Spin glasses with many components

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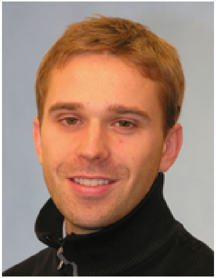

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Outline

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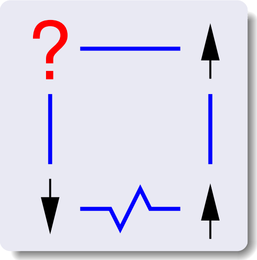
The EA model

Simplify to the essential properties, disorder and frustration to yield the Edwards-Anderson (EA) model,

$$\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij}\, \boldsymbol{s}_i \cdot \boldsymbol{s}_j, \quad |\boldsymbol{s}_i| = \sqrt{m}$$

where J_{ij} are *quenched*, random variables.



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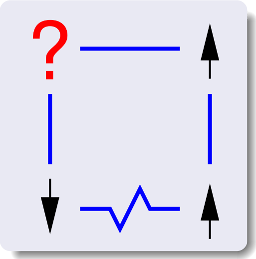
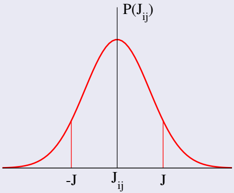
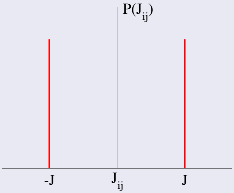


Coupling distributions

Gaussian



bimodal



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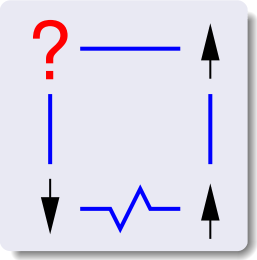


Has been investigated for $\approx$ 30 years, however no agreement on general case. Mean-field model with

$$J_{ij} = \frac{\pm 1}{\sqrt{N}},$$

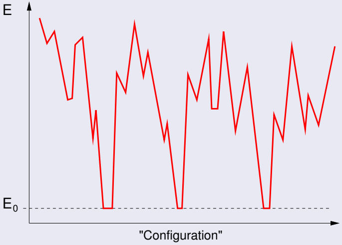
known as Sherrington-Kirkpatrick (SK) model can be solved in the framework of "replica-symmetry breaking" (RSB) (Parisi *et al.*, 1979/80).

The “pictures”

What happens in finite dimensions?

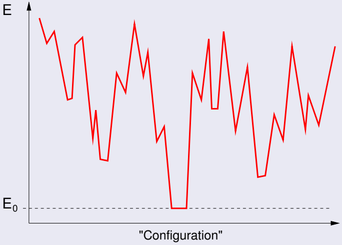
RSB picture

(Parisi, Mezard, ...)



Droplet picture

(D. Fisher, Huse, Bray, Moore, ...)



- many pure states
- global (gapless) excitations
- non-self-averaging and continuous distribution of $P(q)$

- only two pure states
- global excitations cost an infinite energy
- $P(q)$ is self-averaging

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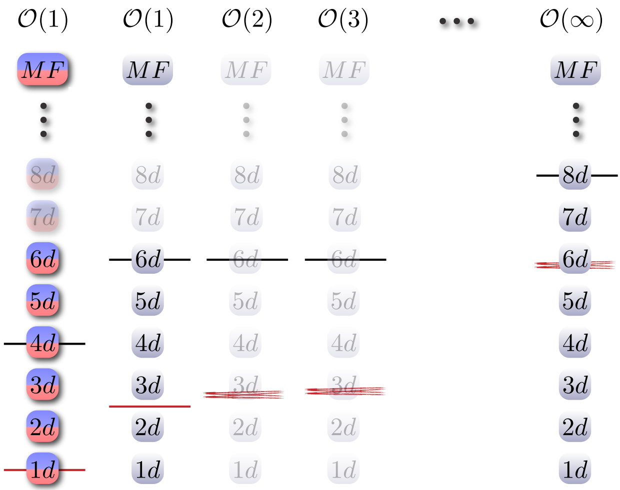
Critical dimensions

Consider lower and upper critical dimensions for the O(m) EA model:



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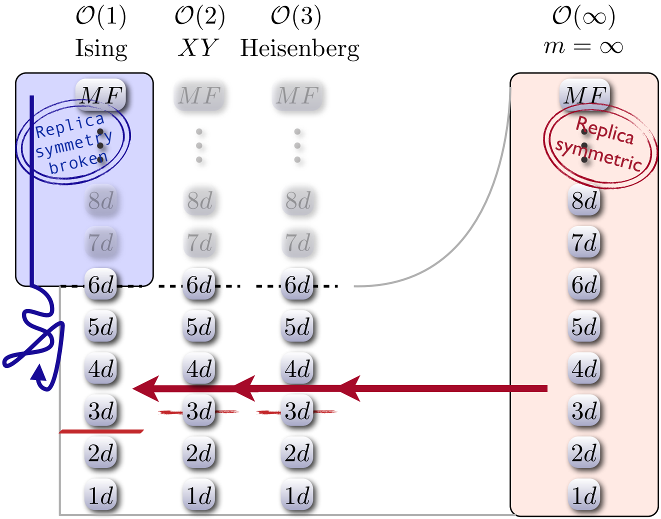
Replica symmetry

How can a well-behaved perturbative approach to “real” spin glasses be found?



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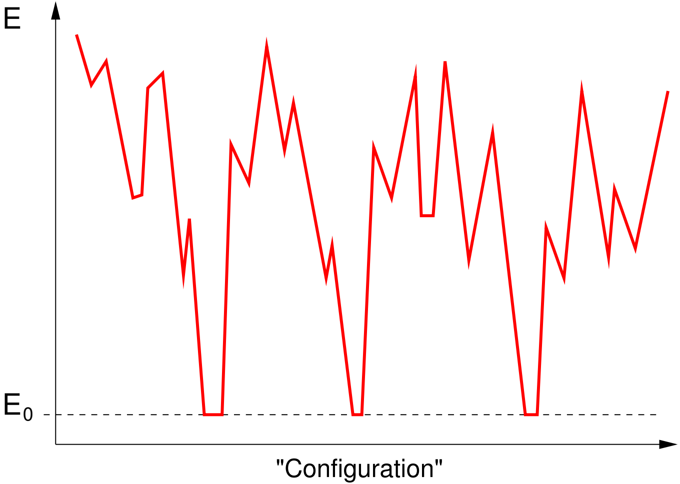
Slow dynamics

Dynamics is slow in the spin-glass phase due to trapping of the system in local energy minima separated by barriers $\Longrightarrow$ system is out of equilibrium at all (human) time scales



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1. simulations suffer from extremely slow relaxation due to the rugged free-energy landscape
2. the results are afflicted by rather strong finite-size corrections, making it hard to extrapolate to the thermodynamic limit

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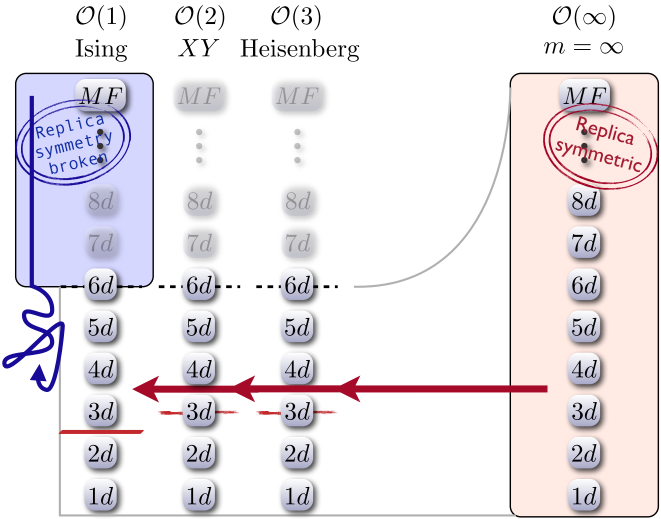

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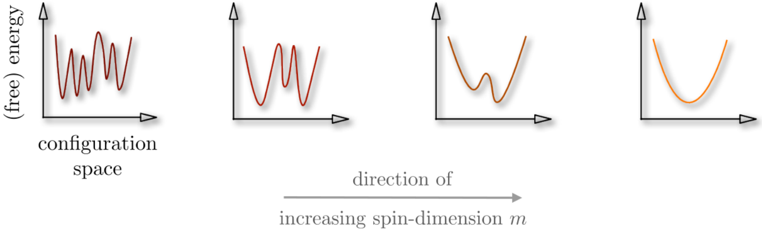
- the system lacks metastability and has a unique ground state, enabling efficient numerical ground-state calculations

Metastability

Metastability gradually disappears as m is increased.



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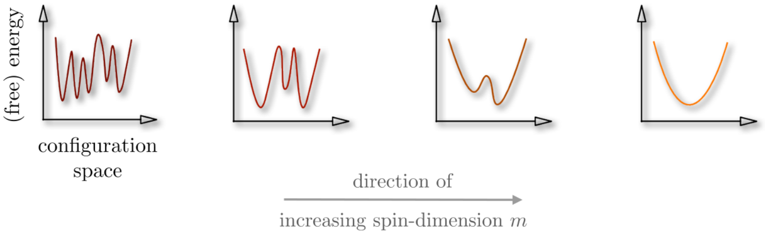

The ground state for N spins occupies an $m^*(N) \leq m_{\max}(N)$ dimensional sub-space,

$$m_{\max}(N) = \left\lfloor \left(\sqrt{8N+1} - 1 \right) /2 \right\rfloor \sim N^\mu, \ \mu = 1/2.$$

(Hastings, 2000)

For each N, a *finite* number of spin components is sufficient to arrive in the $m = \infty$ limit.

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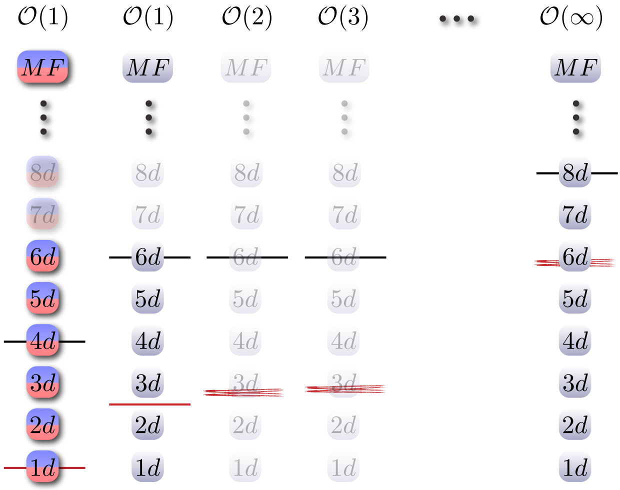

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Hence it is hard to reach the regime $6 \lesssim d < 8$ of non-trivial critical behavior in numerical work.

Spin stiffness and zero-temperature scaling

Edwards-Anderson model: $\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij}\, \boldsymbol{s}_i \cdot \boldsymbol{s}_j, \quad \boldsymbol{s}_i \in \mathrm{O}(n)$

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Ferromagnet

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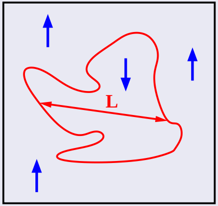
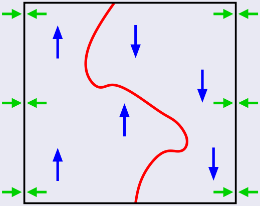


$\Delta E \sim L^{d-1}$ resp. L^{d-2}



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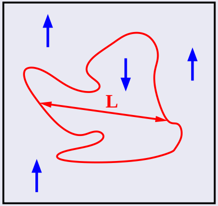

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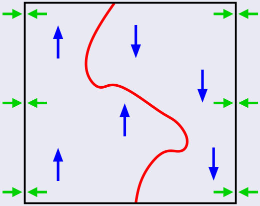

Spin glass (Bray/Moore, 1987)

Distribution of couplings evolving under RG transformations, asymptotic width scales as

$$J(L) \sim J L^{\theta(d)}.$$

Spin-stiffness exponent θ determines lower critical dimension. For $\theta < 0$,

$$\xi \sim T^{-\nu}, \quad \nu = -1/\theta.$$

Numerically, θ can be determined from inducing droplets or domain walls with a change of *boundary conditions*,

$$\Delta E = |E_{\mathrm{AP}} - E_{\mathrm{P}}| \sim L^{\theta}.$$

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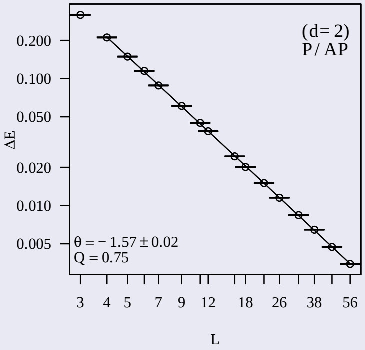
Defect energies

Periodic boundary conditions:



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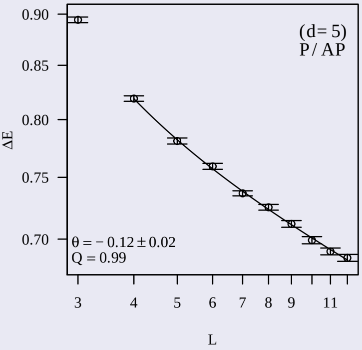

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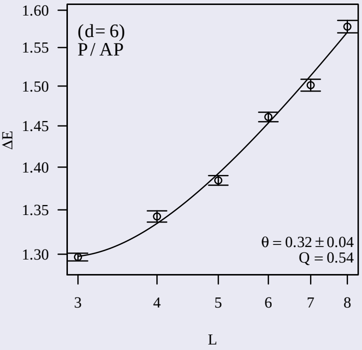

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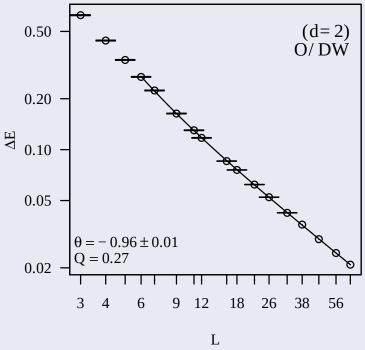

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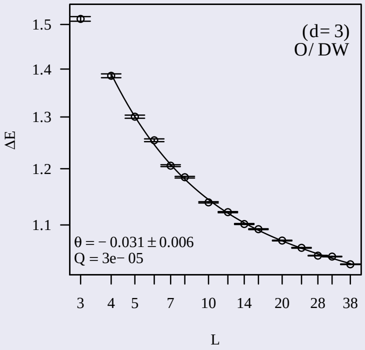

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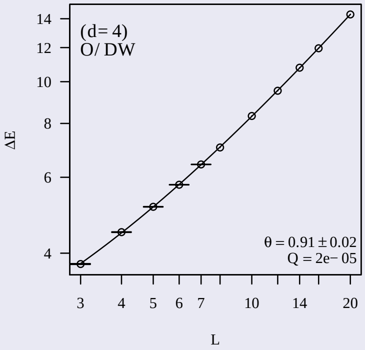

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Periodic boundary conditions:

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- consistent with $d_l = 6$ estimates by field theory
- upper and lower critical dimensions are distinct

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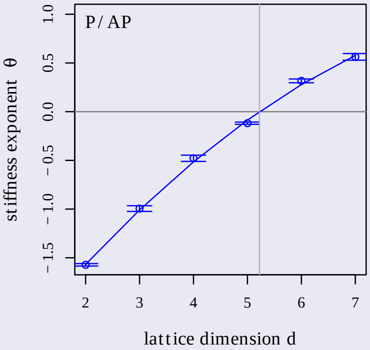

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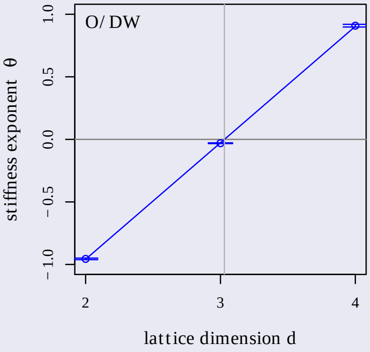
Open/domain-wall boundary conditions:

- lower critical dimension $d_l \approx 3$
- subtleties with limits $m \to \infty$ and $N \to \infty$
- possibly probes finite-m behaviour

Defect energies

Open/domain-wall boundary conditions:



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Considering a 1d EA model with long-range interactions,

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- $T_{\mathrm{SG}} = 0$ for $\sigma > \sigma_u = 1$; σ_u corresponds to the lower critical dimension
- a non-trivial spin-glass transition with $T_{\mathrm{SG}} > 0$ for $2/3 = \sigma_l < \sigma < 1$; $\sigma_l = 2/3$ corresponds to the upper critical dimension

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it is possible by tuning σ to mimic the behavior on hypercubic lattices of variable dimension d (Kotliar *et al.*, 1983). Since the system is 1d, however, one can treat much larger systems.

For the $m = 1$ Ising case, one finds

- $T_{\mathrm{SG}} = 0$ for $\sigma > \sigma_u = 1$; σ_u corresponds to the lower critical dimension
- a non-trivial spin-glass transition with $T_{\mathrm{SG}} > 0$ for $2/3 = \sigma_l < \sigma < 1$; $\sigma_l = 2/3$ corresponds to the upper critical dimension
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This correspondence has been used by Katzgraber, Leuzzi, Moore, Parisi, Young, and others in recent years to study Ising, Potts, p-spin and Heisenberg spin glasses.

Phase diagram

We are able to show that through dimensional reduction which for the $m = \infty$ long-range model takes the form $(d - \Theta)\nu = 2 - \alpha$ with

$$\Theta = \begin{cases} 2\sigma - 1 = 2/d_{\mathrm{eff}}, & 5/8 \leq \sigma, \\ 1/4, & 1/2 \leq \sigma < 5/8. \end{cases}$$

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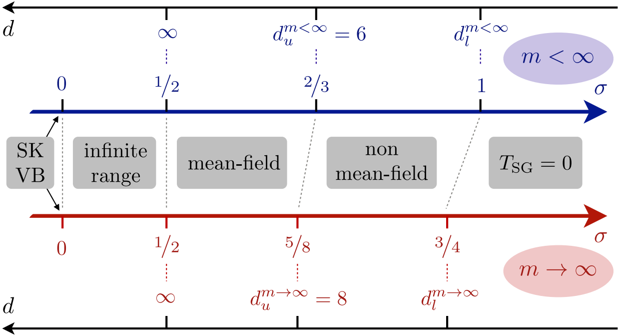

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One can set up an approximate dictionary between the hypercubic short-range and the 1d long-range systems as

$$d_{\mathrm{eff}} = \frac{2}{2\sigma - 1},$$

or, somewhat more precisely,

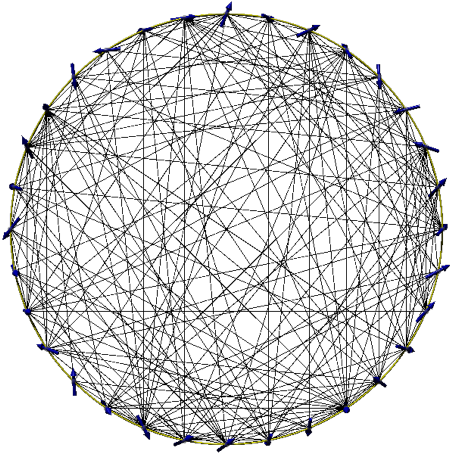
$$d_{\mathrm{eff}} = \frac{2 - \eta(d_{\mathrm{eff}})}{2\sigma - 1},$$

where $\eta(d_{\mathrm{eff}})$ is the exponent of the corresponding short-range model.

Outline

Diluted model

An alternative diluted model with the bond-existence probability falling $\propto 1/r^{2\sigma}$ has also been suggested (Leuzzi *et al.*, 2008).



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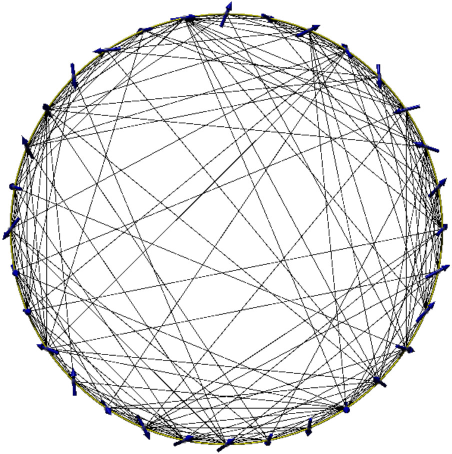

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Determine defect energies from ground-state calculations for periodic and antiperiodic boundaries, $\Delta E = |E_{\mathrm{AP}} - E_{\mathrm{P}}|$. The L dependence is expected to be $E_{\mathrm{def}} \propto L^{\theta}$.

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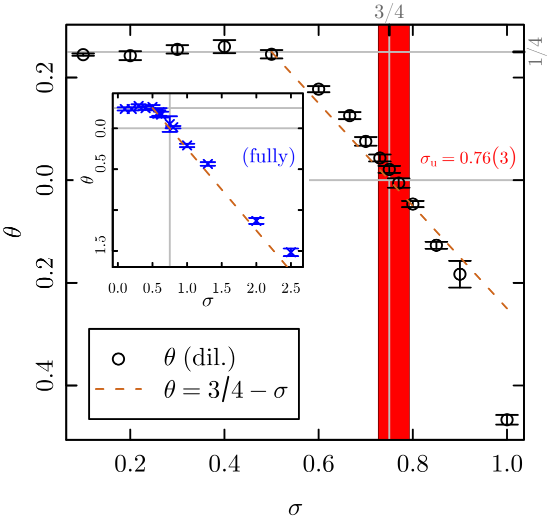

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For the diluted model, however, a (previously missed) breakdown of universality is observed for $\sigma > 1$, where the graphs become 1d short-range due to a percolation transition.

Distribution of ground-state energies

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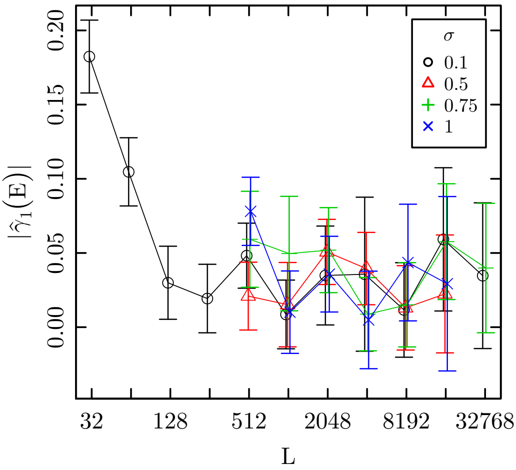

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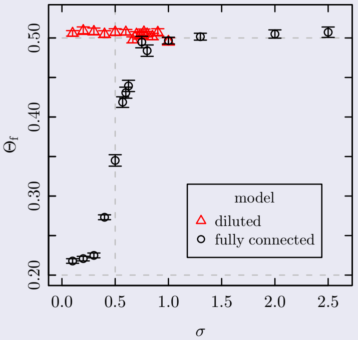

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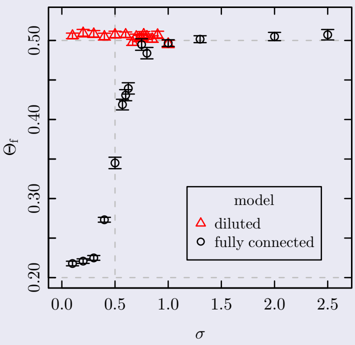

This shows another instance of non-universality between the two models. The result for the fully connected model approaches $\Theta_f = 1/5$ expected for the SK model

(Aspelmeier and Braun, 2010).

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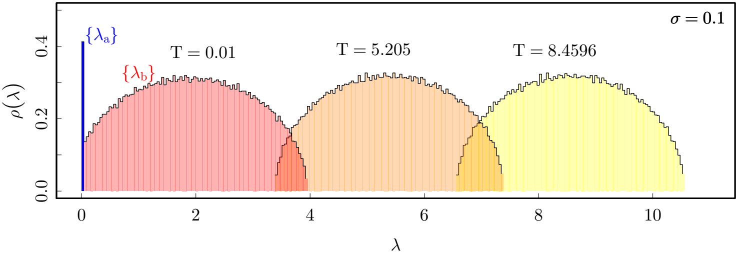

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Then, the order parameter is determined by the zero eigenvalues λ_a,

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while the spin-glass susceptibility defined from the *connected correlation function* contains the non-zero eigenvalues λ_b,

$$\begin{aligned} \chi_{\mathrm{SG}} &= \frac{1}{Nm^2} \sum_{i,j} [\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle - \langle \mathbf{S}_i \rangle \cdot \langle \mathbf{S}_j \rangle]^2 \\ &= \frac{T^2}{N} \sum_b \frac{1}{\lambda_b^2}, \end{aligned}$$

(Aspelmeier and Moore, 2004)

Correlation length

An estimate of the finite-size correlation length can be extracted from the spin-glass susceptibility,

$$\xi_L = \frac{1}{2\sin(k_{\mathrm{min}}/2)}\left[\frac{\chi_{\mathrm{SG}}^0(0)}{\chi_{\mathrm{SG}}^0(k_{\mathrm{min}})} - 1\right]^{1/(2\sigma-1)}.$$

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Above the UCD, finite-size scaling should work with L replaced by $\zeta_L \sim L^{d/d_u}$,

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For $\sigma > 5/8$ McMillan's expansion around the LCD,

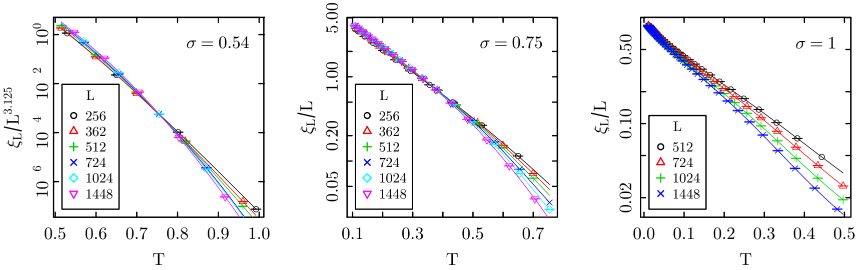
$$\frac{\mathrm{d}T}{\mathrm{d}\ln L} = -\theta T + cT^3 + \ldots,$$

with the conjectured $\theta = 3/4 - \sigma$ yields

$$\nu = \frac{1}{2\theta} = \frac{2}{3-4\sigma}$$

and $T_{SG} \propto \sqrt{3-4\sigma}$.

Correlation length



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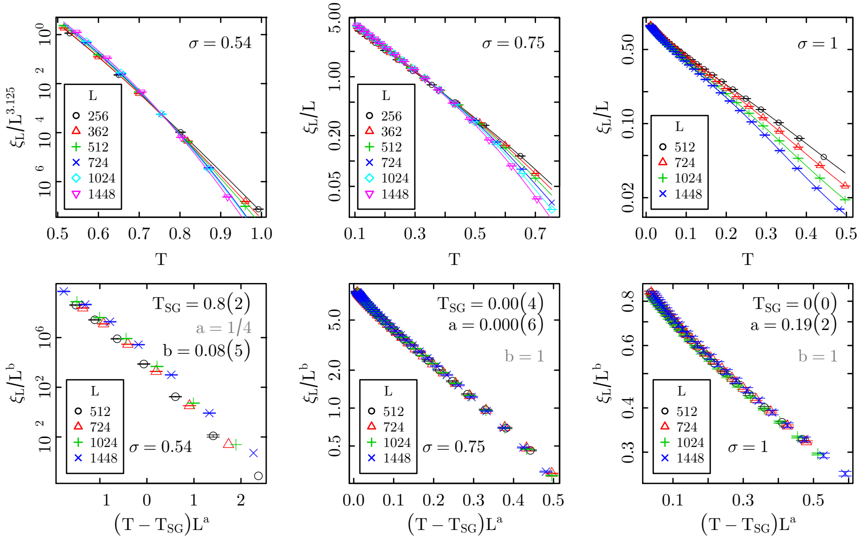

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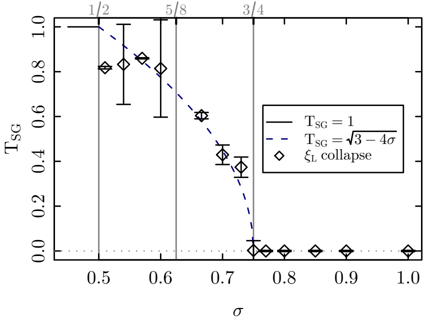
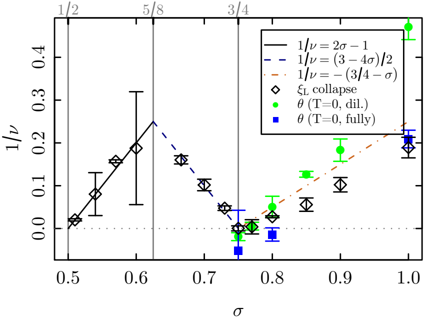
Using an elaborate collapsing technique and a jackknife/resampling analysis of statistical errors, we find:





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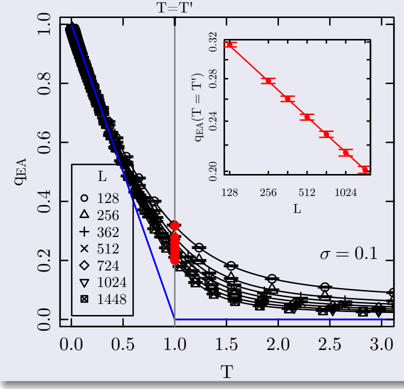

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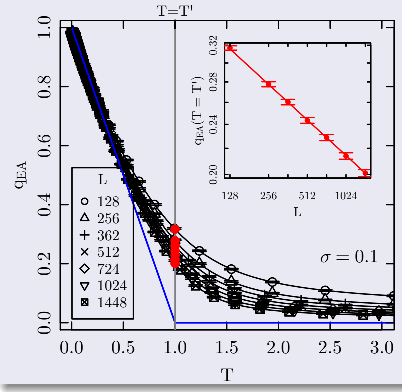

Our scaling arguments give

$$q_{\mathrm{EA}} \sim \begin{cases} L^{-1/4}\mathcal{Q}(tL^{1/4}), & \sigma \leq 5/8, \\ L^{-\beta/\nu}\mathcal{Q}(tL^{1/\nu}), & \sigma > 5/8. \end{cases}$$

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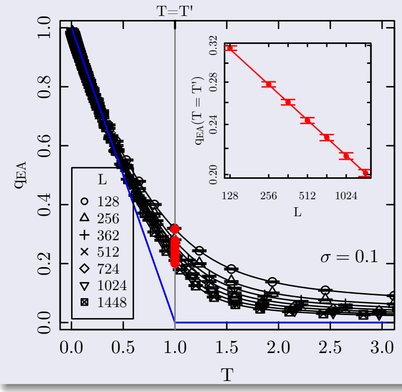

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From the conjectured form of the violation-of-hyperscaling exponent Θ we expect

$$\beta/\nu = (3 - 4\sigma)/2,$$

such that with the approximate $\nu = 2/(3 - 4\sigma)$ the exponent β stays close to its mean-field value $\beta = 1$.

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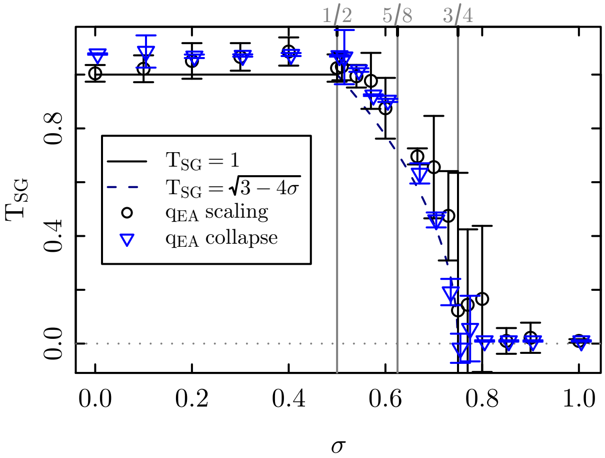

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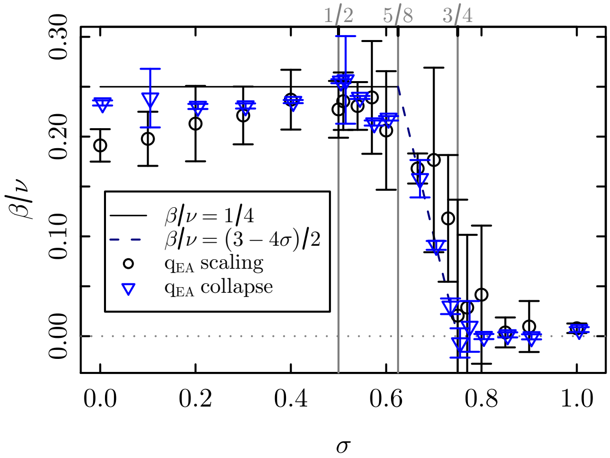

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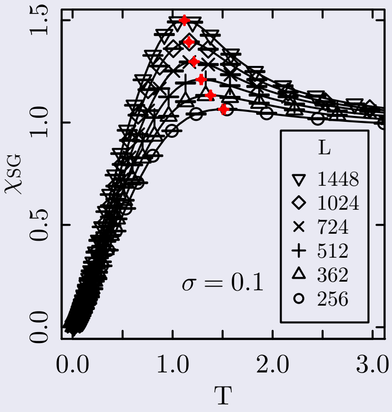

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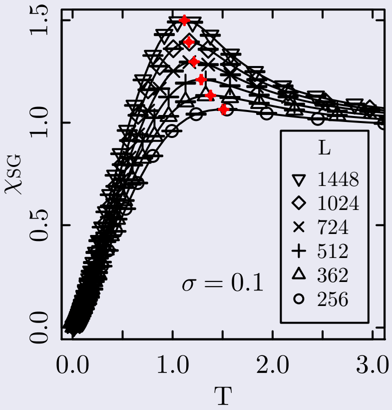

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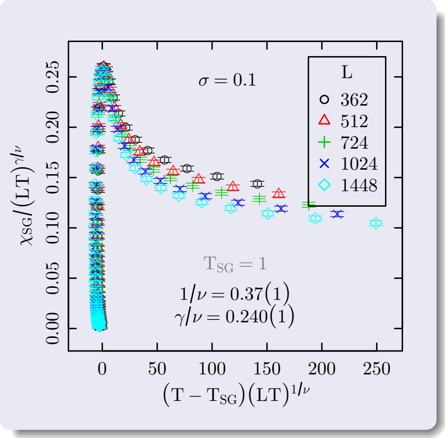

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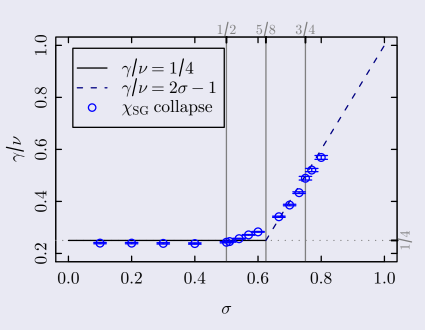

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References:

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