Mean-field behavior of the negative-weight percolation model on random regular graphs

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Outline

- Introduction
- Percolation problem
  - $2d$-setup (more intuitive)
  - random graphs
- Results
- Summary

Model

- $L \times L$ lattice, fully periodic boundary conditions
- Undirected edges, weight (cost) distribution:

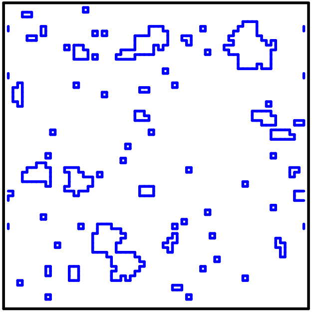
$$P(\omega) = \rho\,\delta(\omega+1) \;+\; (1-\rho)\,\delta(\omega-1)$$

- Allows for loops $\mathcal{L}$ with negative weight $\omega_{\mathcal{L}}$

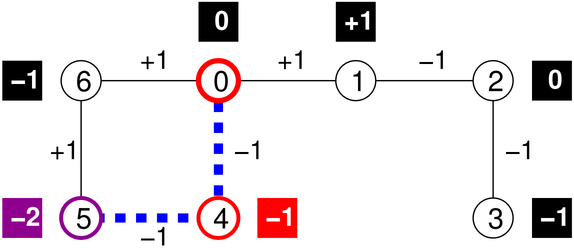
- Agent on lattice edges: pay/receive resources
- Configuration $\mathcal{C}$ of non-intersecting loops, with

$$E \equiv \sum_{\mathcal{L}\in\mathcal{C}} \omega_{\mathcal{L}} \stackrel{!}{=} \min$$

- Obtain $\mathcal{C}$ through mapping to minimum weight perfect matching problem



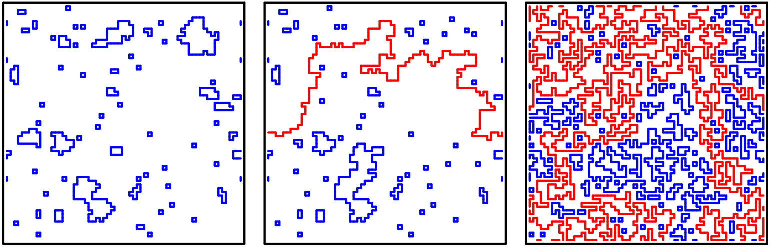
Minimal distances



- $d_{si} = \min_{j \in N(i)} \left[d_{sj} + \omega_{ij} \right]$ not fulfilled (*source*: $s = 0$)
- Standard minimum-weight path algorithms, e.g. Dijkstra, Bellman-Ford, Floyd-Warshall, don't work
- Minimum-weight path problem requires matching techniques

  [R.K. Ahuja, T.L. Magnanti and J.B. Orlin, *Network flows*]

Loop percolation



($L = 64$ at $\rho < \rho_c, \ \rho \approx \rho_c, \ \rho > \rho_c$)

- Observe system spanning loops above critical ρ
- Disorder induced, geometric transition
- Characterize loops using observables from percolation theory (finite-size scaling (FSS) analysis)

[D. Stauffer, A. Aharony, *Introduction to Percolation Theory*]

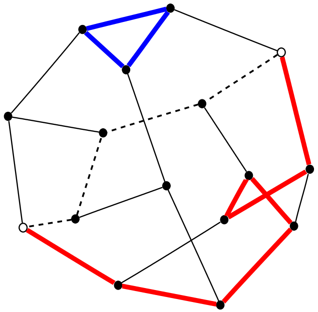
Previous studies

- critical points and exponents in $d = 2 \ldots 7$:

d	ρ_c	ν	β	γ	d_f	τ
2	0.340(1)	1.49(7)	1.07(6)	0.77(7)	1.266(2)	2.59(3)
3	0.1273(3)	1.00(2)	1.54(5)	-0.09(3)	1.459(3)	3.07(1)
4	0.0640(2)	0.80(3)	1.91(11)	-0.66(5)	1.60(1)	3.55(2)
5	0.0385(2)	0.66(2)	2.10(12)	-1.06(7)	1.75(3)	3.86(3)
6	0.0265(2)	0.50(1)	1.92(6)	-0.99(3)	2.02(1)	4.00(2)
7	0.01977(6)	3/7 (?)			2.01(2)	4.50(1)

- results [OM, L. Apolo, and AKH, PRE 2010]:
  - *upper critical dimension* $d_u = 6$
  - for $d \geq d_u$ one expects mean field (MF) exponents ($\nu^{MF} = 0.5$, $d_f^{MF} = 2$)
- random graphs (RGs): direct access to MF exponents
- here: support $d_u = 6$ by computing MF exponents on RGs

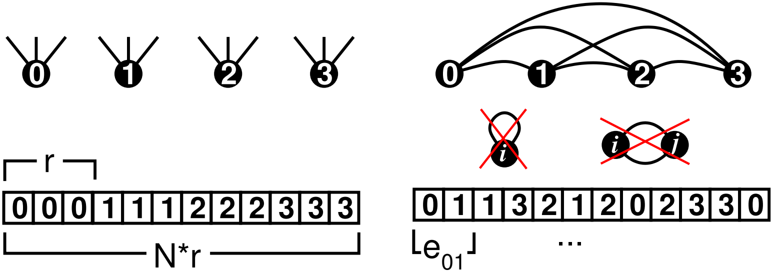
r-regular random graphs (r-RRGs)



- graph $G_{N,r} = (V, E)_{N,r}$
  - V = set of N nodes $i \in V$
  - E = set of $rN/2$ edges
    $e_{ij} = i, j \in E$
  - fixed degree $\deg(i) = r$
- distance d_{ij} = # of edges in min. length path betw. $i, j \in V$
  [obtain all d_{ij} via BFS/DFS in time $O(N^2)$]
- diameter $R = \max_{i,j \in V} d_{ij}$
  [sparse random graphs: $R \propto c \log(N) + O(\log(N))$]
- example: 3-RRG with $N = 16$ and $R = 4$

r-regular random graphs (r-RRGs)

- construct r-RRGs using *stub-model* (here: $r = 3$)



- left: nodes with "stubs" (top), list of edge-ends (bottom)
- right: permutation of edge-ends and corresponding 3-RRG
  [uniform random permutation via "randomize-in-place" in $O(N)$ time]
- rejection algorithm: discard instances with self/multi-edges

Results – path weight

- $P_N^\omega(\rho)$ = probability that path weight is negative
- finite-size fluctuations:

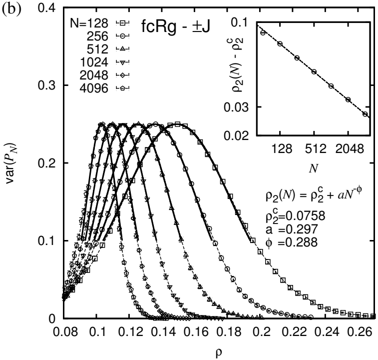
$$\mathrm{var}(P_N^\omega) = \langle (P_N^\omega)^2 \rangle - \langle P_N^\omega \rangle^2$$

- peak locations:

$$\rho_2(N) = \rho_2^c + aN^{-\phi}$$

- scaling parameters:

$$\rho_2^c = 0.0758(9)$$
$$\phi = 0.288(5)$$

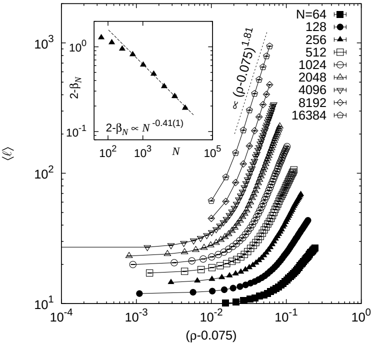


- effective scaling exponent $\nu^*_{\mathrm{eff}} = 1/\phi = 3.47(6)$
- including corrections to scaling: $\nu^* \approx 3$ $(= d_u \nu)$

Results – average path length

- order-parameter: relative path length $\langle \ell \rangle / N$
- order-parameter exp. β:

$$\langle \ell \rangle / N \sim (\rho - \rho_c)^{\beta}$$



- effective scaling exponents:

$$\beta(N) = \beta + aN^{-b}$$

- scaling parameters:

$$\beta = 2.0(1)$$
$$b = 0.41(1)$$

- result compares well to $\beta = 1.92(6)$ on $d = 6$ hypercubic lattices

Results – scaling at the critical point

- path-length:

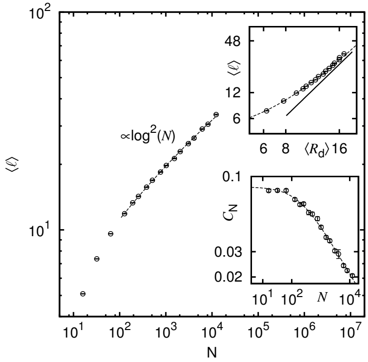
$$\langle \ell \rangle \sim \log^{d_f}(N) + c_1$$

yields $d_f = 2.1(1)$ and $c_1 = 5.2(5)$

- order parameter suszept.

$$C_N = N^{-1}\mathrm{var}(\ell)$$

$$C_N \propto N^{\gamma/\nu^*}(1 + c_1/N^{c_2})$$



- agreement among results for $d = 6$ and 3-RRGs:

$$\text{RRGs}: \quad \nu^* = 3.00(6) \quad \beta = 2.0(1) \quad d_f = 2.1(1) \quad \gamma = -1.02(2)$$
$$6d: \quad d\nu = 3.00(1) \quad \beta = 1.92(6) \quad d_f = 2.00(1) \quad \gamma = -0.99(3)$$

[OM, AKH, and MM, PRE 2011]

Summary

- Negative-weight percolation model on RRGs
- Distinct from random bond/site percolation
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- Thank you!