Coupled order parameter systems on scale−free networks

Christian von Ferber[⊙•] Reinhard Folk[•] Ralph Kenna[⊙]
Vasyl Palchykov[•] Yurij Holovatch[••]
[⊙] Applied Mathematics Research Centre, Coventry University
[•]Physikalisches Institut, Albert-Ludwig University, Freiburg
[•] Johannes Kepler University, Linz
[•] Institute for Condensed Matter Physics,
Ukranian Academy of Sciences, Lviv

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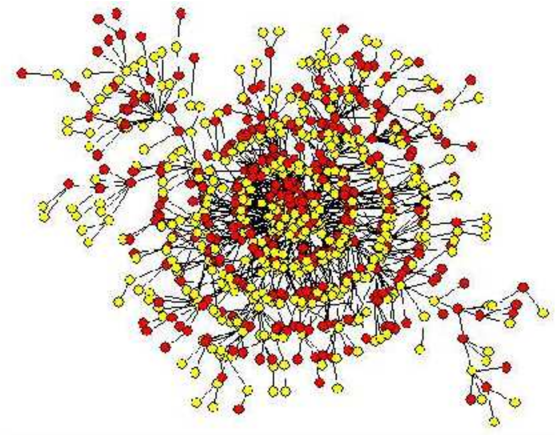
Motivations to study phase transitions on complex networks

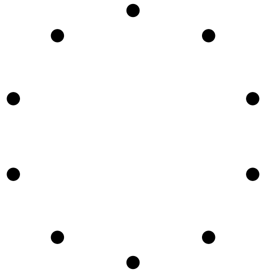
- Academic interest. Unusual features of the Ising (Leone et al.'02, Dorogovtsev et al.'02, Tadić et al.'05), XY (Kwak et al.'07), Potts (Iglói, Turban'02) models



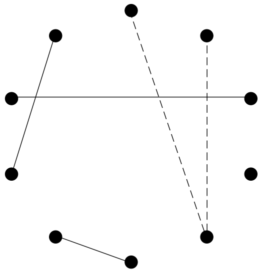
- Social networks as models for opinion formation (Galam'99 - '09, Sznajd'00, Sznajd-Weron'05, Staufer, Solomon'07, Kulakowski'08)

- Integrated nanosystems with nontrivial architecture (Moriarty'01, Wang et al.'03, Archer et al.'07)

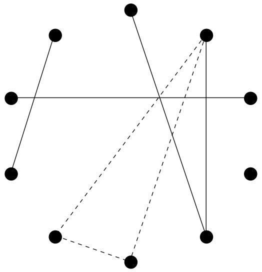
Complex network = more complex, than a random graph



p=0



p=0.1

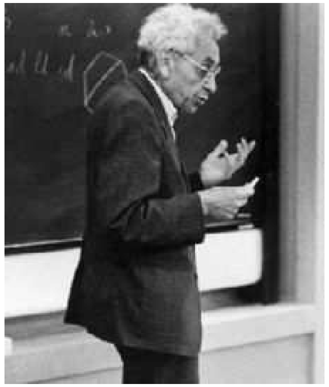


p=0.15

N nodes, connection probability p.

Average node degree:

$$< k >= pN.$$



Paul Erdös

Alfréd Rényi

classical random graph: 1959

Node degree distribution:

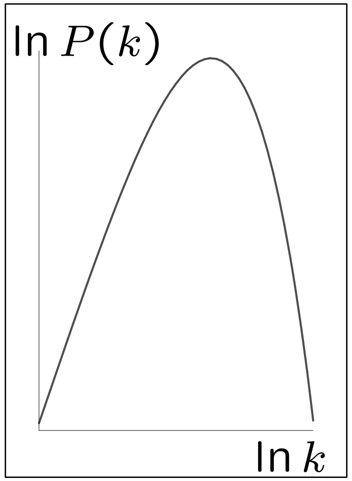
$$P(k_i = k) = C^k_{N-1} p^k (1-p)^{N-1-k}.$$

$N \to \infty$: a Poisson distribution

(with scale $< k >$):

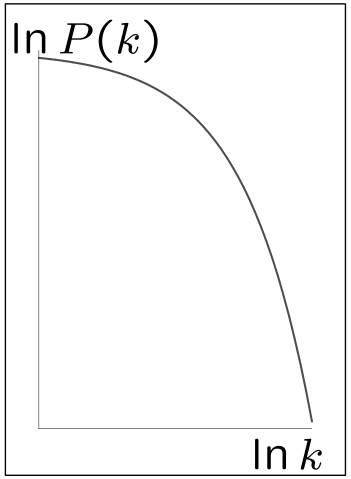
$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}.$$

Node degree distribution



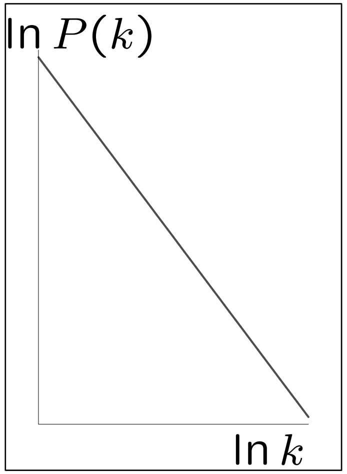
A.

$$P(k) = e^{-\langle k\rangle}\frac{\langle k\rangle^k}{k!}$$



B.

$$P(k) \sim e^{-k/\langle k\rangle}$$

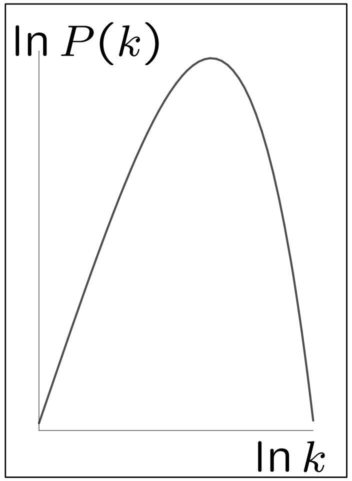


C.

$$P(k) \sim 1/k^{\lambda}, k \neq 0$$

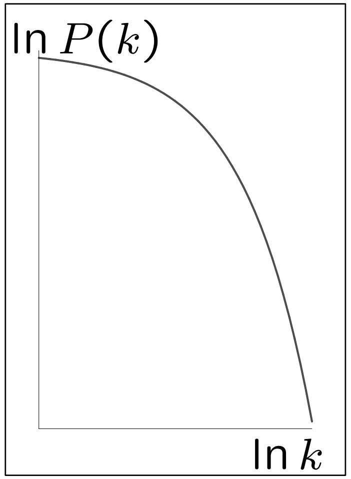
- A, B: typical scale $\langle k\rangle$;
- C: scale-free;

Node degree distribution



A.

$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$



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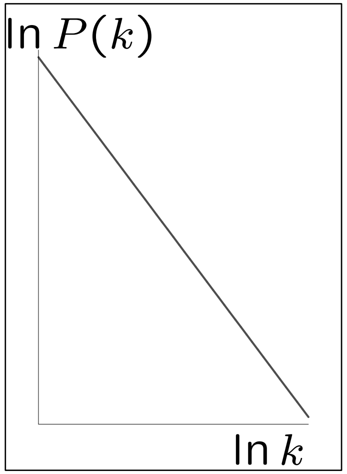

C.

$P(k) \sim 1/k^{\lambda}, k \neq 0$

- A, B: typical scale $\langle k \rangle$; $M_m = \sum_{k=0}^{\infty} k^m P(k) < \infty$.
- C: scale-free; moments with $m \geq \lambda - 1$ diverge.
- C: social, protein reaction, sexual partners, computer, etc.

Coupled order parameters on a scale-free network

Anisotropic free energy:

$$\Phi(\vec{x},T) = \Phi_0(\vec{x}) + \frac{a}{2}(T-T_c)|\vec{x}|^2 + \frac{b}{4}|\vec{x}|^4 + \frac{c}{4}x_1^2x_2^2$$

describes different types of ordering with coupled order parameters x_1, x_2, $\vec{x} = \{x_1, x_2\}$ (Watanabe'85, Imry'75).

Our task: to generalize coupled order parameters model(s) for the case of a scale-free network using:

- phenomenological (Landau) approach;
- microscopic (Ising) approach.

Microscopic (mean field) approach

The spin Hamiltonian

$$H = -\sum_{\langle i,j \rangle} S_i \cdot S_j - h \sum_i S_i \ .$$

The mean-field approach:

- The mean value σ_i of the k_i spins surrounding the i-th node

$$\sigma_i := \frac{1}{k_i} \sum_{\langle i,j \rangle} S_j = \sigma$$

  - Terms of order $(\Delta S)^2$ are neglected, $\Delta S_i = S_i - \sigma$

The Helmholtz free energy per site:

$$F(\sigma, T) = \sum_{k=1}^{k_{\text{max}}} P(k) \left[-T \ln(2\pi) + \frac{1}{2} k \sigma^2 - T \ln I_0(k\sigma/T) \right].$$

$I_0(z)$: modified Besselfunction of the first kind; (asymptotics):

$$I_0(z) \sim 1 + \frac{z^2}{4}, \ \ z \to 0; \qquad I_0(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \ \ z \to \infty.$$

Phenomenology: Landau theory

L.D. Landau
1908-1968

Gibbs potential: $d\Phi(T,h) = -SdT - Mdh.$

Landau free energy (ЖЭТФ **7** (1937) 19):

$$\Phi(T,h,M) = \Phi_0(T) + \frac{a(T-T_c)}{2}M^2 + \frac{b}{4}M^4 - hM.$$

Minimum of Φ: $\boxed{\frac{\partial\Phi(T,h,M)}{\partial M}\Big|_{T,h} = 0.}$

$$\mathbf{M} = \begin{cases} 0, & T > T_c, \quad h = 0, \\ (h/b)^{1/3}, & T = T_c, \\ \sqrt{\frac{a}{b}}(T_c - T)^{1/2}, & T < T_c, \quad h = 0. \end{cases}$$

$$\chi_{\mathbf{T}} = \begin{cases} \frac{1}{a}(T - T_c)^{-1}, \\ \frac{1}{3b^{1/3}}h^{-2/3}, \\ \frac{1}{2a}(T_c - T)^{-1}. \end{cases}$$

Assumptions of Landau theory on networks

- The free energy depends of the node degree distribution (Goltsev et al.'03)

$$\Phi(\vec{x},T)=\int_1^{k_{max}} dkP(k)f(\vec{x},k\vec{x}).$$

- $f(\vec{x},k\vec{x})$ is an analytic function

$$f(\vec{x},k\vec{x})=f_0+\sum_{i=0}^{2} a_i k^i |\vec{x}|^2+\sum_{i=0}^{4} b_i k^i |\vec{x}|^4+\sum_{i=0}^{4} c_i k^i \sum_{\mu=1}^{2} x_\mu^4+\ldots$$

- The free energy is finite for finite $\langle k\rangle$, $|\vec{x}|$

$$f(\vec{x},k\vec{x})\sim k|\vec{x}|, \qquad k|\vec{x}|\to\infty.$$

Landau free energy

$\lambda > 5$

$$\Phi(\vec{x}, T) = \Phi_0 + \frac{a}{2}(T - T_c)|\vec{x}|^2 + \frac{b^{(\lambda)}}{4}|\vec{x}|^4 + \frac{c^{(\lambda)}}{4}x_1^2 x_2^2.$$

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$\lambda = 5$

$$\Phi(\vec{x}, T) = \Phi_0 + \frac{a}{2}(T - T_c)|\vec{x}|^2 + \frac{b^{(\lambda)}}{4}|\vec{x}|^4 \ln\frac{1}{|\vec{x}|} + \frac{c^{(\lambda)}}{4}x_1^2 x_2^2 \ln\frac{1}{|\vec{x}|}.$$

Landau free energy

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$$\Phi(\vec{x},T) = \Phi_0 + \frac{a}{2}(T-T_c)|\vec{x}|^2 + \frac{b^{(\lambda)}}{4}|\vec{x}|^4 + \frac{c^{(\lambda)}}{4}x_1^2x_2^2.$$

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$3 < \lambda < 5$

$$\Phi(\vec{x},T) = \Phi_0 + \frac{a}{2}(T-T_c)|\vec{x}|^2 + \frac{b^{(\lambda)}}{4}|\vec{x}|^{\lambda-1} + \frac{c^{(\lambda)}}{4}\frac{x_1^2x_2^2}{|\vec{x}|^4}|\vec{x}|^{\lambda-1}.$$

Landau free energy

$\lambda > 5$

$$\Phi(\vec{x},T) = \Phi_0 + \frac{a}{2}(T-T_c)|\vec{x}|^2 + \frac{b^{(\lambda)}}{4}|\vec{x}|^4 + \frac{c^{(\lambda)}}{4}x_1^2x_2^2.$$

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$\lambda = 3$

$$\Phi(\vec{x},T) = \Phi_0 + C|\vec{x}|^2 - D|\vec{x}|^2 \ln\frac{1}{|\vec{x}|} + E\frac{x_1^2x_2^2}{|\vec{x}|^4}|\vec{x}|^2.$$

Landau free energy

$\lambda > 5$

$$\Phi(\vec{x},T) = \Phi_0 + \frac{a}{2}(T-T_c)|\vec{x}|^2 + \frac{b^{(\lambda)}}{4}|\vec{x}|^4 + \frac{c^{(\lambda)}}{4}x_1^2x_2^2.$$

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$\lambda = 3$

$$\Phi(\vec{x},T) = \Phi_0 + C|\vec{x}|^2 - D|\vec{x}|^2 \ln\frac{1}{|\vec{x}|} + E\frac{x_1^2x_2^2}{|\vec{x}|^4}|\vec{x}|^2.$$

$2 < \lambda < 3$

$$\Phi(\vec{x},T) = \Phi_0 + C'|\vec{x}|^2 + D'|\vec{x}|^{\lambda-1} + E'\frac{x_1^2x_2^2}{|\vec{x}|^4}|\vec{x}|^{\lambda-1}.$$

Landau free energy minimum

- $$\frac{\partial\Phi(\vec{x},T)}{\partial x_1} = 0, \qquad \frac{\partial\Phi(\vec{x},T)}{\partial x_2} = 0.$$

- $$\omega_{\mu\nu} = \frac{\partial^2\Phi(\vec{x},T)}{\partial x_\mu \partial x_\nu}, \qquad \mu,\nu = 1,2$$

  $$\mathrm{Re}(\omega_{\mu\mu}) > 0, \quad \det(\omega_{\mu\nu}) > 0, \quad \mu,\nu = 1,2.$$

Phase diagram

Two types of ordering: [1,0] and [1,1]

$\lambda > 5$

$$x_1, x_2 \sim (T_c - T)^\beta, \qquad \beta = \frac{1}{2}$$

$\lambda = 5$

$$x_1, x_2 \sim \frac{(T_c - T)^\beta}{\ln(T_c - T)^{-1/2}}, \qquad \beta = \frac{1}{2}$$

$3 < \lambda < 5$

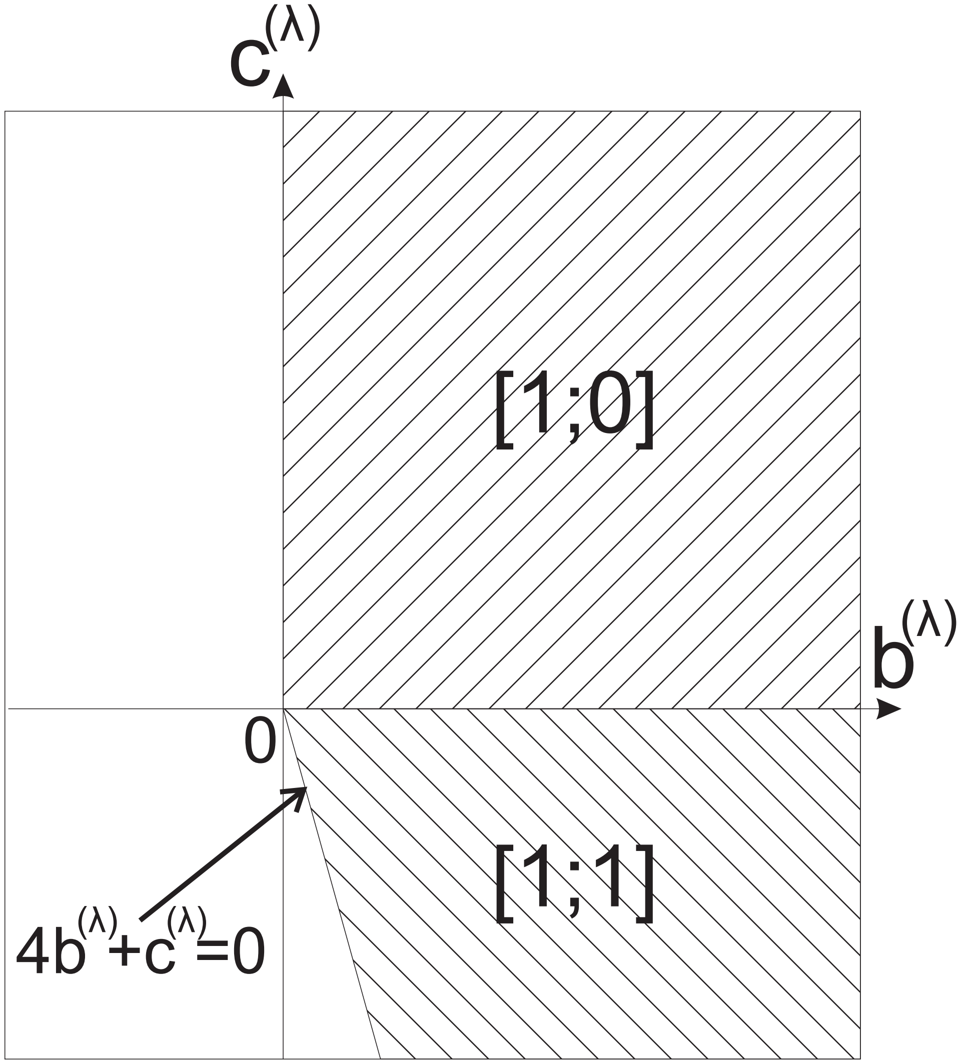
$$x_1, x_2 \sim (T_c - T)^\beta, \qquad \beta = \frac{1}{\lambda - 3}$$

$\lambda = 3$

$$x_1, x_2 \sim e^{-\eta T}$$

$2 < \lambda < 3$

$$x_1, x_2 \sim T^{-\frac{1}{3-\lambda}}$$



Isothermal suscepibility

Susceptibility tensor:

$$\chi_{\mu,\nu} = (\partial x_\mu / \partial h_\nu)_T = \delta_{\mu\nu}\chi_\parallel + (1-\delta_{\mu\nu})\chi_\perp, \qquad \mu,\nu = 1,2.$$

Critical exponents, $\lambda > 3$:

$$T > T_c: \qquad \chi_\parallel = \chi_\perp \sim (T-T_c)^{-1}, \qquad \gamma = 1$$

$$T < T_c: \qquad \chi_\parallel \sim (T-T_c)^{-1}, \qquad \lim_{c\to 0}\chi_\perp \to \infty.$$

Amplitude ratios:

$\lambda > 5:$ $\quad \Gamma_+/\Gamma_- = 2;$ $\qquad$ $3 < \lambda \leq 5:$ $\quad \Gamma_+/\Gamma_- = \lambda - 3.$

Heat capacity

$$c_h = T(\partial S/\partial T)_h$$

Behaviour in the vicinity of T_c:

$\lambda > 5$

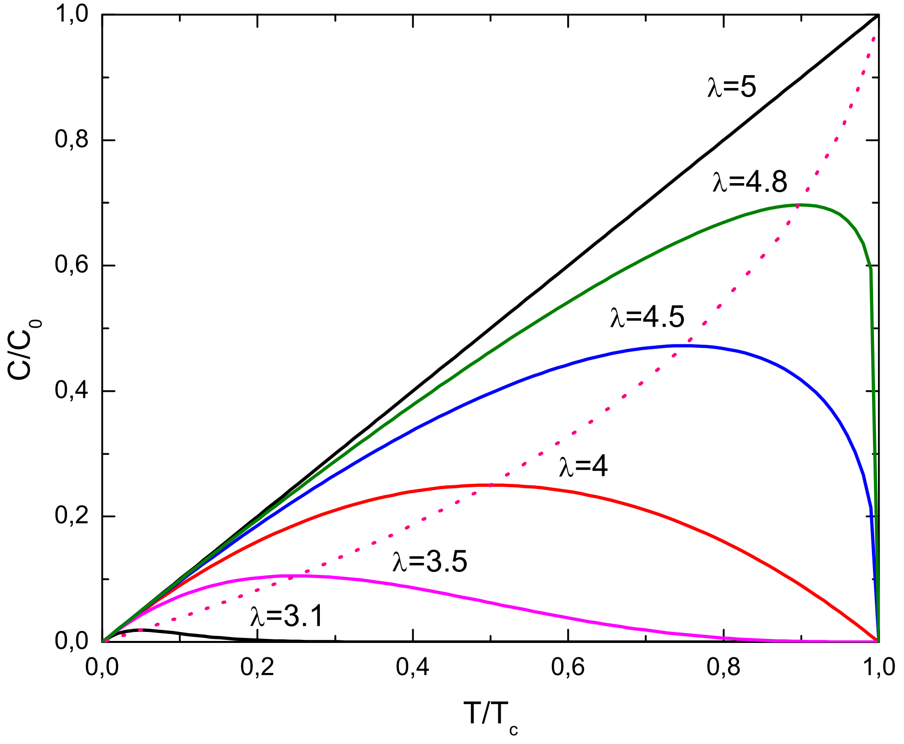
$$\delta c_h = \frac{a^2}{2b^{(\lambda)}} T_c$$

$\lambda = 5$

$$\delta c_h \sim \frac{T}{\ln(T_c - T)^{-1}}$$

$3 < \lambda < 5$

$$\delta c_h \sim T(T_c - T)^{(5-\lambda)/(\lambda-3)}$$



$$c_h \sim \begin{cases} T^2 e^{-\zeta T}, & \lambda = 3 \\ T^{-\frac{\lambda-1}{3-\lambda}}, & 2 < \lambda < 3 \end{cases}$$

$$T_0 = \frac{\lambda-3}{2} T_c$$

Logarithmic corrections

At 'upper critical' value $\lambda = 5$ the scaling behaviour reads:

$\underline{h = 0:}$

$$\begin{aligned} M &\sim \tau^{\beta} |\ln \tau|^{\widehat{\beta}}, \\ \chi &\sim \tau^{-\gamma} |\ln \tau|^{\widehat{\gamma}}, \\ C_h &\sim \tau^{-\alpha} |\ln \tau|^{\widehat{\alpha}}. \end{aligned}$$

$\underline{\tau = 0:}$

$$\begin{aligned} M &\sim h^{1/\delta} |\ln h|^{\widehat{\delta}}, \\ \chi &\sim h^{-\gamma_c} |\ln h|^{\widehat{\gamma}_c}, \\ C_h &\sim h^{-\alpha_c} |\ln h|^{\widehat{\alpha}_c}. \end{aligned}$$

Logarithmic correction-to-scaling exponents are found:

$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\delta}$	$\widehat{\alpha}_c$	$\widehat{\gamma}_c$
-1	$-1/2$	0	$-1/3$	-1	$-1/3$

Logarithmic correction-to-scaling exponents obey the following relations:

$$\hat{\beta}(\delta-1) = \delta\hat{\delta}-\hat{\gamma}, \qquad \hat{\alpha} = 2\hat{\beta}-\hat{\gamma}, \qquad \hat{\gamma}_c = \hat{\delta},$$

$$\hat{\alpha}_c = \frac{(\gamma+2)(\hat{\beta}-\hat{\gamma})}{\beta+\gamma} + \hat{\gamma}.$$

Recall:

$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\alpha}_c$	$\hat{\gamma}_c$
-1	$-\frac{1}{2}$	0	$-\frac{1}{3}$	-1	$-\frac{1}{3}$

Compare with:

$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	
-1	$-\frac{1}{8}$	$\frac{3}{4}$	$-\frac{1}{15}$	$q=4$ Potts, $d=2$
$\frac{4-N}{N+8}$	$\frac{3}{N+8}$	$\frac{N+2}{N+8}$	$\frac{1}{3}$	$O(N)$, $d=4$

Conclusions

- A general theory, not restricted by specific properties of any model.

- Critical behaviour is governed by:
  - structure of a network;
  - symmetry of the OP.

- Types of ordering (phases): [1,1], [1,0], [0,1], [0,0].

- Role of the high-degree vertices (hubs):
  - Rapid decay ($\lambda > 5$): usual Landau theory;
  - "Upper critical dimension"$\lambda = 5$: logarithmic corrections (scaling relations hold);
  - $3 < \lambda < 5$: non-trivial λ-dependence of the exponents and amplitude ratios;
  - $2 < \lambda \leq 3$: $T_c = \infty$.