Corner contribution to percolation cluster numbers

Ferenc Iglói (Budapest)

in collaboration with

John Cardy (Oxford) and *István Kovács* (Budapest)

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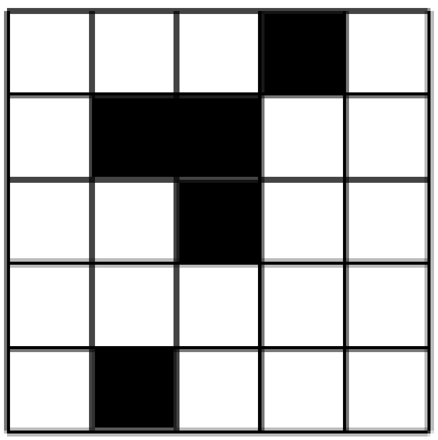
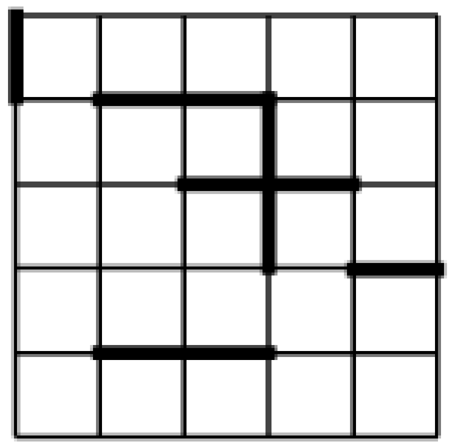
Percolation

site percolation



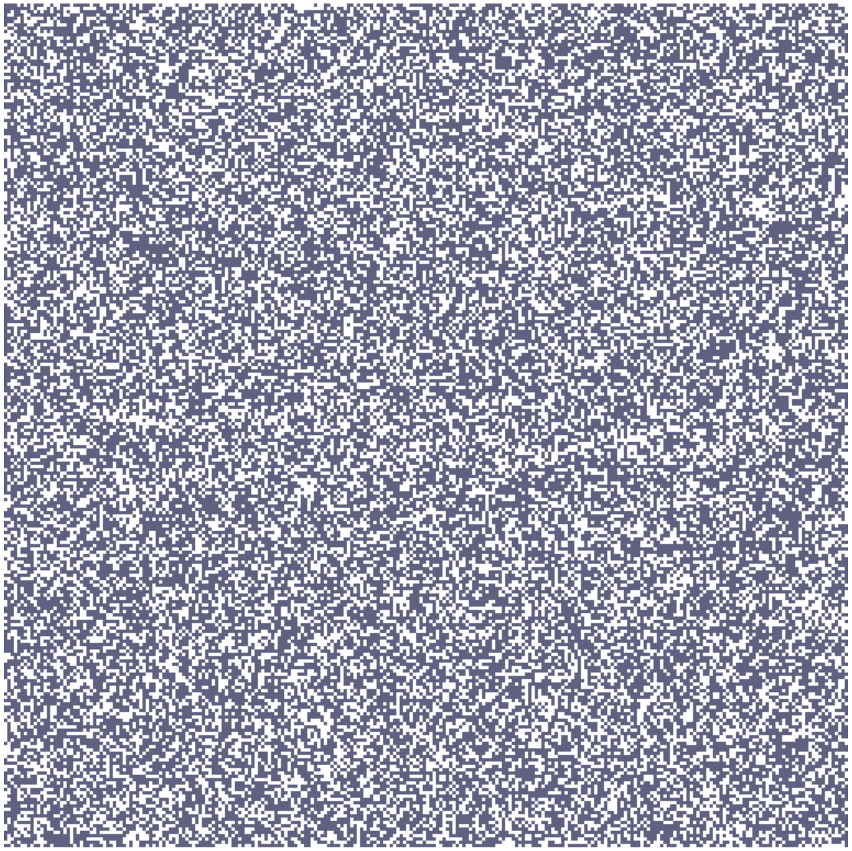
bond percolation



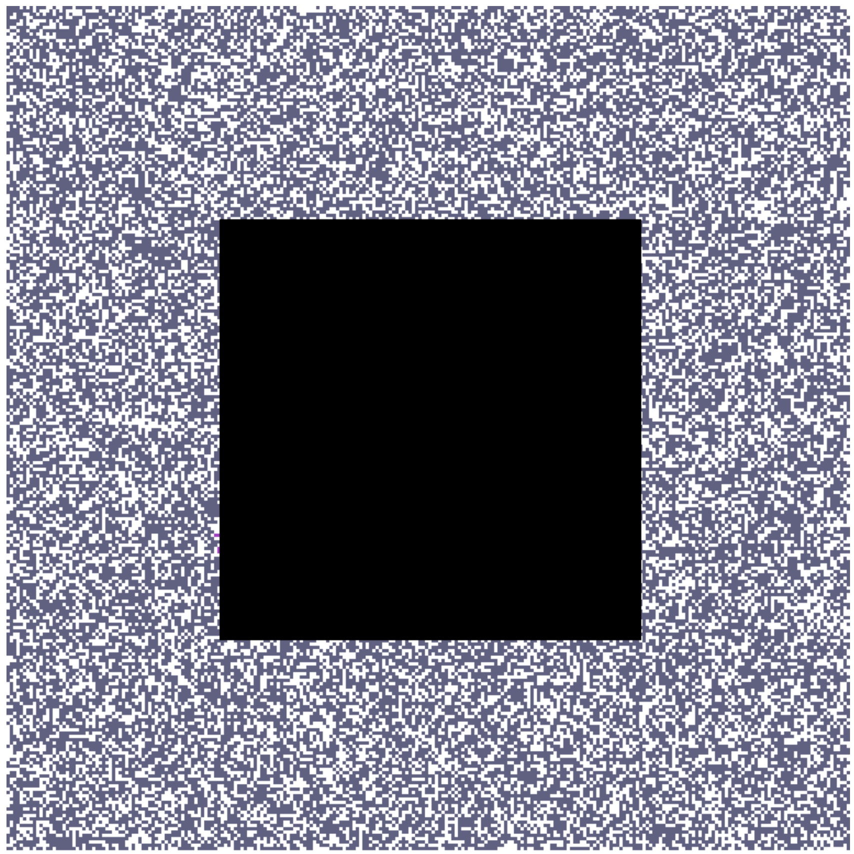
2D: $p_c \approx 0.592746$ $p_c = 0.5$

3D: $p_c \approx 0.3116$ $p_c \approx 0.24881$

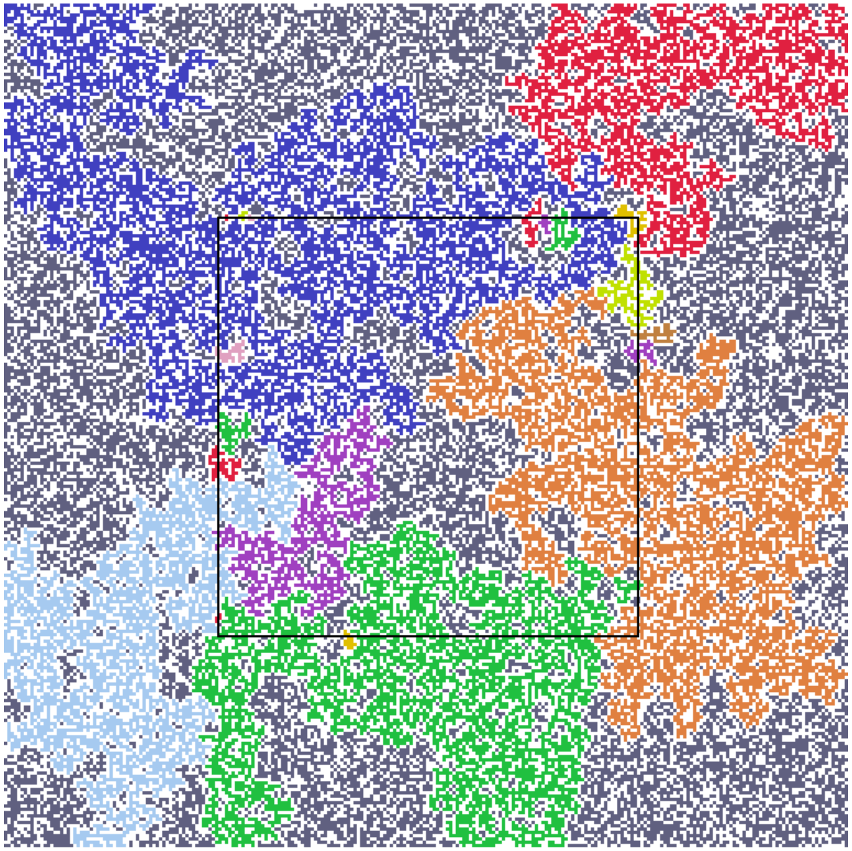
A critical system



Square subsystem

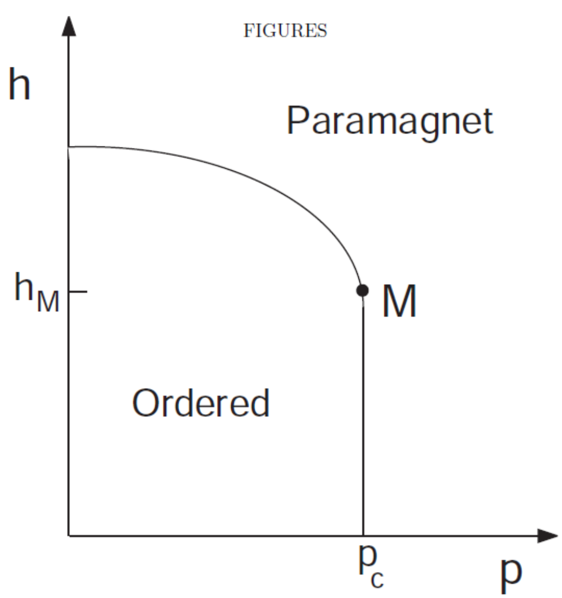


What is the number of crossed clusters?



Entanglement entropy of the diluted quantum Ising model

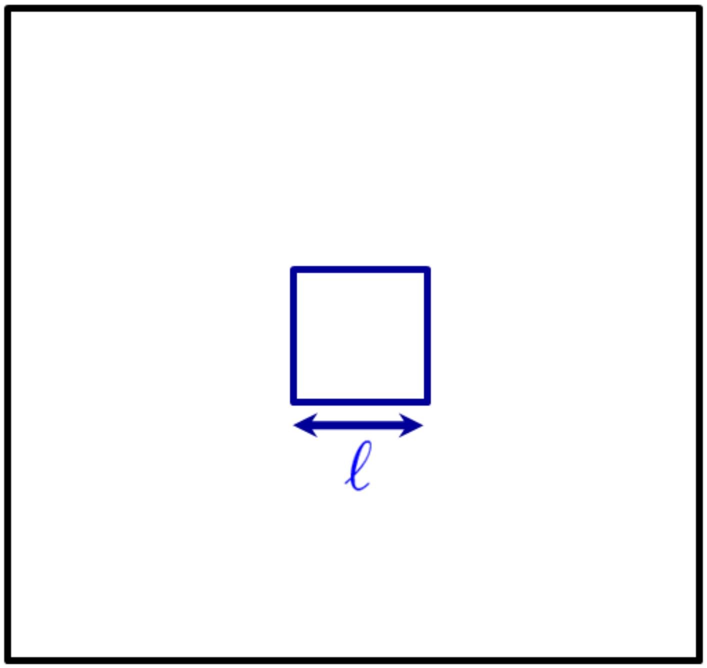
- Hamiltonian:
$$\mathscr{H} = -\sum_{\langle ij \rangle} J_{ij}\sigma_i^x\sigma_j^x - \sum_i h\sigma_i^z$$
- with diluted couplings
$J_{ij} = 0$ with proba. p
$J_{ij} = J > 0$ with proba. $1-p$
- phase diagram:



- for small h percolation transition at p_c
- ground-state of the system:
each connected cluster with c number of spins is in a GHZ-state:
$$\frac{1}{\sqrt{2}}(|\underbrace{\uparrow\uparrow\cdots\uparrow}_{c \text{ times}}\rangle + |\underbrace{\downarrow\downarrow\cdots\downarrow}_{c \text{ times}}\rangle)$$
- entanglement entropy between a subsystem: A and the enviroment: B, having a boundary: Γ:
$$\mathscr{S}_\Gamma = -\mathrm{Tr}(\rho_A \log_2 \rho_A)$$
- $\rho_A = \mathrm{Tr}_B|0><0|$: reduced density matrix
- $|0>$ ground state of $A\bigcup B$
- for the GHZ-state the entanglement entropy is $\mathscr{S}_{GHZ} = 1$
- each crossed cluster gives 1 contribution to $\mathscr{S}_\Gamma$
- asymptotically $\mathscr{S}_\Gamma$ and N_Γ are equivalent

Expectation: area law

$$\mathcal{S}(\ell) \sim \ell^{d-1}$$



How does it grow with the subsystems size?

Expectation: area law $\mathcal{S}(\ell) \sim \ell^{d-1}$

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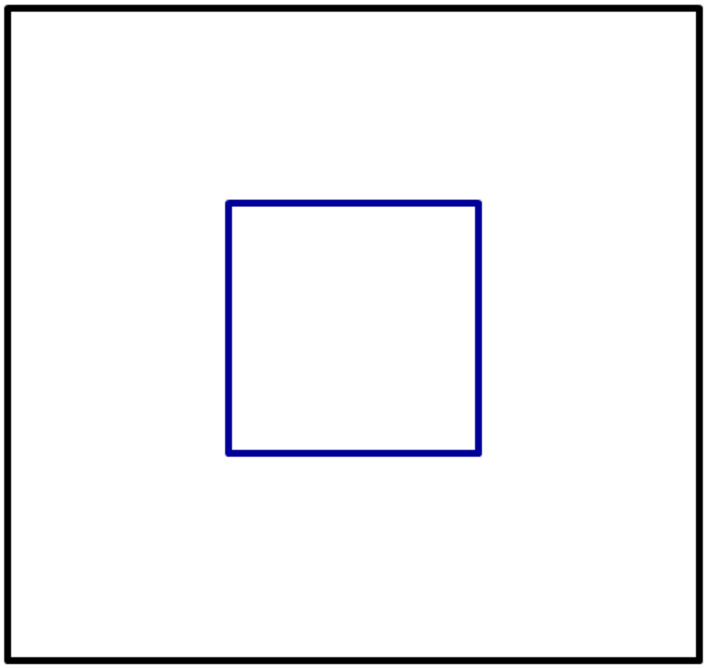

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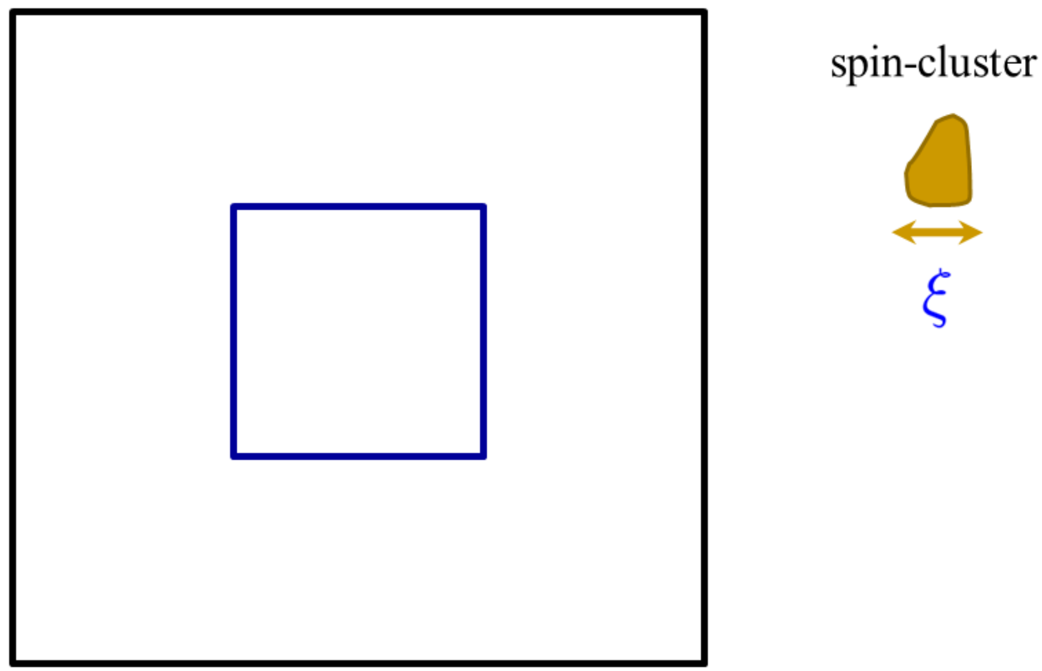

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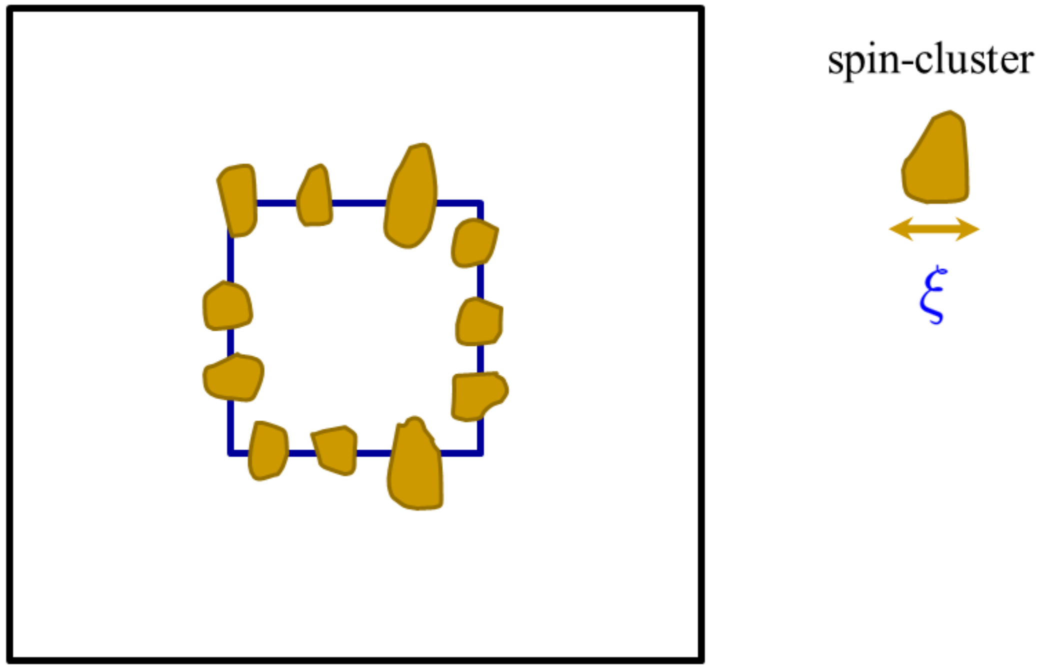
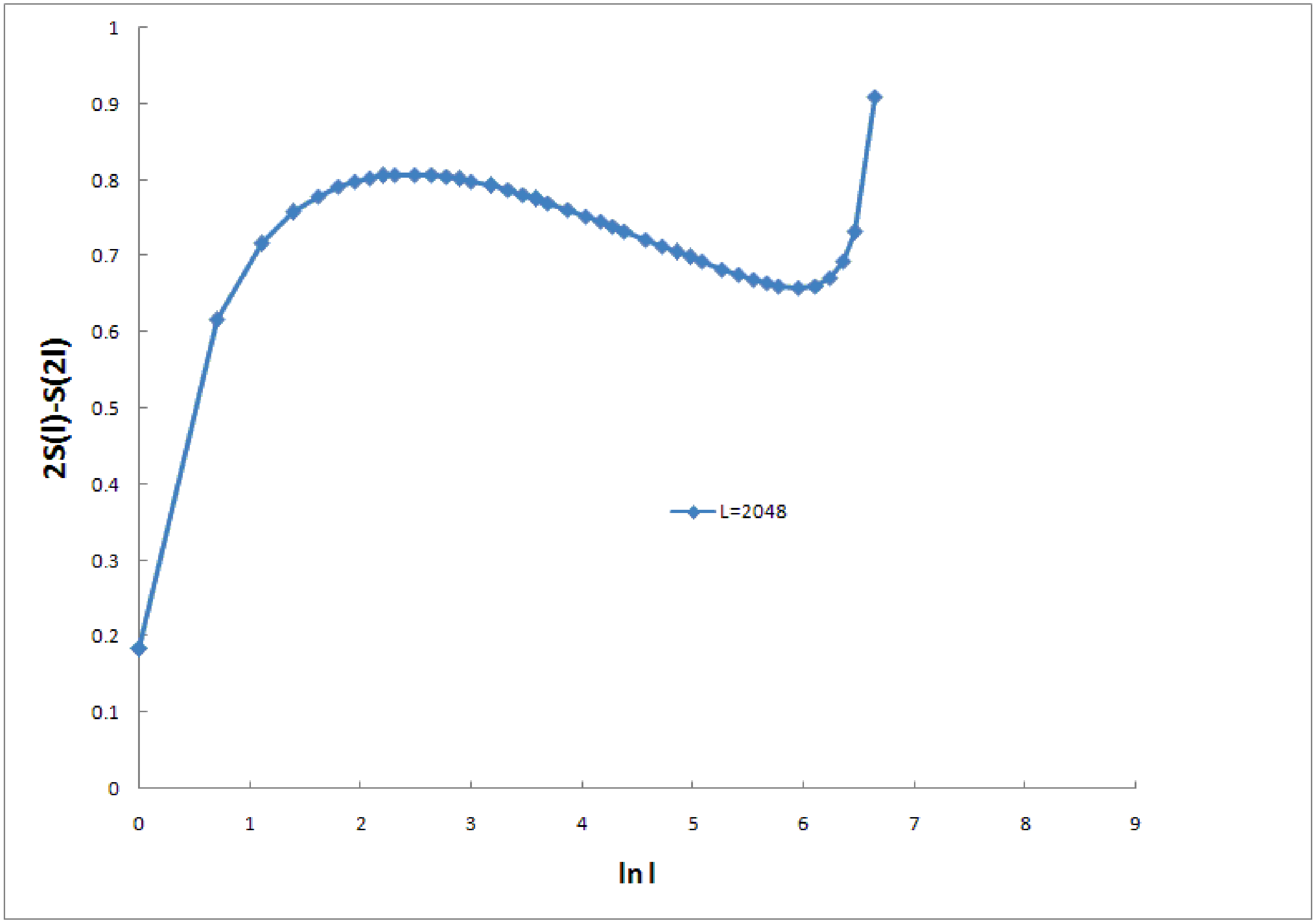


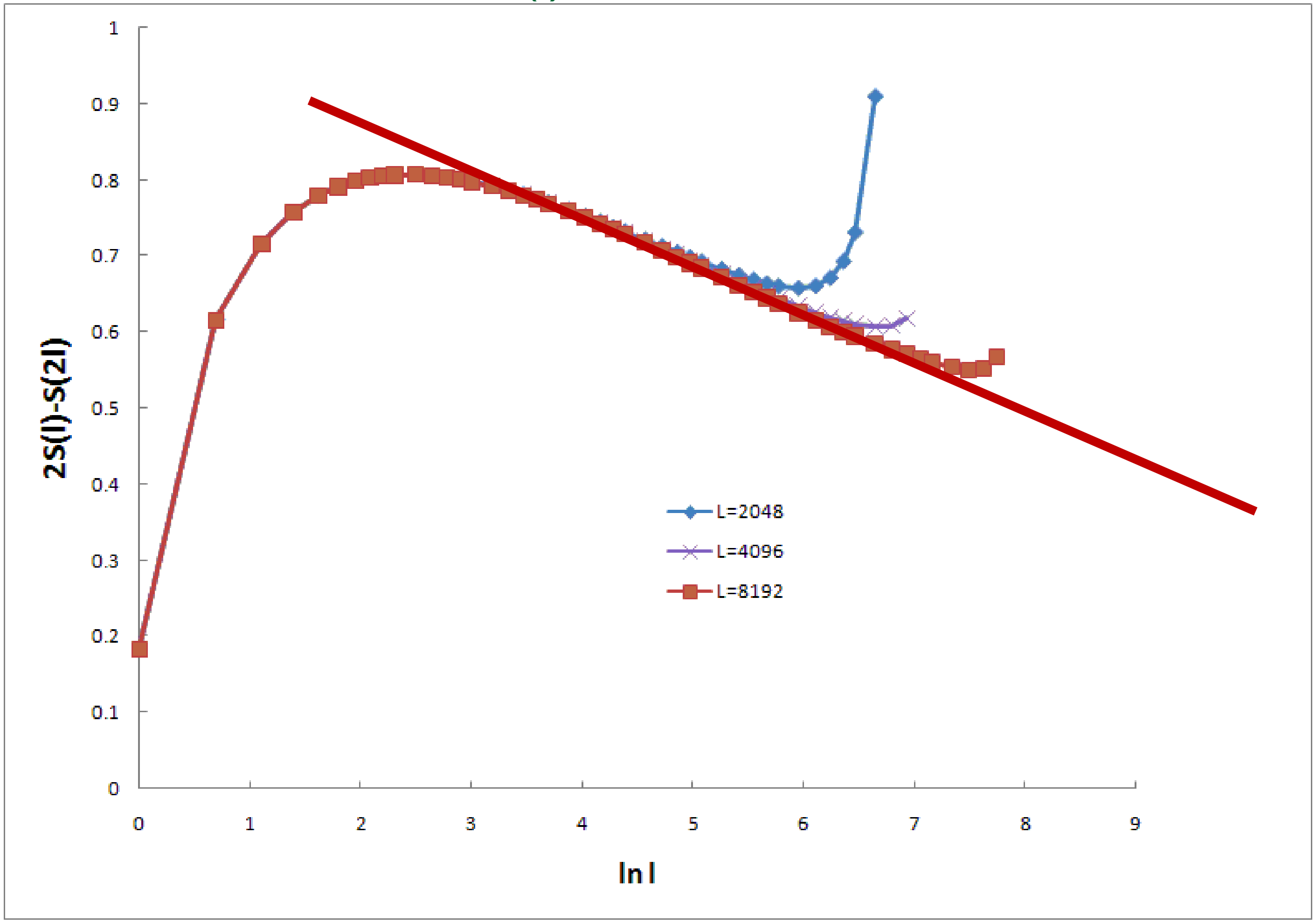
How does it grow with the subsystems size? $\mathcal{S}_{2\mathrm{D}}(\ell) \sim \ell$

Corrections?



First study: Yu et al. Phys. Rev. B 77, 140402 (2008)

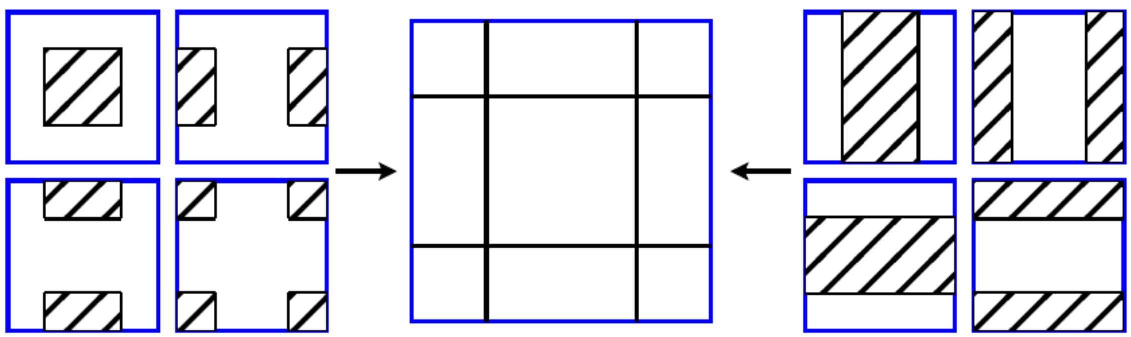
S(l)=al+bln l



First study: Yu et al. Phys. Rev. B 77, 140402 (2008)

Idea: it may come from the corners

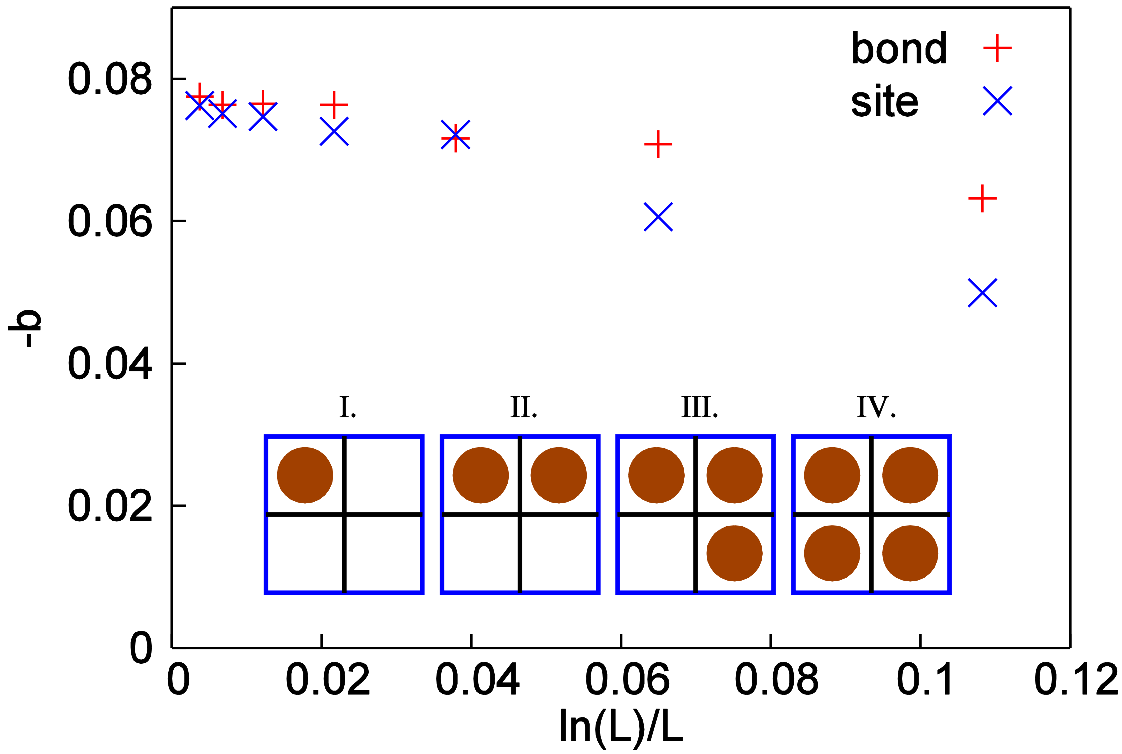
How to check this?



$\ell = L/2$

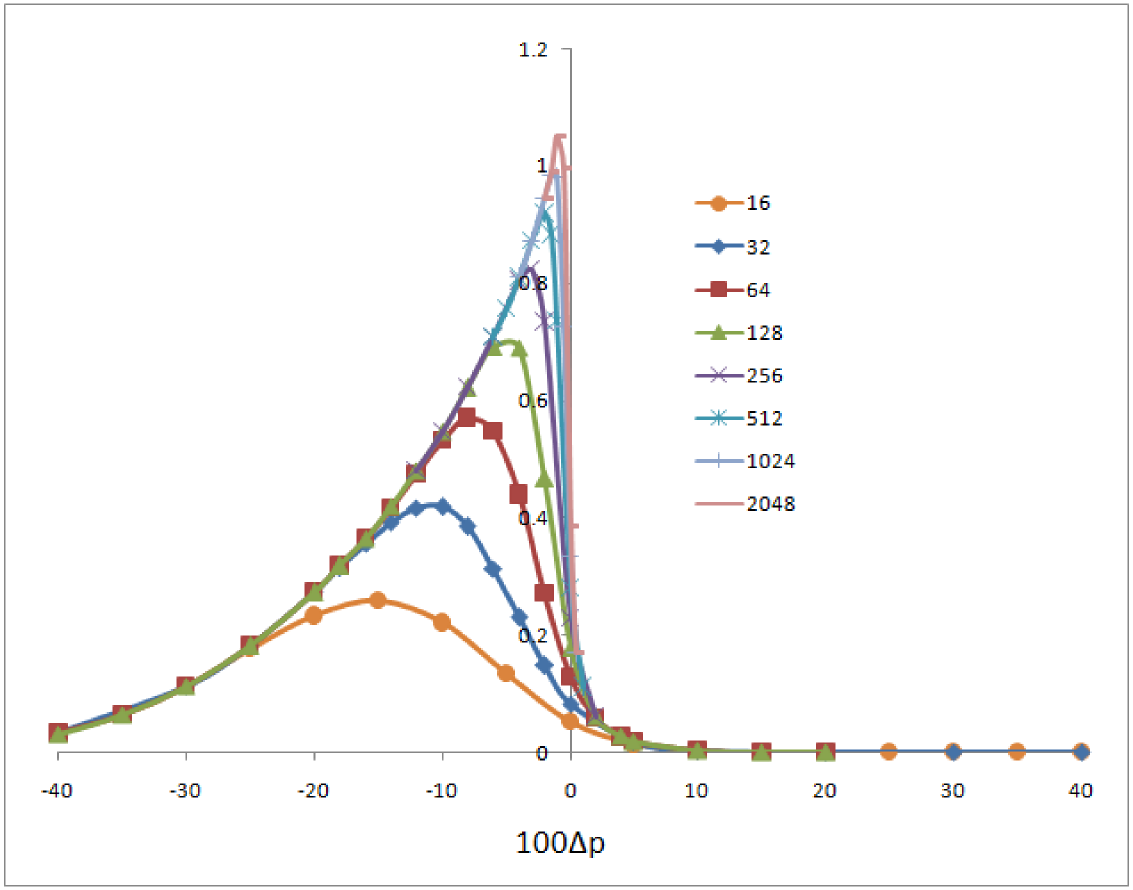
The difference is caused by the corners

The logarithmic correction is universal

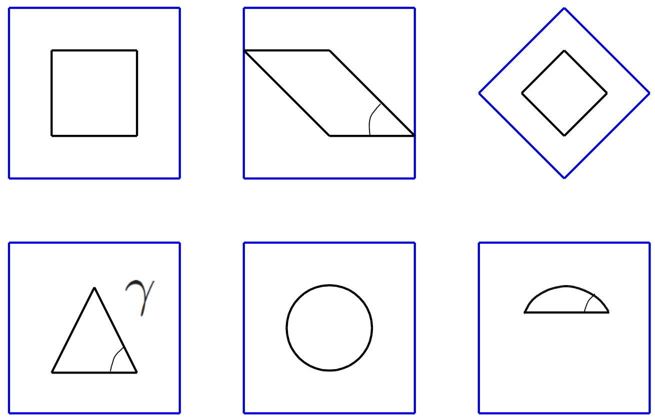


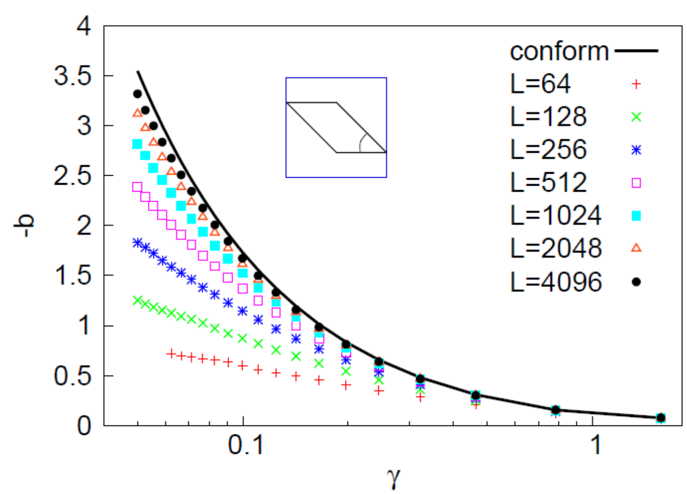
b=-0.077(1)

It is a consequence of diverging correlation length



Other shapes: angle-dependency





Potts model - random cluster representation

- Q-state Potts model
- $s_i = 1, 2, \ldots, Q$
- lattice with $i = 1, 2, \ldots, n$ sites and m edges
- $\langle ij \rangle$: nearest neighbours

$$Z(Q) = \sum_s \prod_{\langle ij \rangle} \exp\left(K \delta_{s_i, s_j}\right)$$

- $\exp\left(K \delta_{s_i, s_j}\right) = 1 + \frac{p}{1-p} \delta_{s_i, s_j}$
- $p = 1 - e^{-K}$

$$Z(Q) = \sum_s \prod_{\langle ij \rangle} \left(1 + \frac{p}{1-p} \delta_{s_i, s_j}\right)$$

- expand the products: the $\frac{p}{1-p} \delta_{s_i, s_j}$ terms define the "F" Fortuin-Kasteleyn clusters
- with $N_{tot}(F)$ number of connected components and $M(F)$ edges

$$\begin{aligned} Z(Q) &= \sum_F Q^{N_{tot}(F)} \left(\frac{p}{1-p}\right)^{M(F)} \qquad (1) \\ &\sim \sum_F Q^{N_{tot}(F)} p^{M(F)} (1-p)^{m-M(F)} \\ &= \left\langle Q^{N_{tot}} \right\rangle \end{aligned}$$

- Q real parameter, $Q \to 1$ percolation
- critical point: $p_c = \frac{1}{2}$
- mean number of clusters:

$$\langle N_{tot} \rangle = \left. \frac{\partial \ln Z(Q)}{\partial Q} \right|_{Q=1} .$$

Number of crossing clusters

- impose a boundary condition
- all the Potts spins are fixed (say, in state 1) on Γ

$$Z_\Gamma(Q) \sim \left\langle Q^{N_{tot}-N_\Gamma} \right\rangle ,$$

where N_Γ is the number of clusters which intersect Γ.

$$\langle N_{tot} - N_\Gamma \rangle = \left. \frac{\partial \ln Z_\Gamma(Q)}{\partial Q} \right|_{Q=1}$$

- at the critical point $p = p_c$

$$\begin{aligned} \ln Z(Q) &\sim A f_b(Q) && (2) \\ \ln Z_\Gamma(Q) &\sim A f_b(Q) + L_\Gamma f_s(Q) + C_\Gamma(Q) \ln L_\Gamma \end{aligned}$$

- $A \propto n$ total area
- L_Γ length of Γ
- f_b: bulk free-energy density
- f_s surface free-energy density
- last term: corner contribution

$$\langle N_\Gamma \rangle = -f_s'(1) L_\Gamma - C_\Gamma'(1) \ln L_\Gamma$$

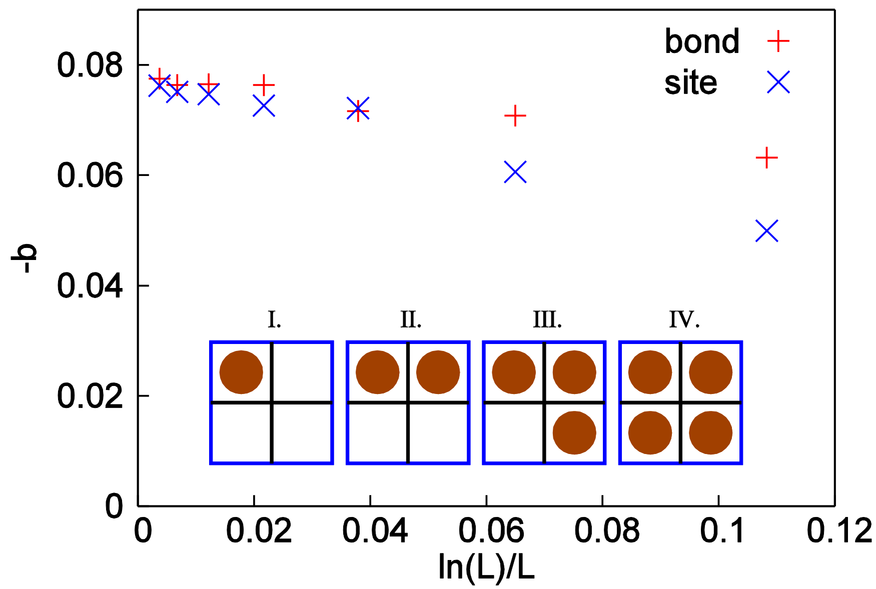
- Cardy-Peschel formula for the corner contribution of conformally invariant systems:

$$\begin{aligned} C_\Gamma(Q) &= \frac{c(Q)}{24} \sum_k \left[\left(\frac{\pi}{\gamma_k} \right) - \left(\frac{\gamma_k}{\pi} \right) \right. && (3) \\ &+ \left. \left(\frac{\pi}{2\pi - \gamma_k} \right) - \left(\frac{2\pi - \gamma_k}{\pi} \right) \right] \end{aligned}$$

- γ_k $(2\pi - \gamma_k)$ interior (exterior) angle of Γ
- $c(Q)$: central charge of the Q-state Potts model

$$c'(1) = \frac{5\sqrt{3}}{4\pi}$$

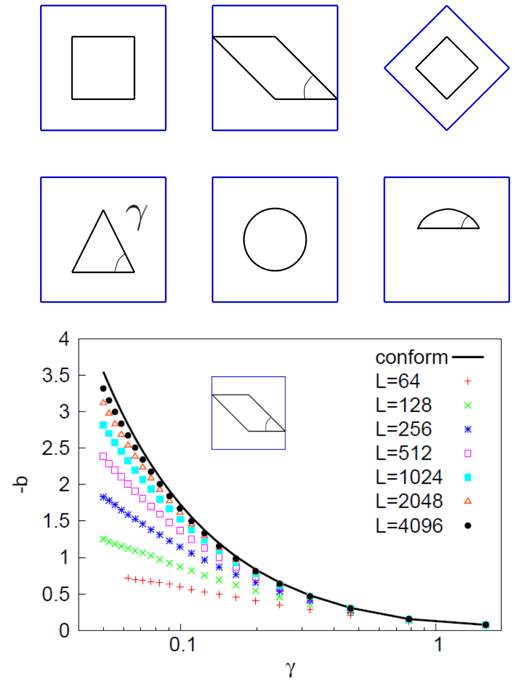
The logarithmic correction is universal



b=-0.077(1)

$$-C'(1) = -5\sqrt{3}/(36\pi) = -0.07657$$

Other shapes: angle-dependency



$$C'(1) = -\frac{c'(1)}{12}\left[4 - \pi\left(\frac{1}{\gamma} + \frac{1}{\pi-\gamma} + \frac{1}{\pi+\gamma} + \frac{1}{2\pi-\gamma}\right)\right]$$

Anisotropic percolation

- different probabilities:
  p_x horizontally, p_y vertically

- at the critical point: $p_x + p_y = 1$

- diagonally oriented square

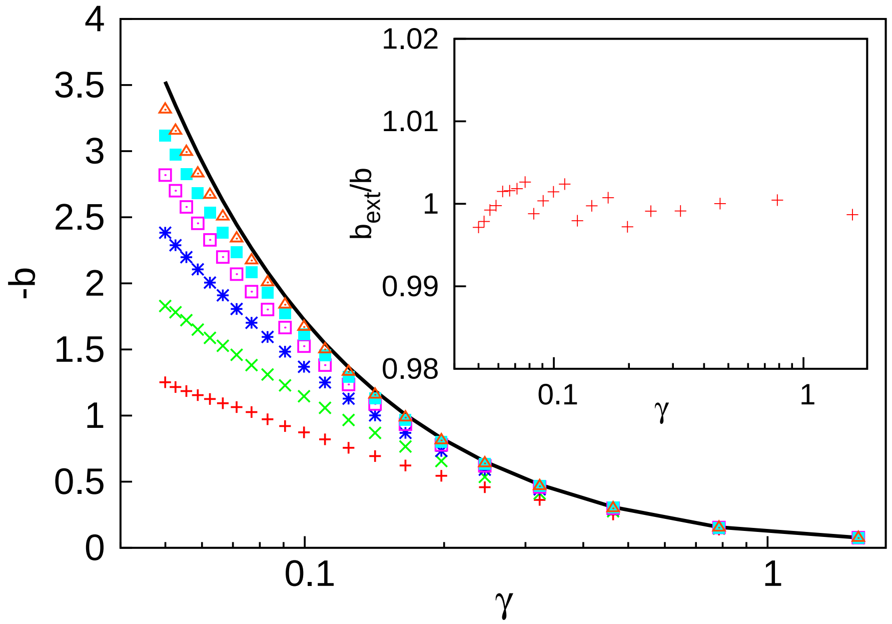
- with symmetric angles:
  $\gamma = \pi - \gamma = \pi/2)$

according to discrete holomorphicity in the scaling limit equivalent to an

- isotropic system: $p_x = p_y = 1/2$

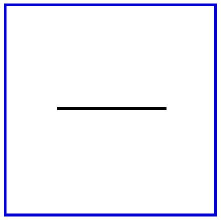
- with asymmetric angles

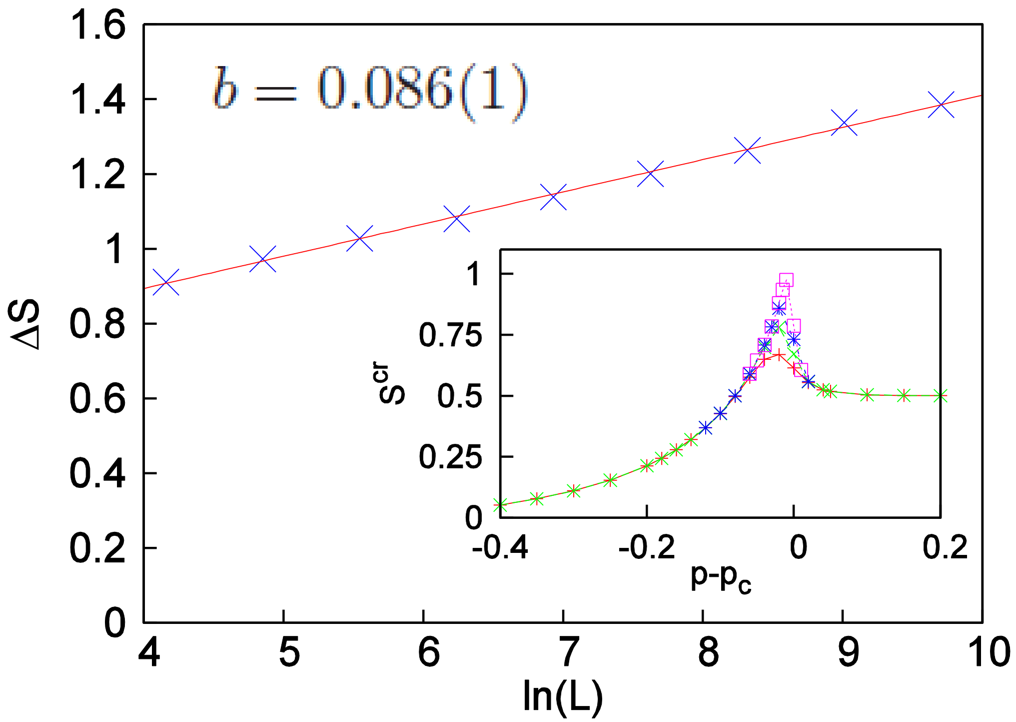
$$\frac{p_y}{p_x} = \frac{\sin((\pi-\gamma)/3)}{\sin(\gamma/3)}$$



In 2D we have a complete understanding…

…it can be any open curve as well!





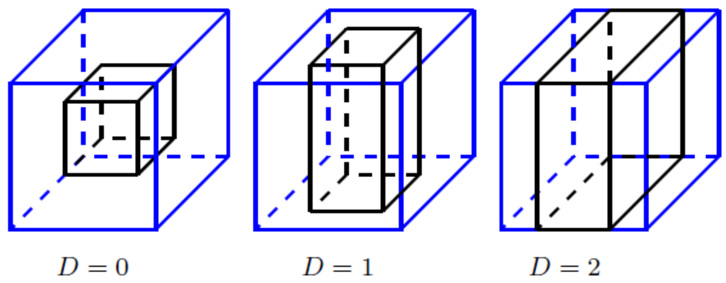
$$-C'_\Gamma(1) = \frac{5\sqrt{3}}{32\pi} \approx 0.08615$$

joint work with John Cardy (Oxford)

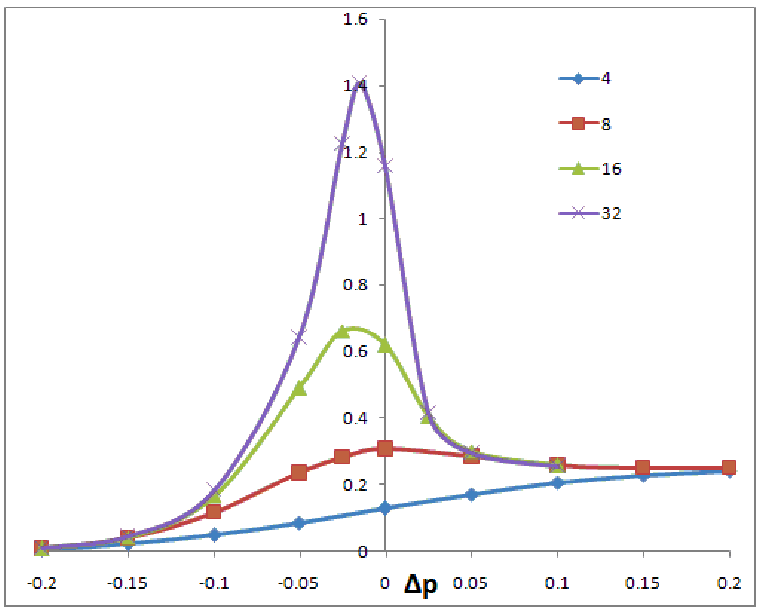
Conclusion - Outlook

- percolation cluster numbers: N_Γ
- entanglement entropy of diluted quantum Ising model: $\mathscr{S}_\Gamma$
- asymptotically: $N_\Gamma \approx \mathscr{S}_\Gamma$
- in $2d$ corner contributions at the critical point are
  - universal
  - exactly known from the Cardy-Peschel formula
- Related problems:
  - percolation cluster numbers in $3d$
  - entanglement entropy of the random quantum Ising model for $d \geq 2$

3D: There is a similar corner contribution

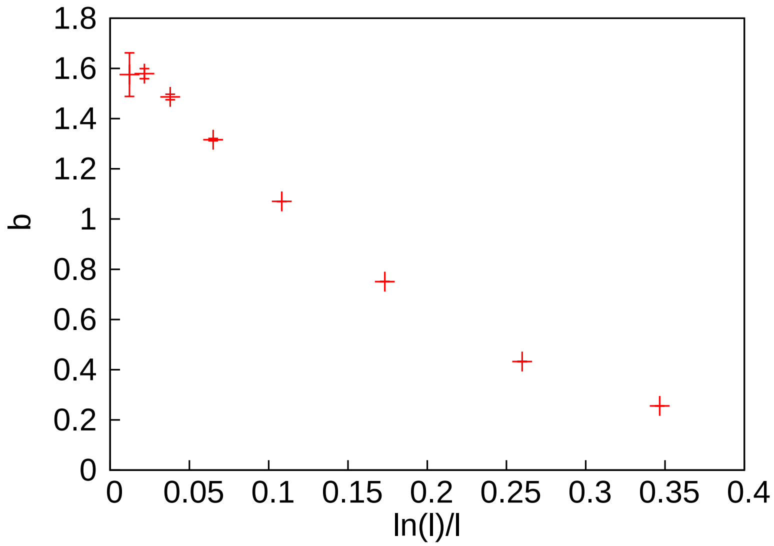


$$\mathcal{S}_0^{(d)}(0) = \sum_{D=0}^{d-1} \left(-\frac{1}{2}\right)^D \binom{d}{D} \mathcal{S}_D^{(d)}$$

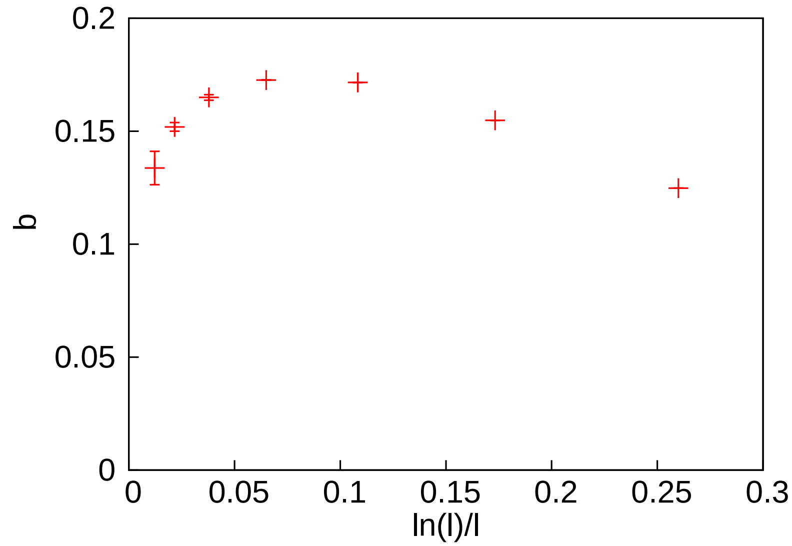


The sign is different!

Is it a log or $\log^2$?



prefactor of $\log l$



prefactor of $\log^2 l$