Frequency-locking and pattern formation in the Kuramoto model with Manhattan delay

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CompPhys11,
24-27.11.2011 Leipzig

Project carried out within the MPD programme "Physics of Complex Systems" of the Foundation for Polish Science and cofinanced by the European Regional Development Fund in the framework of the Innovative Economy Programme

INNOVATIVE ECONOMY
NATIONAL COHESION STRATEGY

FNP
Foundation for Polish Science

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DEVELOPMENT FUND

Outline

1. Examples of synchronization in complex systems.
2. The Kuramoto model.
3. Previous 2D lattice models.
4. Our model.
5. Results, summary.

1. Examples of synchronization in complex systems.

More examples:

clapping audiences

the heart

the brain

many more…

SYNCHRONIZATION: Self-organization phenomenon

- systems of many (possibly *different*) oscillators
- mutual interaction
- emergent, system-size collective behavior
- **frequency-locking** and **phase-locking**

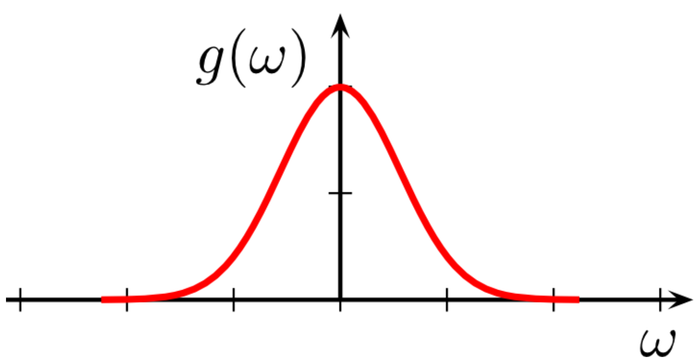
Google: „Steven Strogatz TED talk”

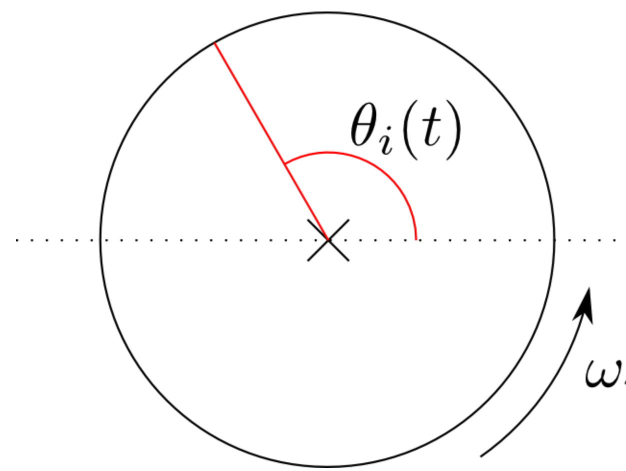
2. The Kuramoto model.

Yoshiki Kuramoto (1984):

- N self-driven oscillators with different intrinsic frequencies
- every oscillator is coupled to all others
- interaction which speeds up oscillators which are **behind** and slows down the ones which are **ahead**

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^{N} sin(\theta_j(t) - \theta_i(t))$$





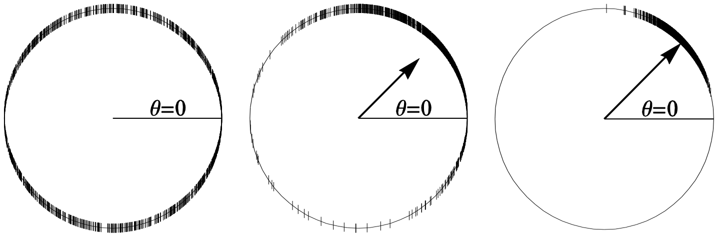
We can decouple:

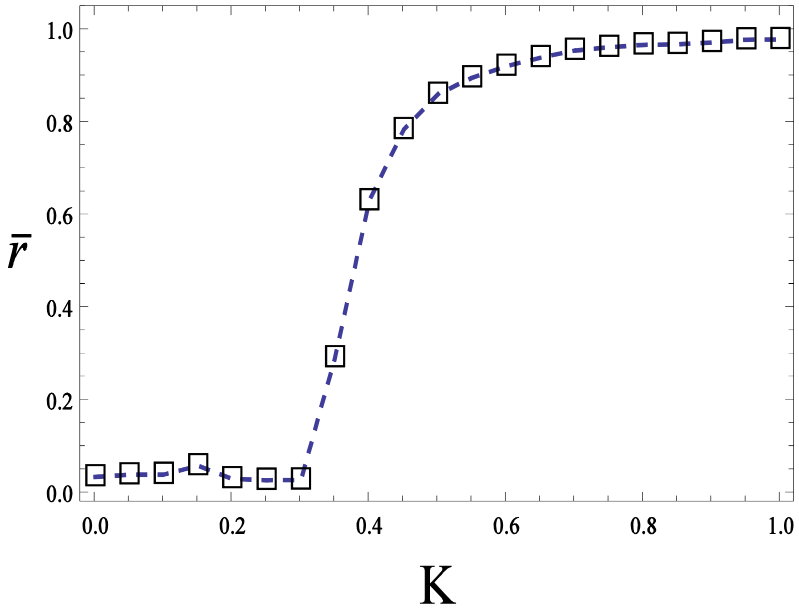
$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N} sin(\theta_j(t) - \theta_i(t))$$

Using:

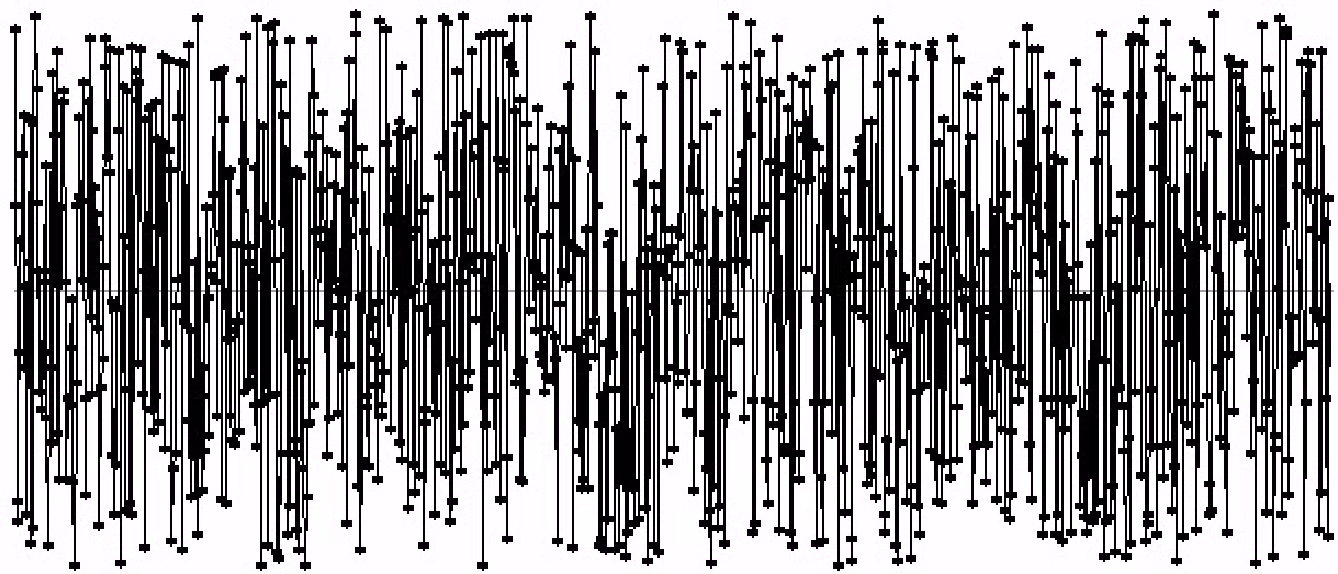
$$r(t)e^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}$$

r - macroscopic order parameter measuring **phase-locking**





t = 0.06



3. Previous 2D lattice models.

PHYSICAL REVIEW E **70**, 065201(R) (2004)

Pattern formation in a two-dimensional array of oscillators with phase-shifted coupling

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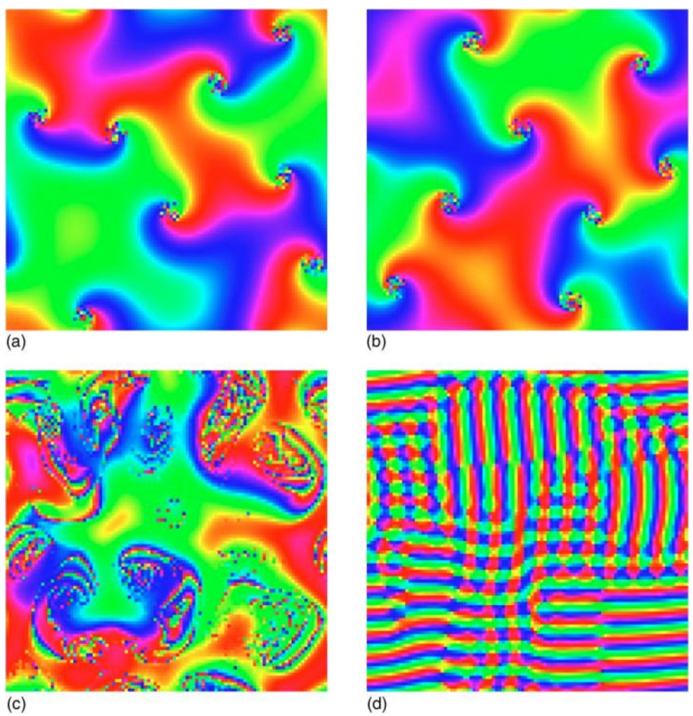
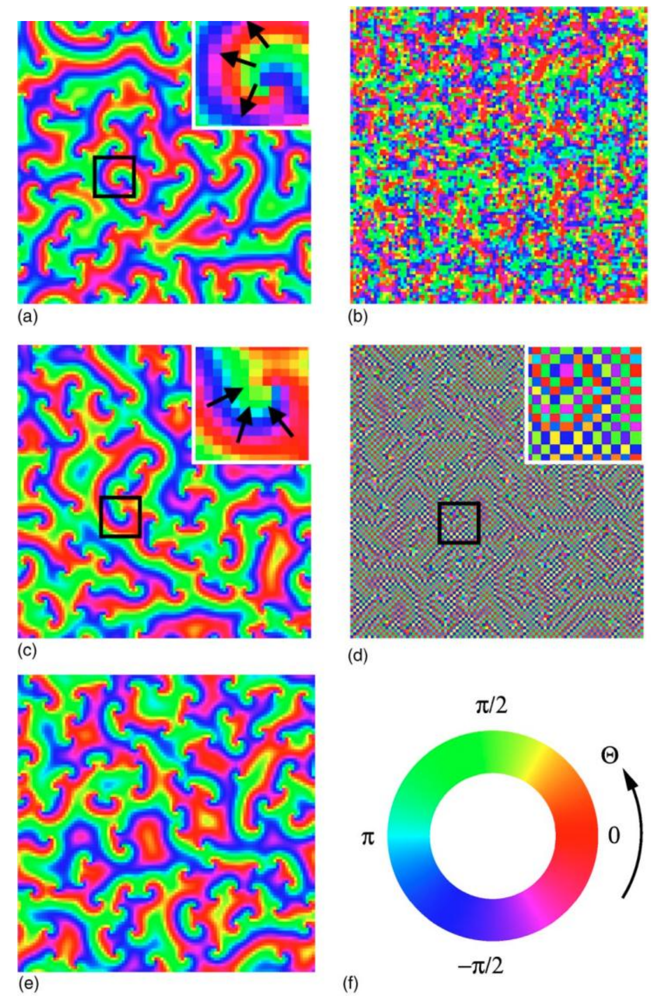
Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon, Korea
(Received 4 June 2004; published 27 December 2004)

$$\dot{\theta}_i(t) = \omega_0 + \frac{K}{z}\sum_{j(i)}^{z} sin(\theta_j(t) - \theta_i(t) - \alpha)$$





Time-Delayed Spatial Patterns in a Two-Dimensional Array of Coupled Oscillators

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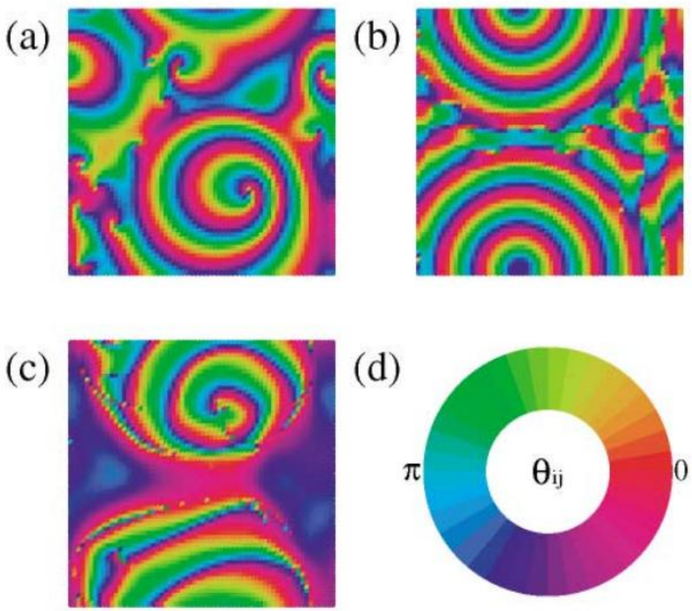
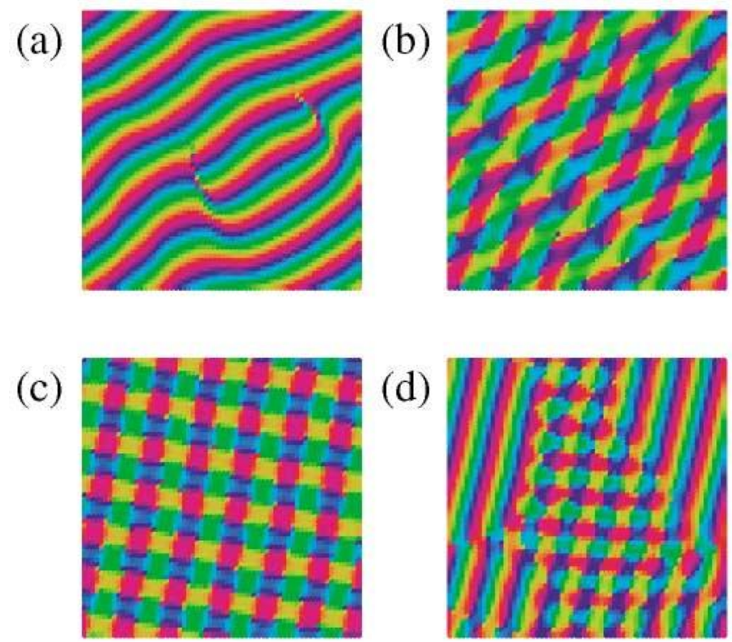
(Received 20 March 2002; published 20 September 2002)

$$\Omega = w + \frac{K}{N(r_0)} \sum_{k,l}^{0<r_{kl,ij}\leq r_0} \frac{1}{r_{kl,ij}} \sin(\phi_{kl} - \phi_{ij} - \Omega r_{kl,ij}/v).$$

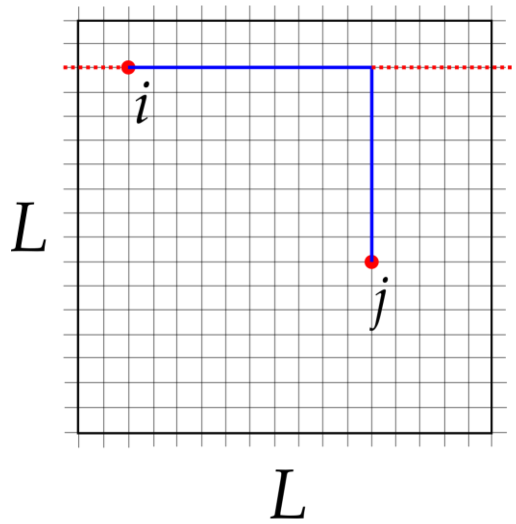
$$r_{kl,ij} = \sqrt{(i-k)^2 + (j-l)^2}.$$





3. Our model.

The connection graph does not change, however it gains a spatial, lattice structure.



$$\tau_{ij} = \tau \frac{d_{ij}}{\langle d \rangle}$$

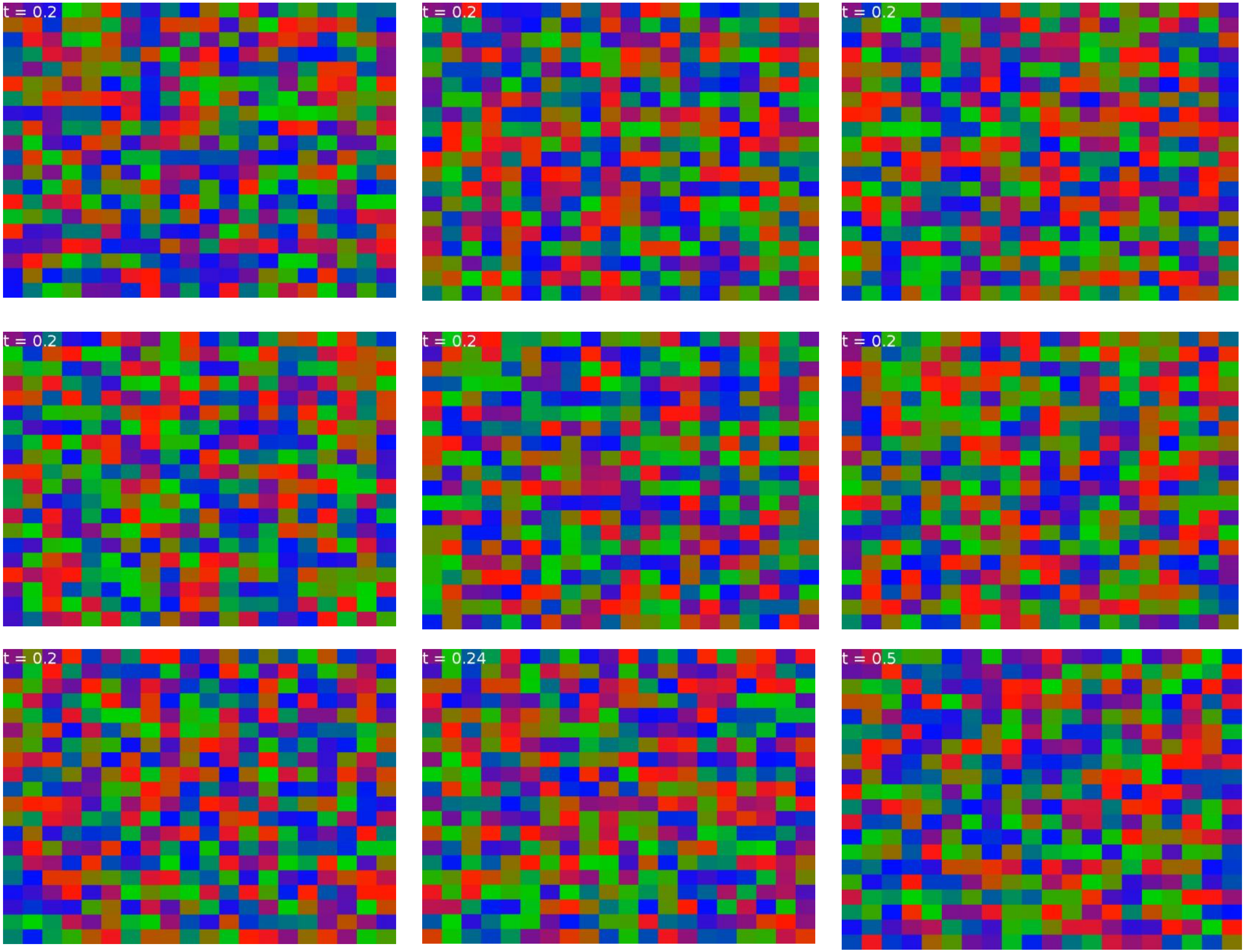
Delay between nodes, proportional to distance

„Manhattan" (AKA „taxi-driver's") metric
(on a periodic lattice)

$$\dot{\theta}_i(t) = \omega_0 + \frac{K}{N} \sum_{j=1}^{N} sin(\theta_j(t - \tau_{ij}) - \theta_i(t))$$

The distance affects only the delay, the coupling remains uniform!

4. Results



Non-trivial frequency-locking

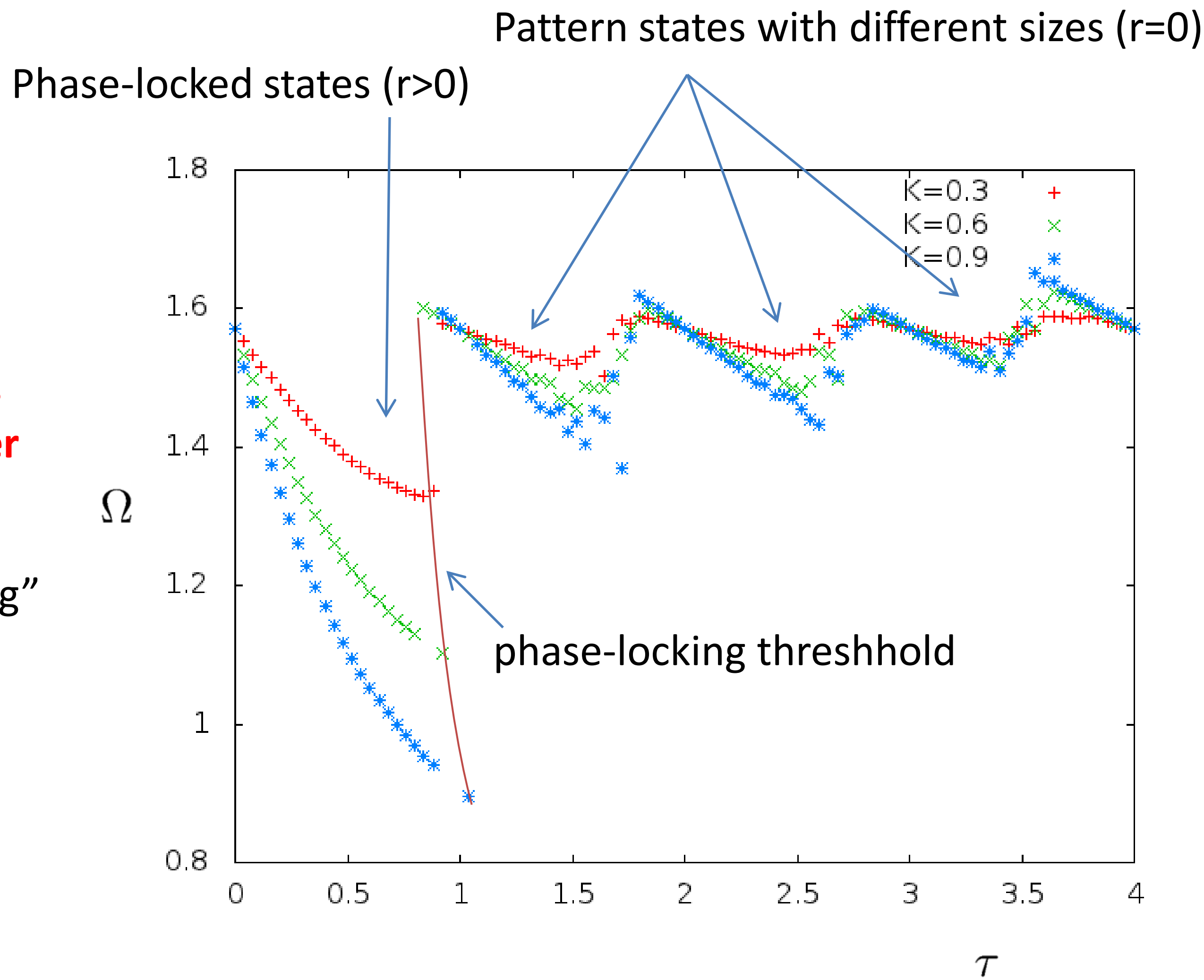
Numerical integration with random initial conditions

$$\theta_i(0) \in (0, 2\pi]$$

$$\omega_0 = \frac{\pi}{2}$$

The patterns change wavelength/size seemingly when the mean frequency jumps from one jag to another

(no „frequency-locking" since oscillators are identical)

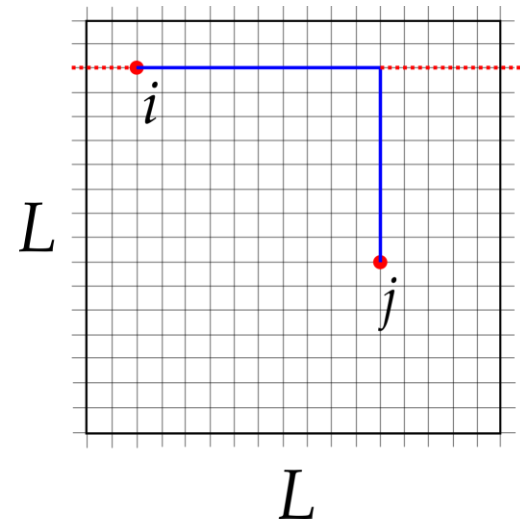


Solution for phase-locked states

$$\dot{\theta}_i(t) = \omega_0 + \frac{K}{N}\sum_{j=1}^{N} sin(\theta_j(t-\tau_{ij}) - \theta_i(t))$$

$$\theta_j(t) = \phi_j + \Omega t$$

$$\boxed{\dot{\theta}_i(t) = \omega_0 + \frac{K}{N}\sum_{j=1}^{N} sin(\phi_j - \phi_i - \Omega\tau_{ij})}$$

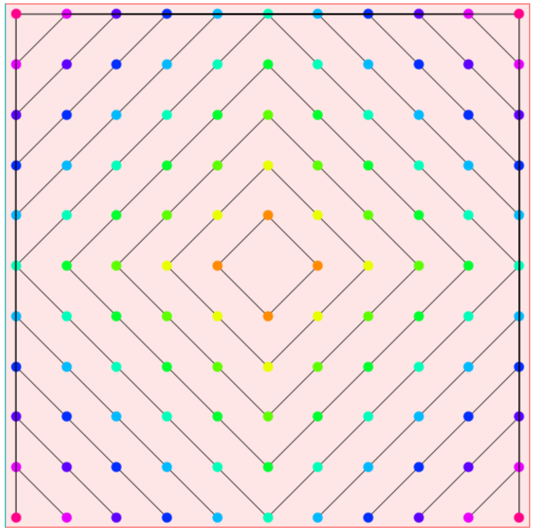


Rewrite the sum as a sum over **distance**

Apply $\frac{1}{N}\sum_{i=1}^{N}$ to both sides

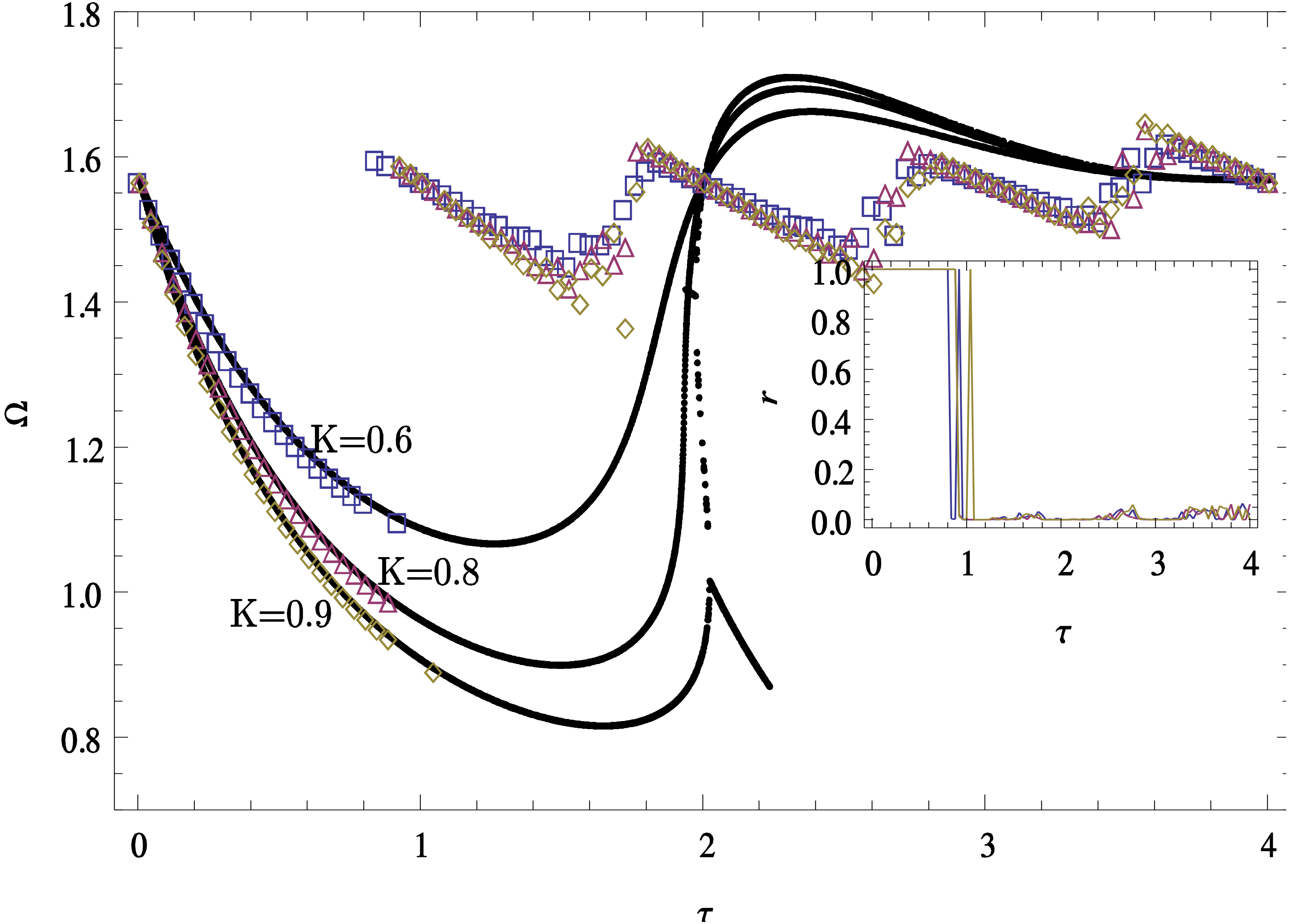
$$n(d) = \begin{cases} 4d & \text{if } d \le \frac{L}{2}; \\ -4(d-1)+4L & \text{if } \frac{L}{2} < d \le L. \end{cases}$$

$$\Omega = \omega_0 - K\sum_{d=1}^{L} n(d) sin(\Omega\tau \frac{d}{L/2})$$



This sum has a limit as N→∞

$$\Omega = \omega_0 - 4K\frac{cos(\Omega\tau)sin^2(\Omega\tau/2)}{\Omega\tau}$$



1.8
1.6
1.4
1.2
1.0
0.8
Ω
0
1
2
3
4
τ
K=0.6
K=0.8
K=0.9
1.0
0.8
0.6
0.4
0.2
0.0
r
0
1
2
3
4
τ

Summary

- We have proposed a model which produces phase patterns without explicitly including local interactions or coupling decreasing with distance and is (at least partially) mathematically tractable

Development outlook

• stability analysis (difference-differential equations, v.v. tedious)
• provide an ansatz for other solutions, test against exp. data
e.g. plane-wave ansatz: $$\phi_j - \phi_i = \frac{\vec{\beta}\cdot\vec{r_{ij}}}{L/2}$$

Further reading

• Kuramoto, Y., *Cooperative Dynamics of Oscillator Community: A Study Based on Lattice of Rings*, Prog. Theor. Phys. Sup. **79**, pp. 223-240 (1984)

• Acebron et al., *The Kuramoto model: A simple paradigm for synchronization Phenomena*, Rev. Mod. Phys. **77**, pp. 137-185 (2005)