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Computational Quantum Field Theory

Multicanonical Analysis of the Gonihedric Ising Model

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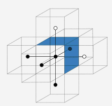
Origin and Motivation

The Gonihedric Ising model arises from bosonic string theory as a possible discretisation of the area swept out by a string worldsheet moving through spacetime (1). The name comprises the greek words gonia (angle) and hedra (face) as a reminder of the origin.

The lattice spin model allows the application of sophisticated generalised ensemble Monte Carlo methods that are tailored to tackle many open questions on the characteristics of phases occurring.

The Gonihedric Ising model

Consider spins $\sigma \in \{-1,+1\}$ that sit on 3-dimensional cube, a surface is then described by a bulk of plaquettes in the dual lattice that separate contiguous spins with opposite sign.



The interaction of these spins are carefully adjusted in such a way, that only the *linear size* of the surface in the dual lattice contributes but not the *surface area* (2), a desired property from the original intention of describing a string theory.

This leads to the Hamiltonian

$$\mathcal{H} = -2\kappa \sum_{\langle i,j\rangle} \sigma_i\sigma_j + \frac{\kappa}{2} \sum_{\langle\langle i,j\rangle\rangle} \sigma_i\sigma_j - \frac{1-\kappa}{2} \sum_{[i,j,k,l]} \sigma_i\sigma_j\sigma_k\sigma_l,$$

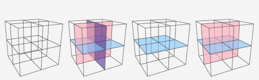
where the sums are carried out over nearest-neighbour spins, next-to-nearest-neighbour spins and plaquettes in the normal lattice.

The degree of self-avoidance of the surface is controlled by the free parameter κ, i.e. the case $\kappa \to \infty$ is the limit of complete self-avoidance, whereas $\kappa = 0$ leads to surfaces that can intersect each other without energetic penalty. The case of $\kappa = 0$, where only the pure plaquette term is considered

$$\mathcal{H} = -\frac{1}{2} \sum_{[i,j,k,l]} \sigma_i\sigma_j\sigma_k\sigma_l, \qquad (1)$$

shows a first order transition and will be discussed in this study.

Considering periodic boundary conditions and supposing that the lattice size is even, the zero-temperature ground state can be constructed by composing elementary surfaces that minimize the energy of a unit cell of the lattice. Starting from a cube with a given spin configuration the adjacent cubes can have the same configuration by reflecting the original cube with respect to the common face, thus the whole lattice can be spanned by cubes with the same energy.



The bulk ground state allows flipping of whole planes parallel to either one of the xy, yz, zx-planes and even diagonal ones, therefore the ground state degeneracy $q = 2^{3L}$, with L being the linear lattice size.

Finite-Size Scaling

For response functions like the heat capacity C or Binder's cumulant U_{B},

$$C = \beta^2\left(\left\langle E^2\right\rangle - \langle E\rangle^2\right), \qquad U_{\mathrm{B}} = 1 - \frac{\langle E^4\rangle}{3\left\langle E^2\right\rangle^2},$$

standard first-order finite-size scaling theory exists. For periodic boundary conditions the inverse temperatures $\beta_0(L)$ where extrema appear are expected to scale like

$$\beta_0(L) = \beta_0^\infty + aL^{-3} + \mathcal{O}\left(L^{-6}\right) \qquad (2)$$

with an amplitude a dependent on the observable.

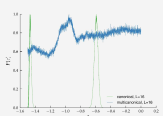
Method

To overcome *supercritical slowing down* observed when conducting canonical simulations near the transition point of a first-order phase transition, the multicanonical algorithm promotes rare states μ with energy $E_\mu = \mathcal{H}\{\mu\}$. These are suppressed by a factor $\propto \exp(-2\beta\sigma L^2)$, where σ is the interface tension, by simulating with an auxiliary distribution,

$$p_\mu^{\mathrm{muca}}(\beta) = \exp(-\beta E_\mu - f(E_\mu)),$$

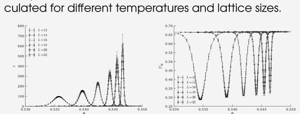
that differs from the Boltzmann distribution by arbitrary weights $f(E_\mu)$.

These weights have to be determined beforehand by clever algorithms and allow to overcome barriers apparent in the system.



Results for the original model $\mathcal{H}(\kappa = 0)$

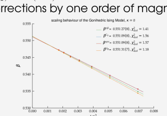
The specific heat and Binder's cumulant have been calculated for different temperatures and lattice sizes.



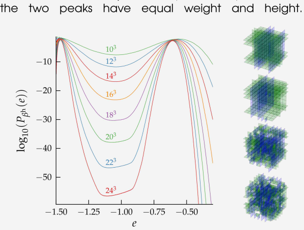
Carefull analysis of the finite-size scaling behaviour revealed better agreement with the scaling ansatz

$$\beta_0(L) = \beta_0^\infty + \hat{a}L^{-2} + \mathcal{O}\left(L^{-6}\right). \qquad (3)$$

The amplitude in Eq. (2) of the original Pirogov-Sinai theory is $a \propto \ln(q) \propto \ln(2^{3L}) \propto L$, which effectively lowers the finite-size corrections by one order of magnitude.



The energy probability densities have been calculated near the transition temperature to obtain estimates on the temperatures β^{eqw} and β^{eqh}, where the two peaks have equal weight and height.



The reduced interface tension $\hat{\sigma} = \beta\sigma$ has been extracted from the canonical distribution at inverse temperature β^{eqh} by calculating

$$\hat{\sigma}(L) = \frac{1}{2L^2} \ln\left(\frac{\max[P_{\beta^{\mathrm{eqh}}}(L)]}{\min[P_{\beta^{\mathrm{eqh}}}(L)]}\right).$$

from the distribution's maxima and minimum. Linear regression of the reduced interface tension for various lattice sizes leads to a value of

$$\hat{\sigma} = 0.1183(6)$$

which would be nearly impossible to retrieve from canonical data with the same accuracy.

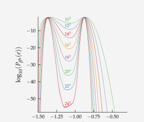
Results for the dual model $\mathcal{H}^{\mathrm{d}}$

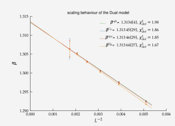
A dual representation of the Hamiltonian in Eq. (1) has been formulated explicitely in terms of two Ising spins σ, τ sitting on each vertex of a three dimensional lattice (3). The dual Hamiltonian is of the form

$$\mathcal{H}^{\mathrm{d}} = -\frac{1}{2} \sum_{\langle i,j\rangle_x} \sigma_i\sigma_j - \frac{1}{2} \sum_{\langle i,j\rangle_y} \tau_i\tau_j - \frac{1}{2} \sum_{\langle i,j\rangle_z} \sigma_i\sigma_j\tau_i\tau_j. \qquad (4)$$

Two orthogonal Ising spin chains interact in a non-trivial way due to the last sum. Canonical Monte Carlo simulations of this Hamiltonian led to a transition temperature $\beta_0 = 0.510(2)$ that did not coincide with earlier results of $\beta_0 = 0.54757(63)$ of the original model (4).

A multicanonical simulation has been conducted for the dual model. For the finite-size scaling ansatz, Eq. (3) improves the quality of the regression comparable to the regression results in the original model. The zero-energy ground state of the dual model has similar degeneracies (3), therefore the unusual scaling ansatz is justified.





The reduced interface tension was estimated to be

$$\hat{\sigma} = 0.1223(6),$$

which is greater than for the original model hence hysteresis effects are stronger.

Agreement of the models

The inverse transition temperature of the dual model β^* is related to the transition temperature β^∞ in the Gonihedric Ising model by the relation

$$\beta^* = -\ln\left(\tanh\left(\frac{\beta^\infty}{2}\right)\right).$$

Applying standard error propagation, the relation suggests a value of $\beta_0^\infty = 0.551\,23(15)$ when using the best regression results for the dual model retrieved by the multicanonical simulations. Therefore the data is consistent.

Conclusions & Outlook

Multicanonical simulations of the Gonihedric Ising model with vanishing energy penalty for intersecting surfaces and a dual model have been conducted. The response functions of the energy show pronounced peaks and their locations have been analysed by finite-size scaling. It was found that the dual model and the original model coincide when one assumes an unusual finite-size scaling ansatz for both models. The regression results then improve significantly. The ansatz was supported by taking the ground state degeneracy under consideration. It would be very interesting to simulate larger lattices to gain further insight into the finite-size scaling. Recently, order parameters for the model have been determined that could significantly improve the estimations on the transition temperature (5).

References

(1) G. K. Savvidy and F. J. Wegner, *Geometrical String and Spin Systems*, Nucl. Phys. B **413**, 605 (1994).

(2) G. K. Savvidy and K. G. Savvidy, *String Fine-Tuning*, Int. J. Mod. Phys. A **8**, 3993 (1993).

(3) D. A. Johnston and R. P. K. C. M. Ranasinghe, *The dual gonihedric 3D Ising model*, J. Phys. A: Math. Theor. **44**, 295004 (2011).

(4) M. Baig, J. Clua, D. A. Johnston, and R. Villanova, *Fixed Boundary Conditions Analysis of the 3D Gonihedric Ising Model with κ=0*, Phys. Lett. B **585**, 180 (2004).

(5) D. A. Johnston, *Gonihedric (and Fuki-Nuke) order*, J. Phys. A: Math. Theor. **45**, 405001 (2012).

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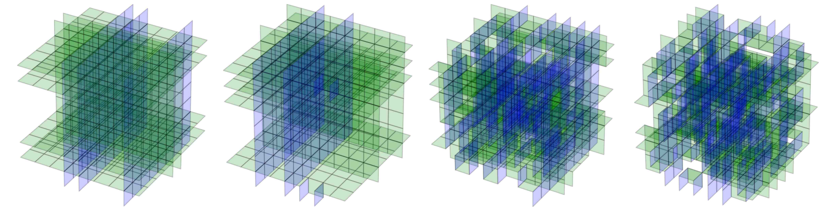

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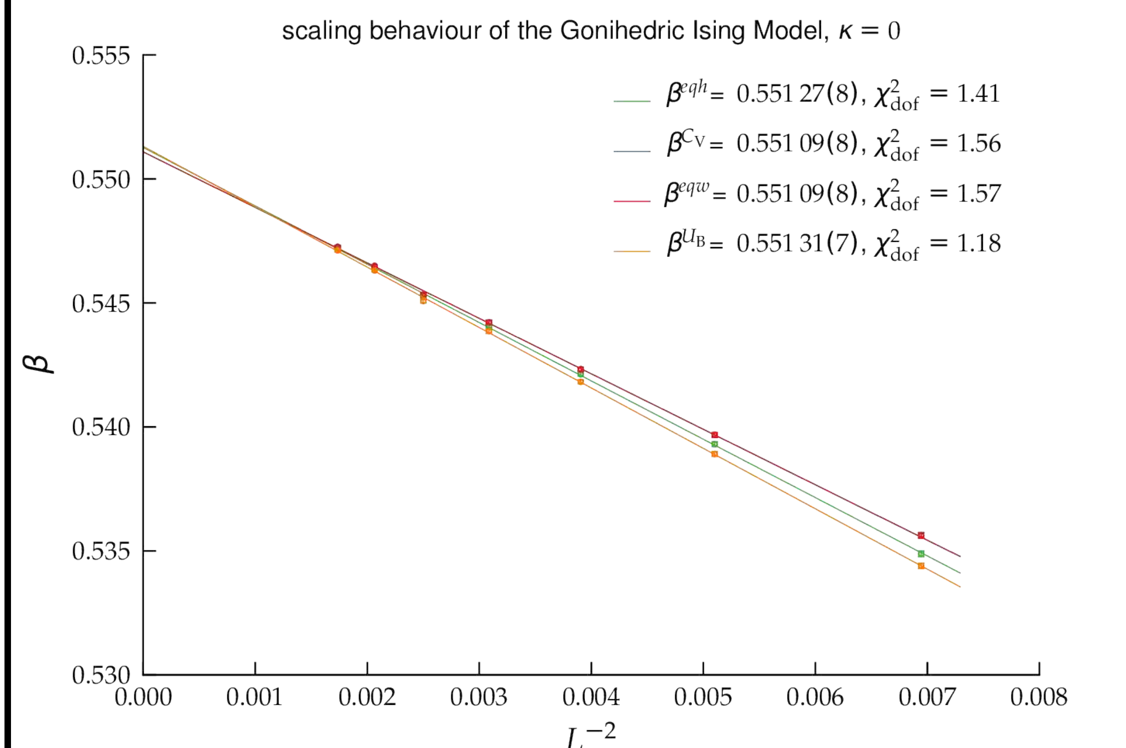

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