

Logarithmic extensions of local scale-invariance

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[arxiv:1009.4139](#), [arxiv:1109.5022](#)

CompPhys 12,
Université de Leipzig (Allemagne), novembre 2011

1. Ageing phenomena

known & practically used since prehistoric times (metals, glasses)
systematically studied in physics since the 1970s

STRIJK '78

occur in widely different systems

(structural glasses, spin glasses, polymers, simple magnets, ...)

Three **defining properties** of **ageing** :

- ① slow relaxation (non-exponential!)
- ② **no** time-translation-invariance (TTI)
- ③ dynamical scaling without fine-tuning of parameters

Most existing studies on 'magnets' : relaxation towards **equilibrium**

Question : what can be learned about intrinsically **irreversible** systems by studying their **ageing behaviour** ?

Non-equilibrium ageing without detailed balance : summary

(A) surface growth : Kardar-Parisi-Zhang (**KPZ**)

(B) critical directed percolation (**DP**)

simple ageing of the correlators and responses :

two-time auto-**correlator** $C(t, s) := \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}) \rangle - \langle \phi(t, \mathbf{r}) \rangle \langle \phi(s, \mathbf{r}) \rangle$

two-time auto-**response** $R(t, s) := \left. \frac{\delta \langle \phi(t, \mathbf{r}) \rangle}{\delta h(s, \mathbf{r})} \right|_{h=0} = \langle \phi(t, \mathbf{r}) \tilde{\phi}(s, \mathbf{r}) \rangle$

Scaling regime : $t, s \gg \tau_{\text{micro}}$ and $t - s \gg \tau_{\text{micro}}$

$$C(t, s) = s^{-b} f_C\left(\frac{t}{s}\right), \quad R(t, s) = s^{-1-a} f_R\left(\frac{t}{s}\right)$$

$$f_C(y) \sim y^{-\lambda_C/z}, \quad f_R(y) \sim y^{-\lambda_R/z} \quad y \gg 1$$

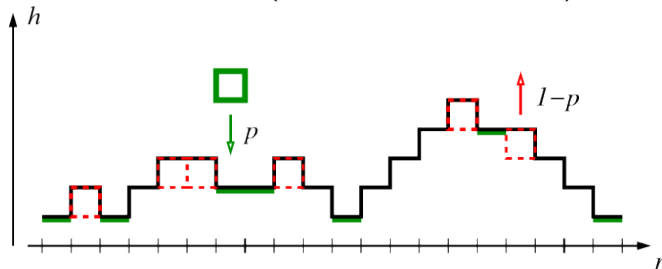
values of the non-equilibrium exponents & scaling relations

KPZ in 1D : $\lambda_C = \lambda_R = 1$, $1 + a = b + \frac{2}{z}$, $b = -2\beta = -\frac{2}{3}$, $z = \frac{3}{2}$

DP : $\lambda_C = \lambda_R = d + z + \frac{\beta}{\nu_{\perp}}$, $1 + a = b = \frac{2\beta}{\nu_{\parallel}}$

2. Surface growth

deposition (evaporation) of particles on a surface \rightarrow height profile $h(t, \mathbf{r})$
generic situation : RSOS (restricted solid-on-solid) model KIM & KOSTERLITZ 89



p = deposition prob.
 $1 - p$ = evap. prob.

here $p = 0.98$

éq. **KPZ** $\partial_t h = \nu \nabla^2 h + \frac{\mu}{2} (\nabla h)^2 + \eta$

KARDAR, PARISI, ZHANG 86

η is a gaussian white noise with $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2\nu T \delta(t - t') \delta(\mathbf{r} - \mathbf{r}')$

Family-Viscek scaling on a spatial lattice of extent L^d : $\bar{h}(t) = L^{-d} \sum_j h_j(t)$

$$w^2(t; L) = \frac{1}{L^d} \sum_{j=1}^{L^d} \left\langle (h_j(t) - \bar{h}(t))^2 \right\rangle = L^{2\zeta} f(tL^{-z}) \sim \begin{cases} L^{2\zeta} & ; \text{if } tL^{-z} \gg 1 \\ t^{2\beta} & ; \text{if } tL^{-z} \ll 1 \end{cases}$$

β : growth exponent, ζ : roughness exponent, $\zeta = \beta z$

two-time correlator :

$$C(t, s; \mathbf{r}) = \langle h(t, \mathbf{r}) h(s, \mathbf{0}) \rangle - \langle \bar{h}(t) \rangle \langle \bar{h}(s) \rangle = s^{-b} F_C \left(\frac{t}{s}, \frac{\mathbf{r}}{s^{1/z}} \right)$$

with ageing exponent : $b = -2\beta$

KALLABIS & KRUG 96

two-time integrated response :

* sample **A** with deposition rates $p_i = p \pm \epsilon_i$, up to time s ,

* sample **B** with $p_i = p$ up to time s ;

then switch to common dynamics $p_i = p$ for all times $t > s$

$$\chi(t, s; \mathbf{r}) = \int_0^s du R(t, u; \mathbf{r}) = \frac{1}{L} \sum_{j=1}^L \left\langle \frac{h_{j+r}^{(A)}(t; s) - h_{j+r}^{(B)}(t)}{\epsilon_j} \right\rangle = s^{-a} F_\chi \left(\frac{t}{s}, \frac{|\mathbf{r}|^z}{s} \right)$$

Effective action of the KPZ equation :

$$\mathcal{J}[\phi, \tilde{\phi}] = \int dt dr \left[\tilde{\phi} \left(\partial_t \phi - \nu \nabla^2 \phi - \frac{\mu}{2} (\nabla \phi)^2 \right) - \nu T \tilde{\phi}^2 \right]$$

⇒ **Very special properties of KPZ in $d = 1$ spatial dimension !**

Exact critical exponents $\beta = 1/3, \zeta = 1/2, z = 3/2, \lambda_C = 1$ KPZ 86; KRECH 97

Special KPZ symmetry in $1D$: let $v = \frac{\partial \phi}{\partial r}, \tilde{\phi} = \frac{\partial}{\partial r} \left(\tilde{p} + \frac{v}{2T} \right)$

$$\mathcal{J} = \int dt dr \left[\tilde{p} \partial_t v - \frac{\nu}{4T} (\partial_r v)^2 - \frac{\mu}{2} v^2 \partial_r \tilde{p} + \nu T (\partial_r \tilde{p})^2 \right]$$

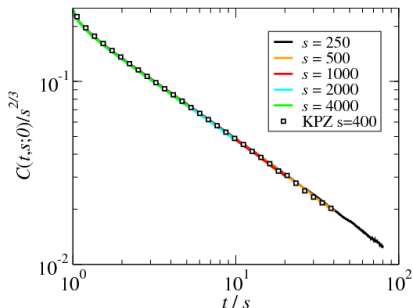
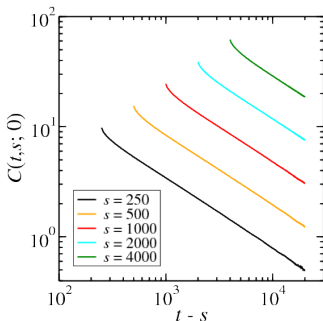
is invariant under **time-reversal** LVOV, LEBEDEV, PATON, PROCACCIA 93; FREY, TÄUBER, HWA 96

$$t \mapsto -t, \quad v(t, r) \mapsto -v(-t, r), \quad \tilde{p} \mapsto +\tilde{p}(-t, r)$$

⇒ **fluctuation-dissipation relation** for $t \gg s$ $TR(t, s; r) = -\partial_r^2 C(t, s; r)$

find ageing exponents : $\lambda_R = \lambda_C = 1, 1 + a = b + \frac{2}{z}$

1D relaxation dynamics, starting from an initially flat interface

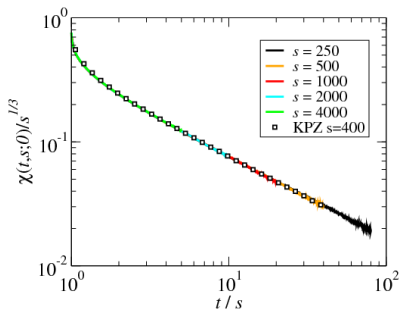
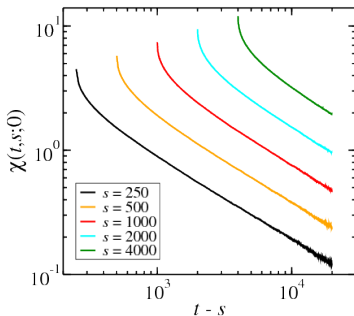


observe all **3** properties of **ageing** : $\left\{ \begin{array}{l} \text{slow dynamics} \\ \text{no TTI} \\ \text{dynamical scaling} \end{array} \right.$

confirm expected exponents $b = -2/3$, $\lambda_c/z = 2/3$

N.B. : this confirmation is out of the stationary state

relaxation of the integrated response, $1D$

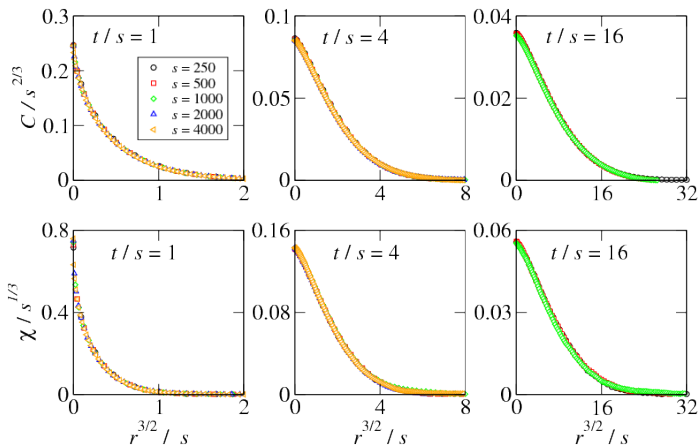


observe all **3** properties of **ageing** : $\left\{ \begin{array}{l} \text{slow dynamics} \\ \text{no TTI} \\ \text{dynamical scaling} \end{array} \right.$

exponents $a = -1/3$, $\lambda_R/z = 2/3$, as expected from FDR

N.B. : numerical tests for 2 models in KPZ class

Simple ageing is also seen in **space-time** observables



correlator $C(t, s; r) = s^{2/3} F_C \left(\frac{t}{s}, \frac{r^{3/2}}{s} \right)$

integrated response $\chi(t, s; r) = s^{1/3} F_\chi \left(\frac{t}{s}, \frac{r^{3/2}}{s} \right)$

non-equilibrium confirmation of expected dynamical exponent $z = 3/2$

3. Form of the scaling functions & LSI

Observation : dynamical scaling generic property of non-equilibrium criticality

Question : can one extend non-equilibrium dynamical scaling ?

analogy : conformal invariance at equilibrium phase transitions, but $z = 1$ there \Rightarrow **other important differences** ?

Schrödinger group, $z = 2$

JACOBI 1842, LIE 1881

$$t \mapsto t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \mathbf{r} \mapsto \mathbf{r}' = \frac{\mathcal{D}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta}, \quad \alpha\delta - \beta\gamma = 1$$

dynamical symmetry of free Schrödinger/diffusion equation

$$\mathcal{S}\phi(t, \mathbf{r}) = 0, \quad \mathcal{S} = 2\mathcal{M}\frac{\partial}{\partial t} - \frac{\partial}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}}$$

infinite-dimensional extension $\mathfrak{sv}(d)$ **Schrödinger-Virasoro algebra** MH 94

! **but** Schrödinger-invariance **cannot** be applied to ageing, since it contains time-translations !

essential : absence of TTI in ageing phenomena !

Transformation $t \mapsto t'$ with

$$t = \beta(t') \quad , \quad \phi(t) = \left(\frac{d\beta(t')}{dt'} \right)^{-x/z} \left(\frac{d \ln \beta(t')}{dt'} \right)^{-2\xi/z} \phi'(t')$$

with $\beta(0) = 0$ and $\dot{\beta}(t') \geq 0$.

out of equilibrium, have **2 distinct** scaling dimensions, x and ξ .

mean-field for magnets : expect $\begin{cases} \xi = 0 & \text{in ordered phase } T < T_c \\ \xi \neq 0 & \text{at criticality } T = T_c \end{cases}$

NB : if TTI (equilibrium criticality), then $\xi = 0$.

physical requirement :

co-variance of **response functions** under local scaling !

⇒ set of linear differential equations for $R(t, s)$

most simple case !

$$R(t, s) = \langle \phi(t) \tilde{\phi}(s) \rangle = s^{-1-a} f_R \left(\frac{t}{s} \right)$$
$$f_R(y) = f_0 y^{1+a'-\lambda_R/z} (y-1)^{-1-a'} \underbrace{\Theta(y-1)}_{\text{causality}}$$

$$a = \frac{1}{z} (x + \tilde{x}) - 1, \quad a' - a = \frac{2}{z} (\xi + \tilde{\xi}), \quad \frac{\lambda_R}{z} = x + \xi$$

magnetic example : 1D Glauber-Ising model at $T = T_c = 0$:

$$a = 0, \quad a' - a = -\frac{1}{2}, \quad \lambda_R = 1, \quad z = 2$$

NB : $\Phi(t, \mathbf{r}) := t^{-\xi} \phi(t, \mathbf{r})$ has standard local scaling,
with $x_\Phi = x_\phi + 2\xi_\phi$, $\xi_\Phi = 0$

4. Logarithmic conformal invariance

generalise conformal invariance \rightarrow doublets $\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$ ROZANSKY & SALEUR 92
GURARIE 93

formally replace exponent by Jordan matrix : $\Delta \mapsto \begin{pmatrix} \Delta & 1 \\ 0 & \Delta \end{pmatrix}$

two-point functions : have $\Delta_1 = \Delta_2$ GURARIE 93, RAHIMI TABAR *et al.* 97...

$$F = \langle \phi_1(w_1)\phi_2(w_2) \rangle = 0$$

$$G = \langle \phi_1(w_1)\psi_2(w_2) \rangle = G_0 |w|^{-2\Delta_1}$$

$$H = \langle \psi_1(w_1)\psi_2(w_2) \rangle = (H_0 - 2G_0 \ln |w|) |w|^{-2\Delta_1}$$
$$= w_2^{-2\Delta_1} (H_0 - 2G_0 \ln |y - 1| - 2G_0 \ln |w_2|) |y - 1|^{-2\Delta_1}$$

with $w = w_1 - w_2$ and $y = w_1/w_2$.

Simultaneous log corrections to scaling and modified scaling function

Logarithmic conformal invariance has been found in, e.g.

- critical $2D$ percolation
- disordered systems
- sand-pile models

CARDY 92, WATTS 96, MATHIEU & RIDOUT 07/08

CAUX *et al.* 96

RUELLE *et al.* 08-10

5. Logarithmic ageing-invariance

Schrödinger-invariance cannot be a dynamical symmetry for ageing, since it contains time-translations X_{-1} !

Go to **ageing algebra** $\text{age}(d) := \langle X_{1,0}, Y_{\pm 1/2}^{(j)}, M_0, R_0^{(jk)} \rangle_{j,k=1,\dots,d}$

Need generalised form of generator

$$X_n = -t^{n+1} \partial_t - \frac{n+1}{2} t^n \mathbf{r} \cdot \nabla_{\mathbf{r}} - \frac{\mathcal{M}}{2} (n+1) n t^{n-1} \mathbf{r}^2 - \frac{n+1}{2} x t^n - n \xi t^n$$

with two independent exponents x and ξ

construct **logarithmic ageing-invariance** by the formal changes :

$$x \mapsto \begin{pmatrix} x & 2x' \\ 0 & x \end{pmatrix}, \quad \xi \mapsto \begin{pmatrix} \xi & \xi' \\ \xi'' & \xi \end{pmatrix}$$

show : we can always arrange for $\xi'' = 0$.

Co-variant two-point functions :

$$\begin{aligned}F &= F(t_1, t_2) := \langle \phi_1(t_1)\phi_2(t_2) \rangle && \text{non-logarithmic} \\G_{12} &= G_{12}(t_1, t_2) := \langle \phi_1(t_1)\psi_2(t_2) \rangle \\G_{21} &= G_{21}(t_1, t_2) := \langle \psi_1(t_1)\phi_2(t_2) \rangle \\H &= H(t_1, t_2) := \langle \psi_1(t_1)\psi_2(t_2) \rangle\end{aligned}$$

co-variance requirement \Rightarrow coupled set of **linear** differ. equations

* scaling variable $y = t_1/t_2 > 1$

* expand solutions into series in $\ln t_2 \Rightarrow$ naturally truncates!

relation with non-equilibrium dynamics :

réponse $R(t, s) = \langle \psi(t)\tilde{\psi}(s) \rangle$ can be written as correlator between the 'order parameter' ψ and its 'reponse field' $\tilde{\psi}$, within the Janssen-de Dominicis formalism of non-equilibrium field-theory.

$$\begin{aligned}
 F(t_1, t_2) &= t_2^{-(x_1+x_2)/2} \mathcal{F}(y) f_0 \\
 G_{12}(t_1, t_2) &= t_2^{-(x_1+x_2)/2} \mathcal{F}(y) \left(g_{12}(y) - \ln t_2 \cdot x_2' f_0 \right) \\
 G_{21}(t_1, t_2) &= t_2^{-(x_1+x_2)/2} \mathcal{F}(y) \left(g_{21}(y) - \ln t_2 \cdot x_1' f_0 \right) \\
 H(t_1, t_2) &= t_2^{-(x_1+x_2)/2} \mathcal{F}(y) \left(h_0(y) - \ln t_2 \cdot (x_1' g_{12}(y) + x_2' g_{21}(y)) \right. \\
 &\quad \left. + \ln^2 t_2 \cdot x_1' x_2' f_0 \right)
 \end{aligned}$$

$$\mathcal{F}(y) := y^{\xi_2 + (x_2 - x_1)/2} (y - 1)^{-(x_1 + x_2)/2 - \xi_1 - \xi_2} ; \quad y = t_1/t_2 > 1$$

with the explicit functions :

$f_0, g_{12,0}, g_{21,0}, h_0$ are normalisations

$$\begin{aligned}
 g_{12}(y) &= g_{12,0} + (x_2' + \xi_2') f_0 \ln \left| \frac{y}{y-1} \right| \\
 g_{21}(y) &= g_{21,0} - (x_1' + \xi_1') f_0 \ln |y-1| - x_1' f_0 \ln |y| \\
 h_0(y) &= h_0 - \left[(x_1' + \xi_1') g_{21,0} + (x_2' + \xi_2') g_{12,0} \right] \ln |y-1| - \left[x_1' g_{21,0} - (x_2' + \xi_2') g_{12,0} \right] \ln |y| \\
 &\quad + \frac{1}{2} f_0 \left[\left((x_1' + \xi_1') \ln |y-1| + x_1' \ln |y| \right)^2 - (x_2' + \xi_2')^2 \ln^2 \left| \frac{y}{y-1} \right| \right]
 \end{aligned}$$

add time-translations \Rightarrow logarithmic Schrödinger-invariance

6. Numerical experiments

- (A) Kardar-Parisi-Zhang (**KPZ**)
- (B) critical directed percolation (**DP**)

simple ageing of the correlators and responses, especially

$$C(t, s) = s^{-b} f_C\left(\frac{t}{s}\right), \quad R(t, s) = s^{-1-a} f_R\left(\frac{t}{s}\right)$$
$$f_C(y) \sim y^{-\lambda_C/z}, \quad f_R(y) \sim y^{-\lambda_R/z} \quad y \gg 1$$

values of the non-equilibrium exponents & scaling relations

KPZ in 1D : $\lambda_C = \lambda_R = 1$, $1 + a = b + \frac{2}{z}$, $b = -2\beta = -\frac{2}{3}$, $z = \frac{3}{2}$

DP : $\lambda_C = \lambda_R = d + z + \frac{\beta}{\nu_{\perp}}$, $1 + a = b = \frac{2\beta}{\nu_{\parallel}}$

what can be said on the **form** of the scaling function of the **auto-response**?

explore applicability of logarithmic ageing-invariance

(A) assumption : $R(t, s) = \langle \psi(t) \tilde{\psi}(s) \rangle$ 1D KPZ equation/RSOS model

good collapse \Rightarrow **no** logarithmic corrections \Rightarrow $x' = \tilde{x}' = 0$

no logarithmic factors for $y \gg 1 \Rightarrow \xi' = 0$

\Rightarrow only $\tilde{\xi}' = 1$ remains

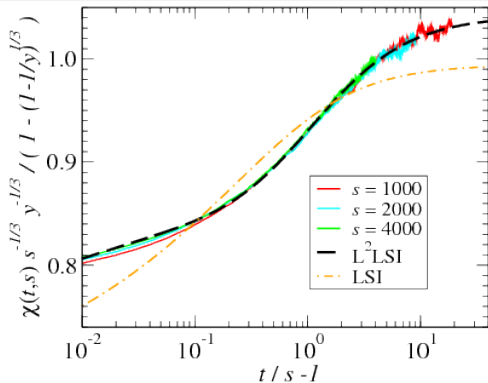
$$f_R(y) = y^{-\lambda_R/z} \left(1 - \frac{1}{y}\right)^{-1-a'} \left[h_0 - g_0 \ln \left(1 - \frac{1}{y}\right) - \frac{1}{2} f_0 \ln^2 \left(1 - \frac{1}{y}\right) \right]$$

use specific values of 1D KPZ class $\frac{\lambda_R}{z} - a = 1$

find integrated autoresponse $\chi(t, s) = \int_0^s du R(t, u) = s^{1/3} f_\chi(t/s)$

$$f_\chi(y) = y^{1/3} \left\{ A_0 \left[1 - \left(1 - \frac{1}{y}\right)^{-a'} \right] + \left(1 - \frac{1}{y}\right)^{-a'} \left[A_1 \ln \left(1 - \frac{1}{y}\right) + A_2 \ln^2 \left(1 - \frac{1}{y}\right) \right] \right\}$$

with free parameters A_0, A_1, A_2 and a'



non-log LSI with $a = a'$:
deviations $\approx 20\%$

non-log LSI with $a \neq a'$:
 works up to $\approx 5\%$

log LSI : works **better**
 than $\approx 0.1\%$

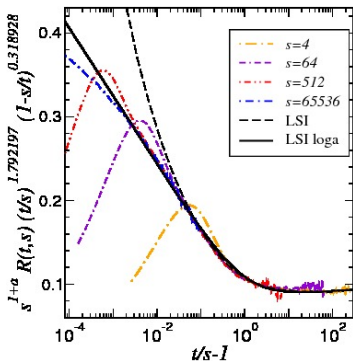
R	a'	A_0	A_1	A_2
$\langle \phi \tilde{\phi} \rangle$ - LSI	-0.500	0.662	0	0
$\langle \phi \tilde{\psi} \rangle$ - L^1 LSI	-0.500	0.663	$-6 \cdot 10^{-4}$	0
$\langle \psi \tilde{\psi} \rangle$ - L^2 LSI	-0.8206	0.7187	0.2424	-0.09087

logarithmic LSI works at least down to $y \simeq 1.01$, with
 $a' - a \approx -0.4873$ (can we make a conjecture?)

(B) assumption : $R(t, s) = \langle \psi(t) \tilde{\psi}(s) \rangle$ 1D critical contact process

good collapse \Rightarrow **no** logarithmic corrections \Rightarrow $x' = \tilde{x}' = 0$

$$h_R(y) = \left(1 - \frac{1}{y}\right)^{a-a'} \left[h_0 - g_{12,0} \tilde{\xi}' \ln(1 - 1/y) - g_{21,0} \xi' \ln(y - 1) - \frac{1}{2} f_0 \tilde{\xi}'^2 \ln^2(1 - 1/y) + \frac{1}{2} f_0 \xi'^2 \ln^2(y - 1) \right]$$



find empirically :
very small amplitude of
 \ln^2 -terms

$$\Rightarrow f_0 = 0$$

require both $\xi \neq 0$, $\tilde{\xi}' \neq 0$

BUT : logarithmic factor for $y \gg 1$?

logar. LSI works at least down to $y \simeq 1.002$, with $a' - a \simeq -0.002$.

7. Conclusions

- physical ageing occurs naturally in many **irreversible** systems relaxing towards **non**-equilibrium stationary states
considered here : absorbing phase transitions & surface growth
- scaling phenomenology essentially the same as in simple magnetic systems
- **but** finer differences in relationships between non-equilibrium exponents
- a **major difference** w/ equilibrium : intrinsic **absence** of time-translation-invariance \Rightarrow **2** scaling dimensions
- shape of scaling functions :
logarithmic local scale-invariance ?
non-logarithmic LSI is quantitatively insufficient for examples studied
performed **numerical experiments** on auto-response function :
(i) 1D KPZ equation (ii) 1D critical directed percolation



Vol. 1 : absorbing phase transitions – co-authors H. Hinrichsen, S. Lübeck 2009

Vol. 2 : ageing & local scaling – co-author M. Pleimling 2010

ISBN : 978-1-4020-8764-6 (vol 1.) & 978-90-481-2868-6 (vol. 2)

