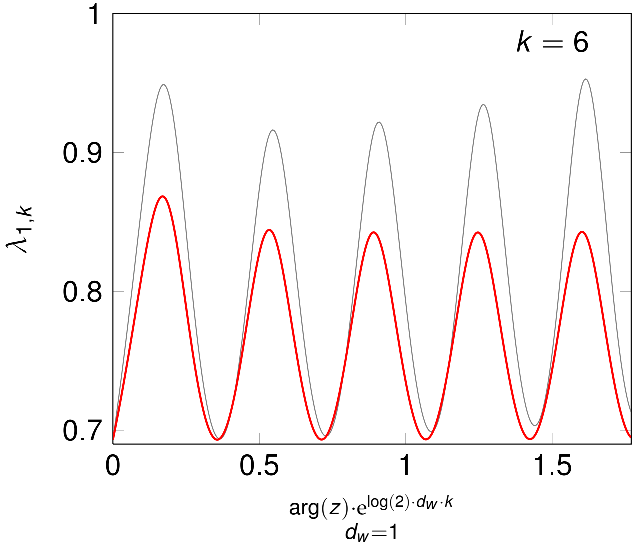

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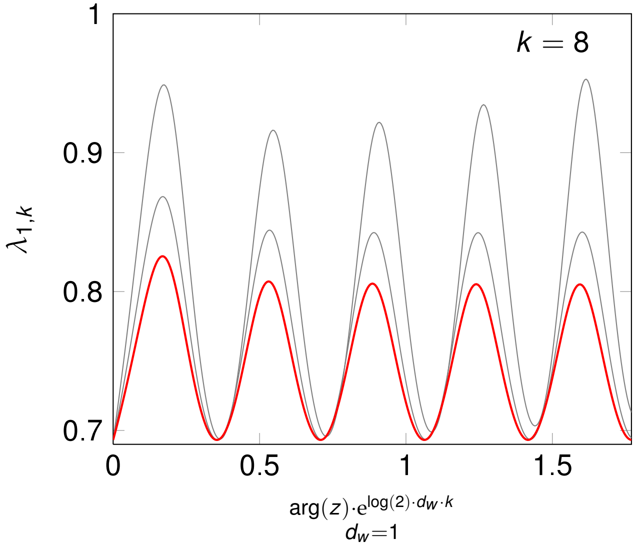

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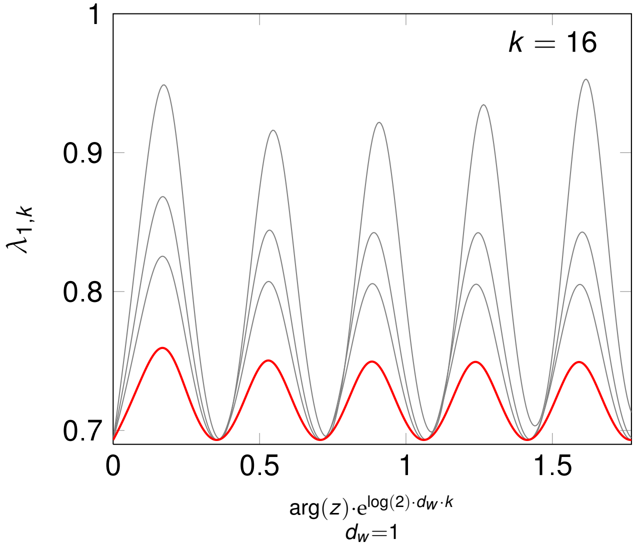

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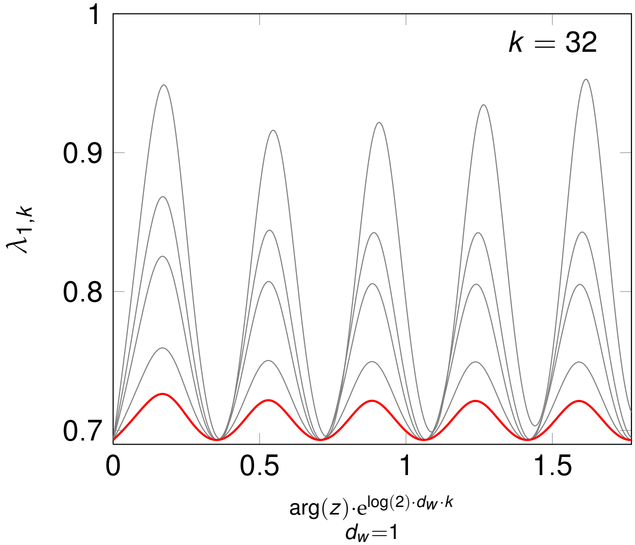

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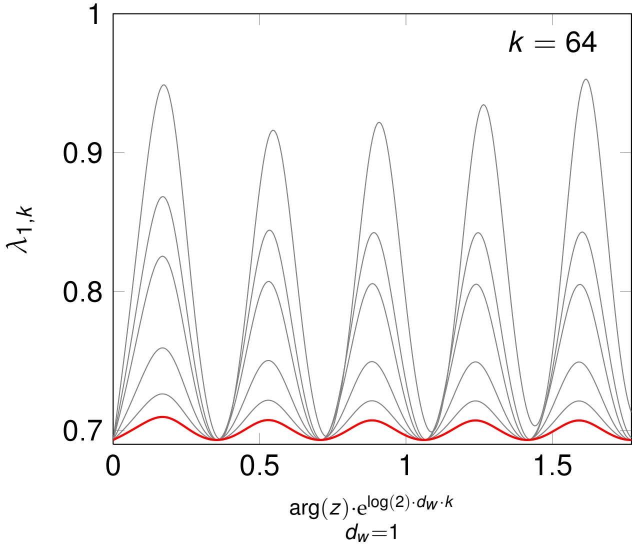
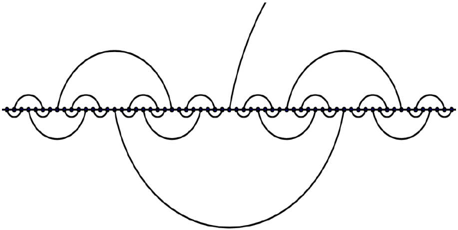


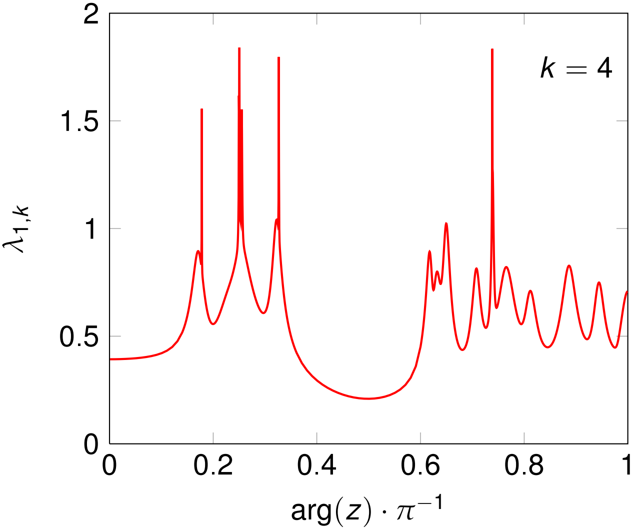
3 regular Hanoi network HN3

- hierarchichal network (Boettcher EPL 2008)
- exact renormalization
- interpolates between lattices and small-world networks



- $d_w^{RW} = 2 - log_2(\phi) = 1.30576\ldots$

Lyapunov exponent for HN3



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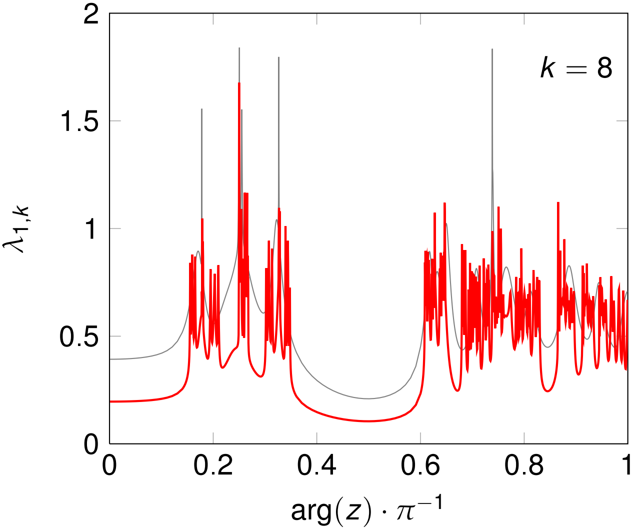

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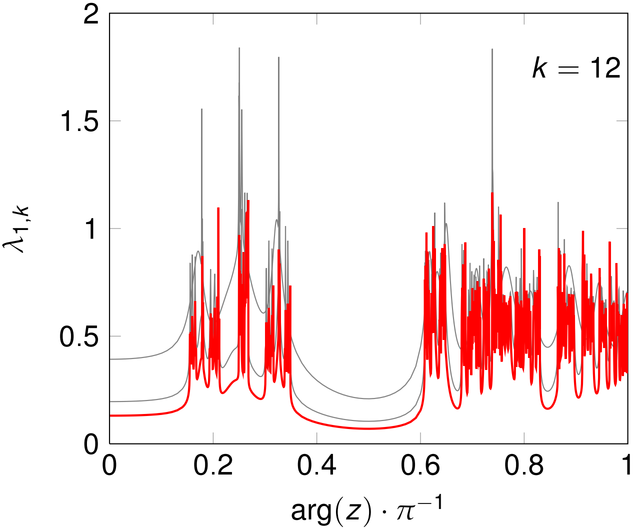

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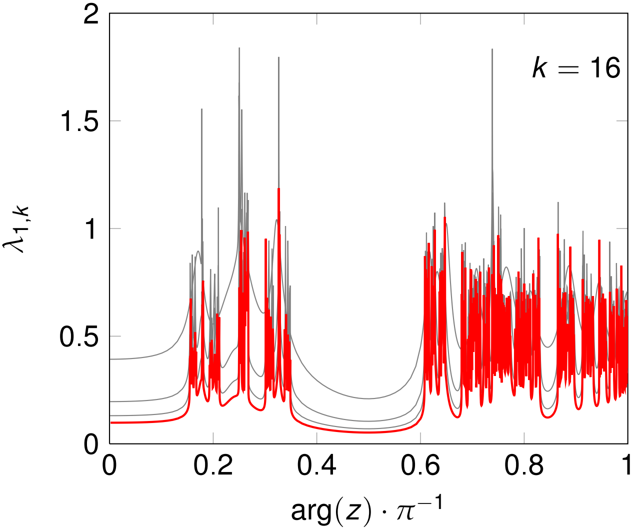

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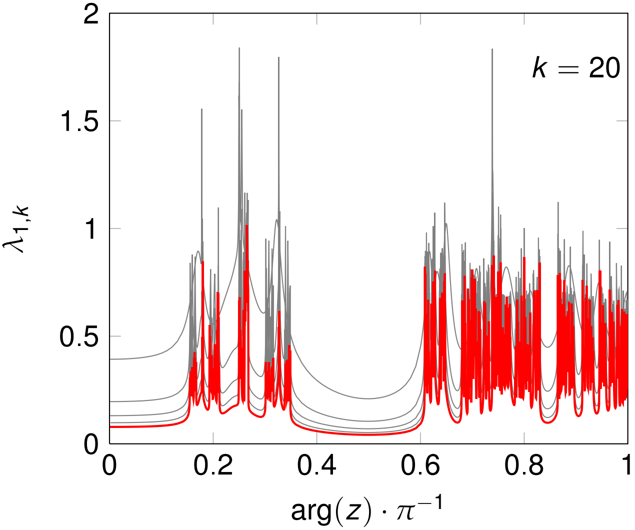

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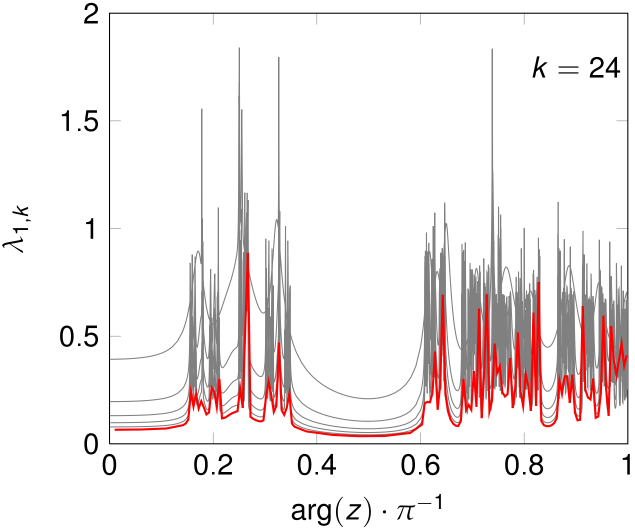
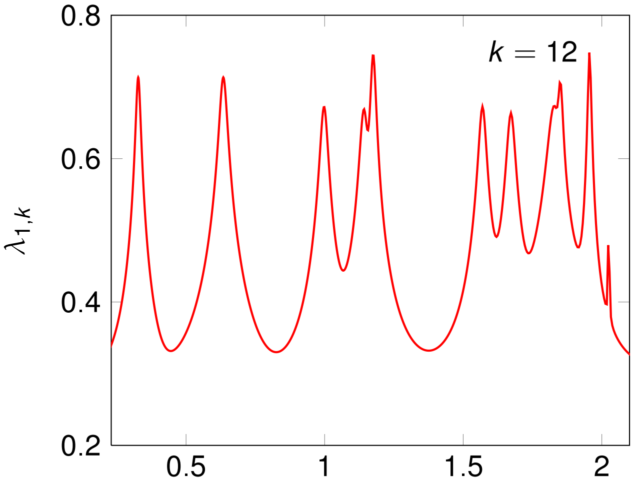


Scaling of the Lyapunov exponent for HN3



$(\pi - \arg(z)) \cdot e^{d_w \log(2) \cdot k}$

$d_w = \frac{1}{2} \cdot (2 - \log_2(\varphi))$

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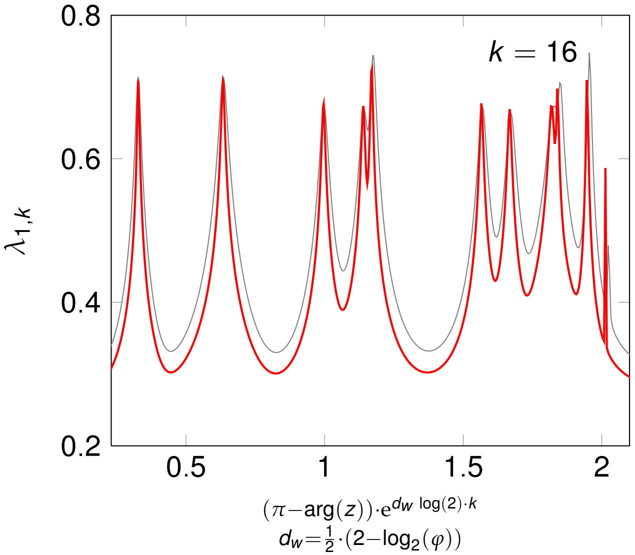

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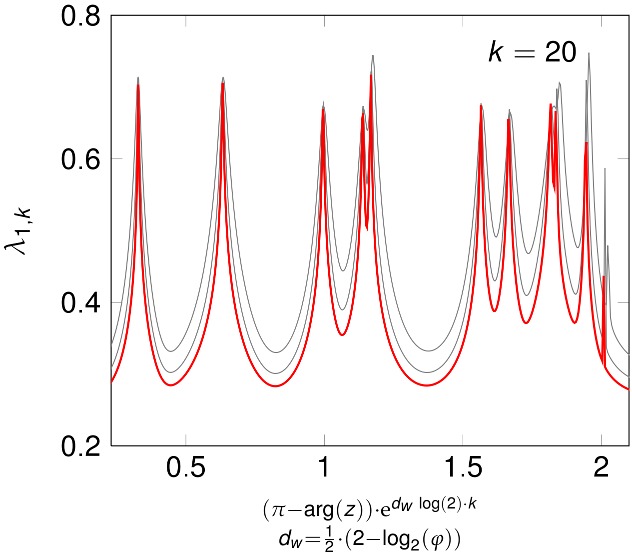

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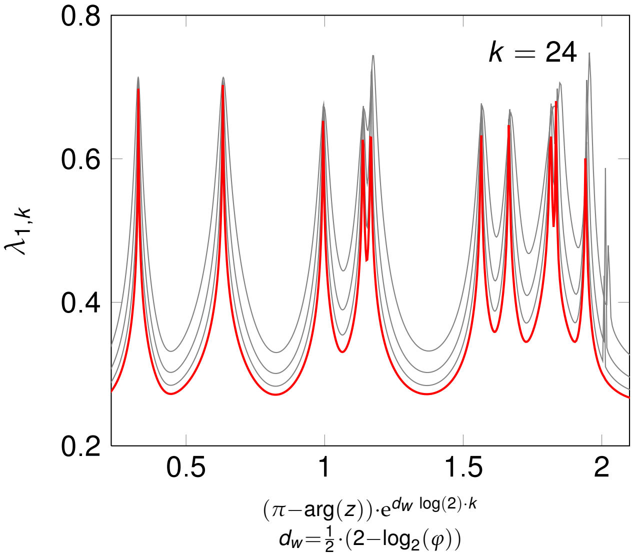

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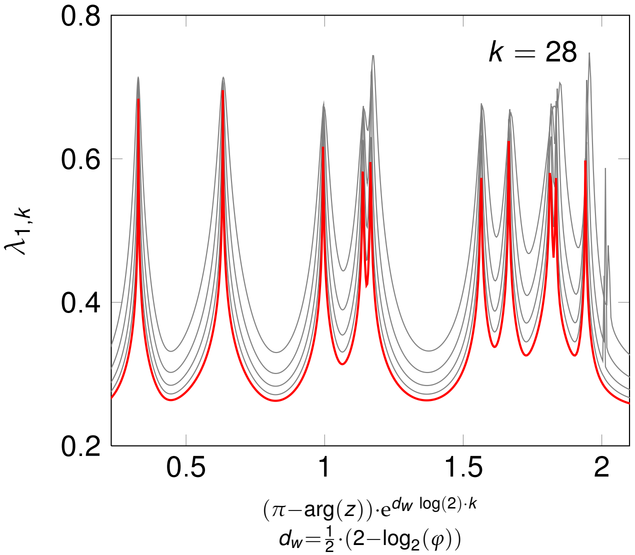

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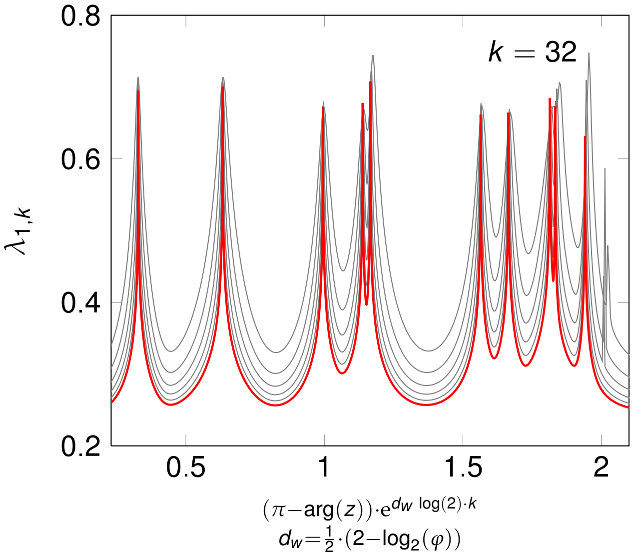

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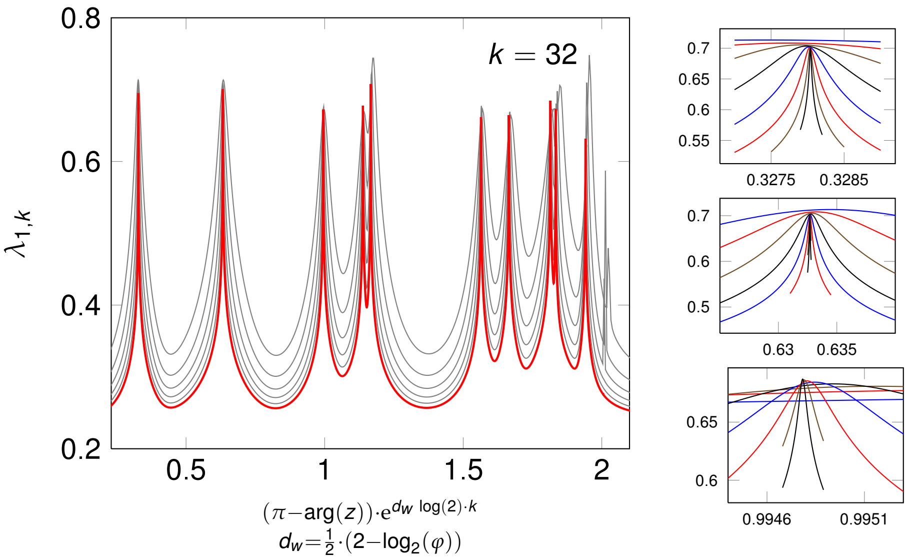
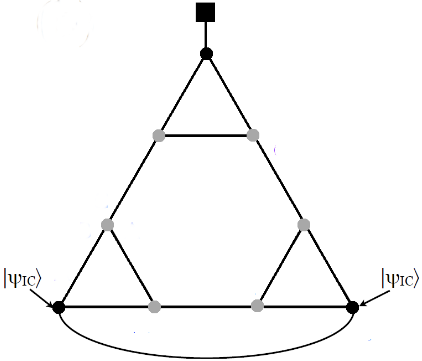
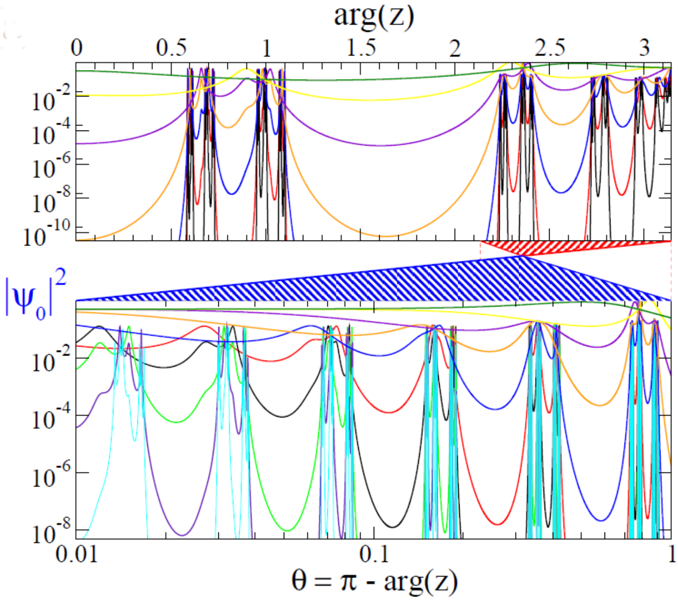


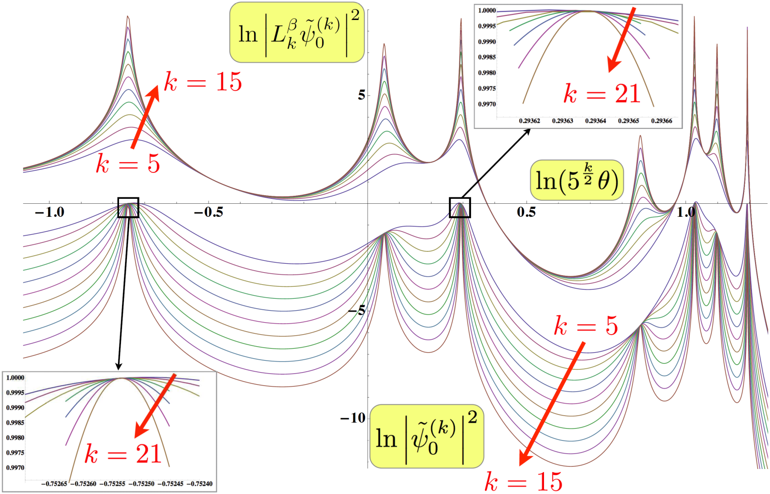
The dual Sierpinsky gasket

- degree 3 fractal
- exact renormalization
- $d_w^{RW} = \log_2 5$
- one absorbing boundary (square)



Lyapunov exponent for the dual Sierpinsky gasket





$\ln \left| L_k^{\beta} \tilde{\psi}_0^{(k)} \right|^2$
$k = 15$
$k = 5$
$\ln(5^{\frac{k}{2}}\theta)$
$k = 21$
$k = 5$
$\ln \left| \tilde{\psi}_0^{(k)} \right|^2$
$k = 15$
$k = 21$
−1.0
−0.5
0.5
1.0
5
−5
−10

Conclusions

- first attempt to study quantum walks on graphs without translational invariance
- renormaliztion group leads to potentially chaotic recursion equations
- scaling determined by non-local properties rather than simple fixed points
- Lyapunov exponents and observables scale similarly
- hints for $d_w^{QW} = d_w^{RW}$

- How to analyze situation where no fixed point on the unit circle exists?
- universality among different coins?
- problem of localization?