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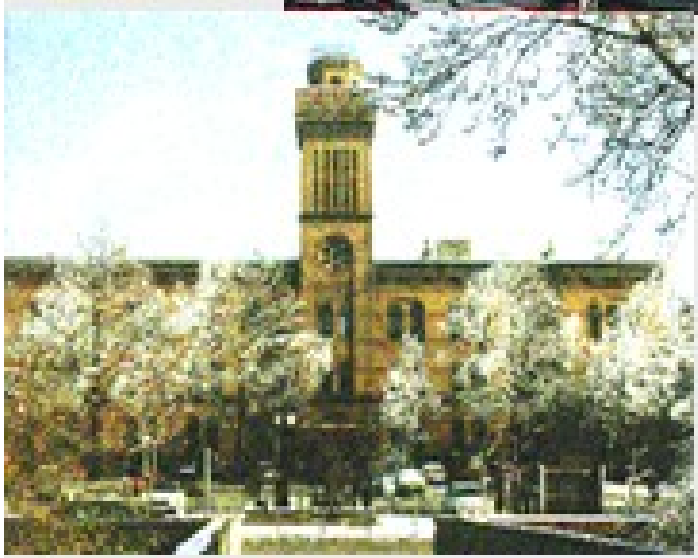

Halle
1200
years

EXACT SOLUTION OF A STOCHASTIC SIR-MODEL

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CompPhys09

MODEL KEY WORDS

- SIR: Susceptible-Infectious-Recovered Model
- SIR: Evolution of this three species
- SIR: Simple tractable model capturing relevant features
- SIR Non-linear model
- SIR Out of equilibrium
- SIR Stochastic description

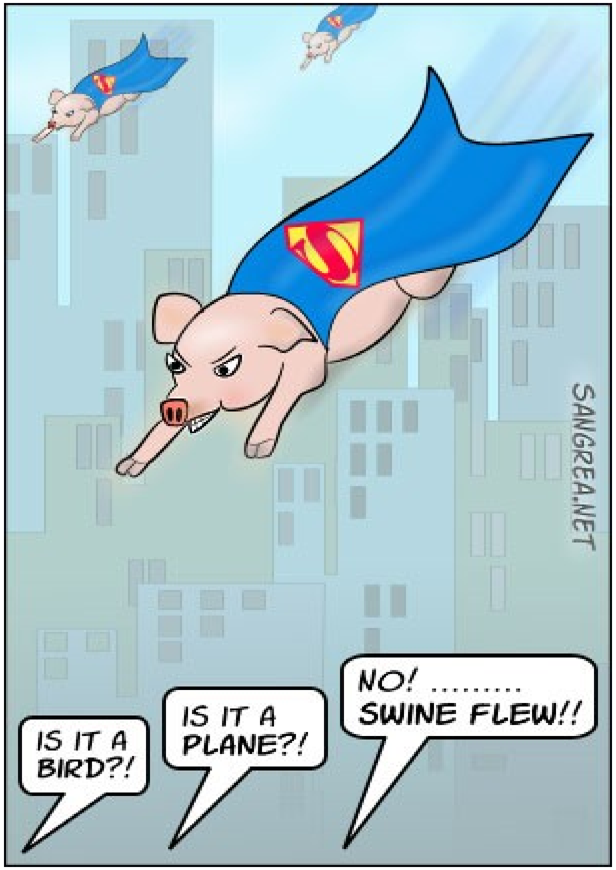
Infection produces further infections

28/09/09
Newspaper Roanoke Time

REGION READY FOR FLU

Halle:
‚Organized' Vaccination

Flew versus Flu



COMPLEX SYSTEMS

Comprise many components

Interacting non-linearly

System far from equilibrium

Natural home of big surprises

Emergent behavior

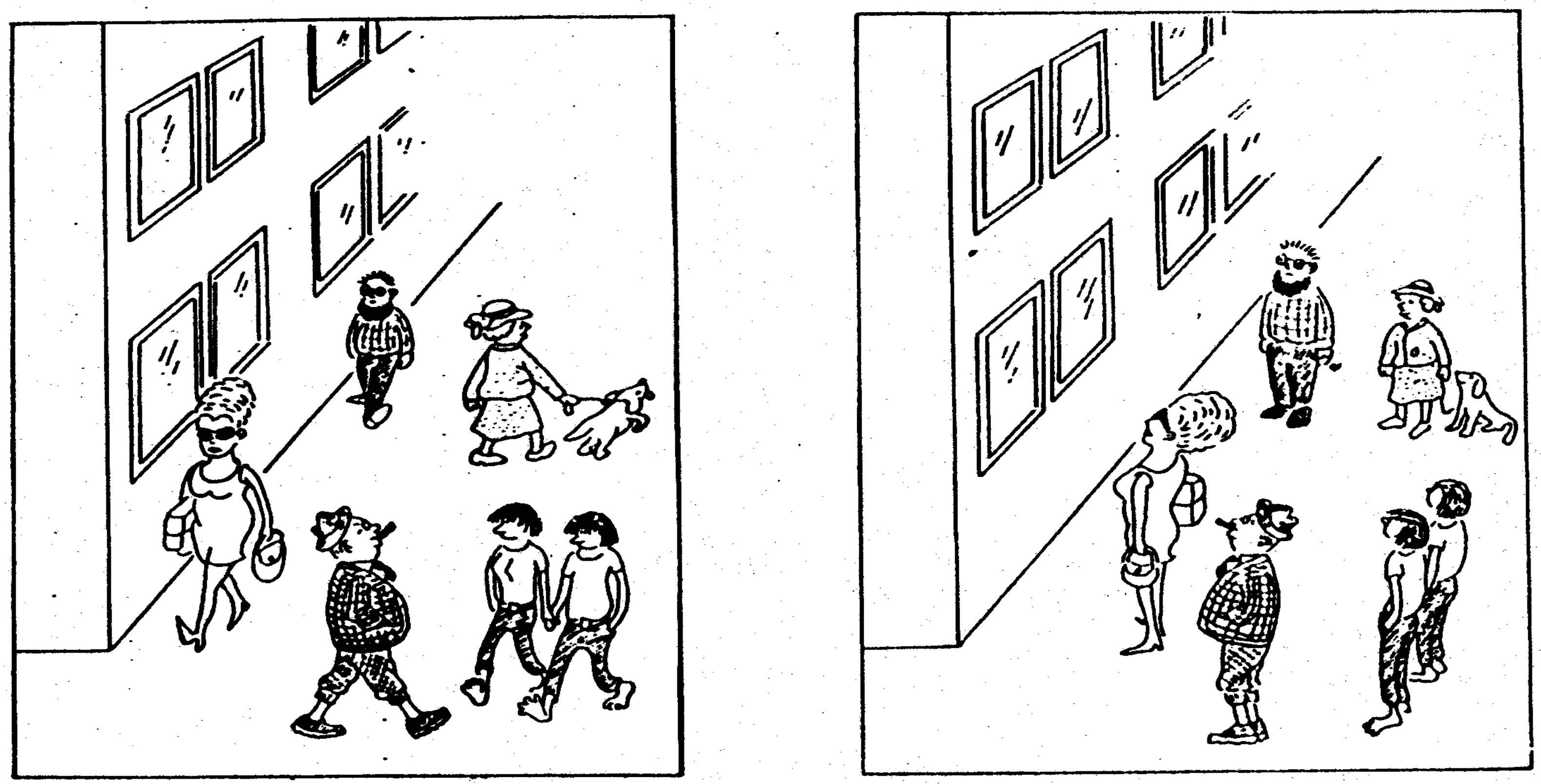
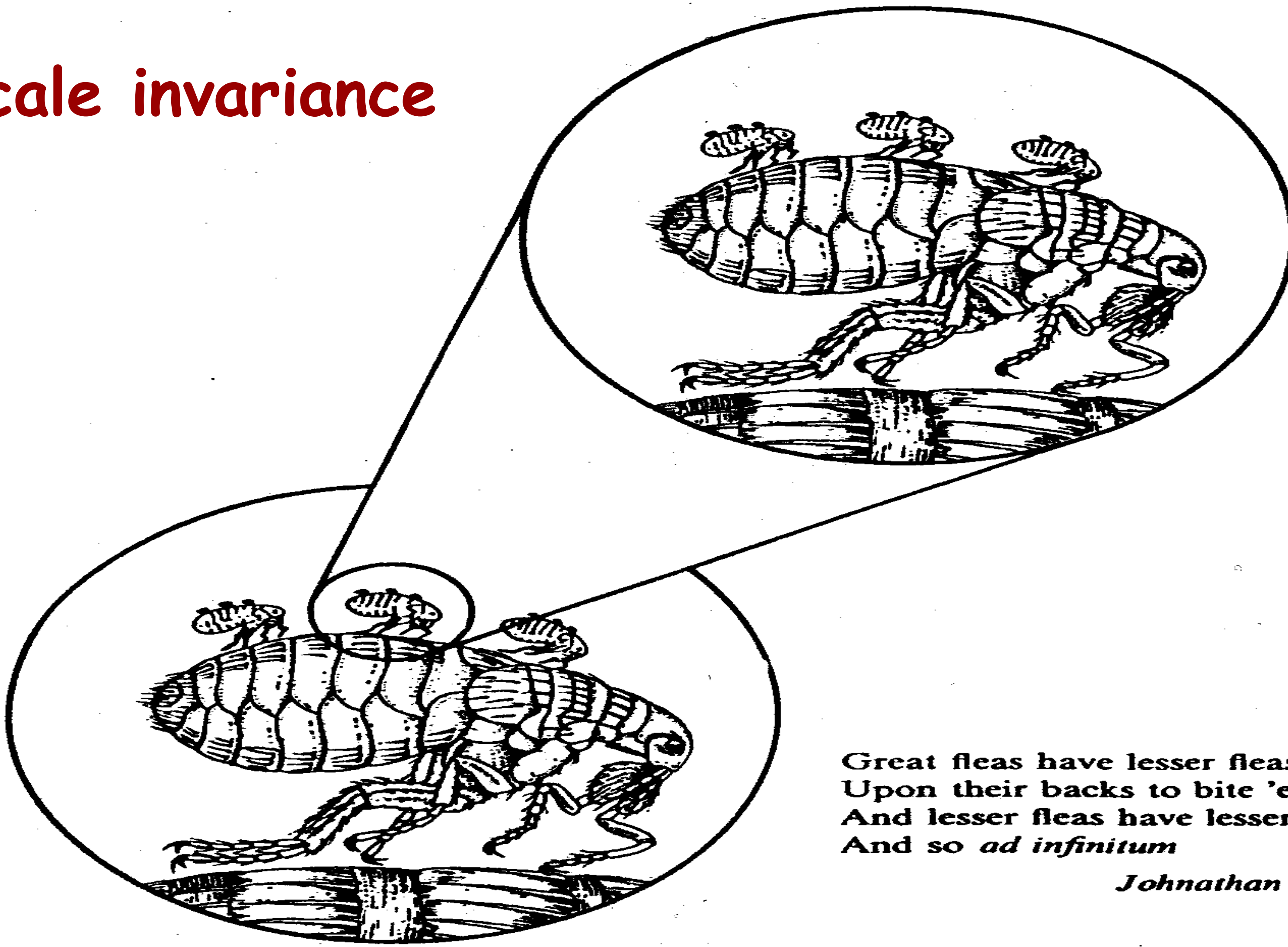


Abbildung 2.8: Ein Phasenübergang in der neugierigen Menge. (a) "Paramagnet"; (b) "Ferromagnet"

Flash Mob

Scale invariance



Great fleas have lesser fleas
Upon their backs to bite 'em.
And lesser fleas have lesser still.
And so *ad infinitum*

Johnathan Swift

Fig. 16.3 Scale invariance.

Collective Behavior

Spreading of Diseases

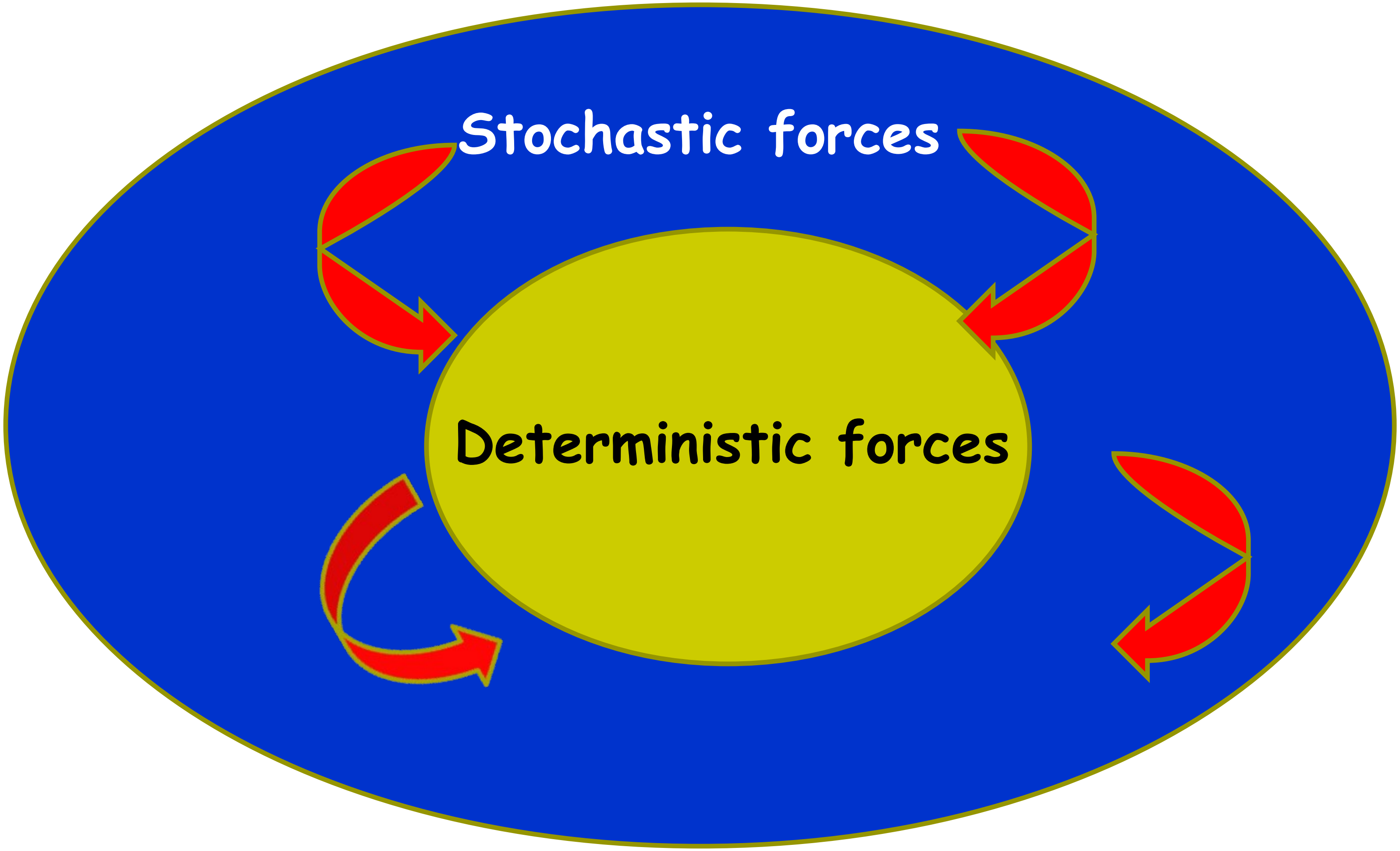
DESCRIPTION

Many degrees of freedom
Not all of them specified by deterministic forces
Different time scales

Stochastic descriptions

- Langevin equation
- Fokker-Planck equation
- **Master equation**
- **Simulations (MC**,MD)

DESCRIPTION



MODEL

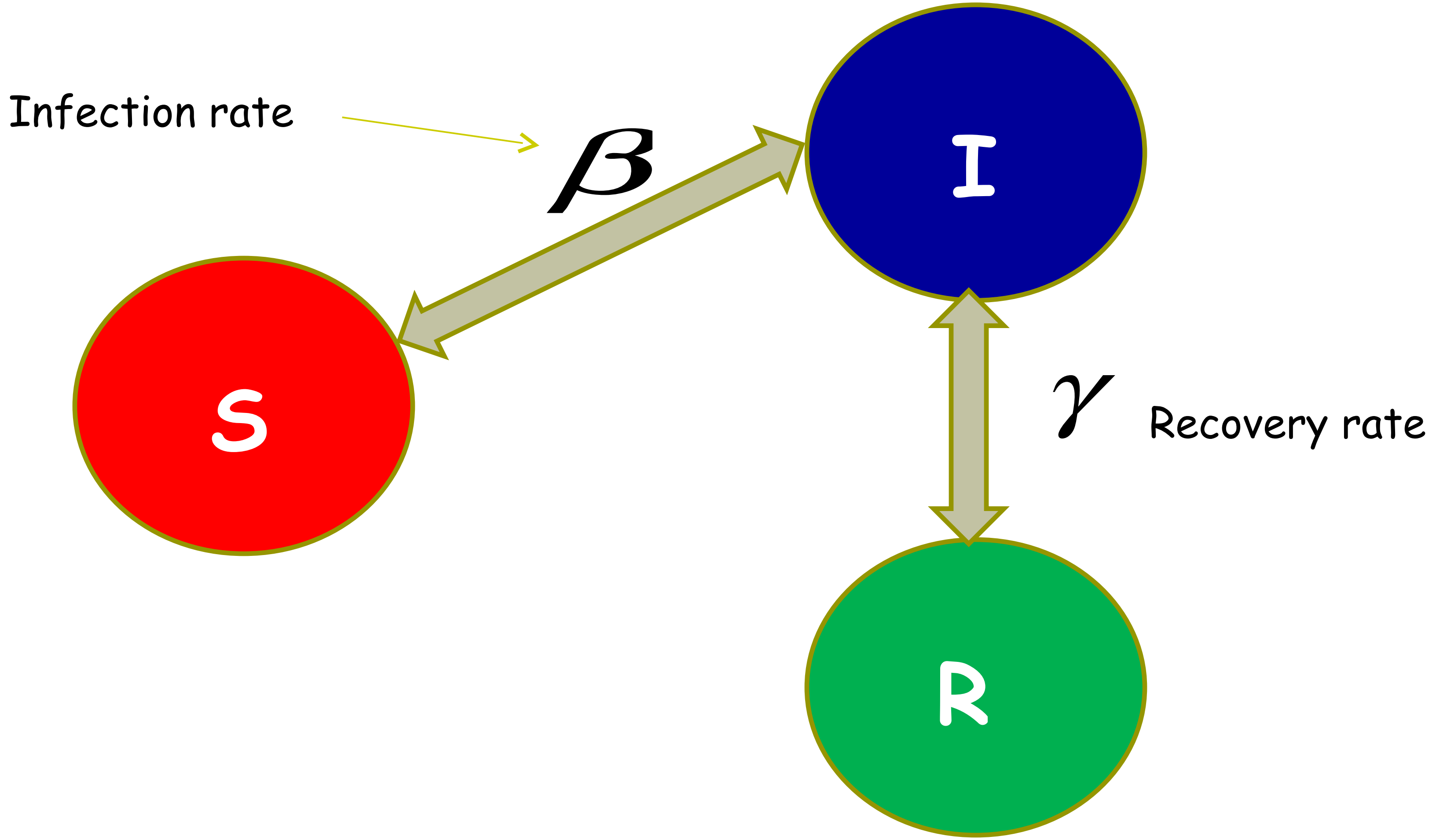
Three state model

S-susceptibles: healthy individuals can catch diseases

I-infectives: can transmit disease provided contacts

R-recovered: immunized or removed out

THREE STATE MODEL



MFA

determinsitic equations

S -> I -> R

S(t)+I(t)+R(t) = N

$$\frac{\partial S(t)}{\partial t} = -\beta S(t) I(t)$$

$$\frac{\partial I(t)}{\partial t} = \beta S(t) I(t) - \gamma I(t)$$

$$\frac{\partial R(t)}{\partial t} = \gamma I(t)$$

STOCHASTIC MODEL

- Deterministic evolution neglect fluctuations
- Individuals are represented by nodes which are either of the three states S I R
- Contact defined by links between nodes, minimal connectivity
- Each node, state variable occupation number n = 0, 1:
- If node i is in the S state: $n_S(i,t)=1$ otherwise n = 0
- Mapping master equation in a quantum formulation

Master-Equation

- PROCESS-DYNAMICS
- CHANGE OF A CERTAIN CONFIGURATION
- MARKOV PROCESS, NO MEMORY
- UNIFORM IN TIME
- CONTINUOUS IN TIME

$$\partial_t P(S_1,..,S_N,t) = \sum_i w(-S_i \to S_i)P(-S_i) - w(S_i \to -S_i)\,P(S_i)$$

→ **GAIN** **LOST**

$$\partial_t P(S) = L\ P(S)$$

Simple Example

$$RATE: \quad \lambda(T, \ldots) \qquad \uparrow \longrightarrow \downarrow$$

$$RATE: \quad \chi(T, \ldots) \qquad \downarrow \longrightarrow \uparrow$$

Description

$p_{\uparrow}(t), \quad p_{\downarrow}(t) \;\; prob$

$Markov\ process$

$p_{\uparrow}(t+\Delta t) = p_{\downarrow}(t)\gamma\Delta t + p_{\uparrow}(t)(1-\lambda\Delta t)$

$$\frac{dp_{\uparrow}}{dt} = \gamma p_{\downarrow} - \lambda p_{\uparrow} \qquad\qquad \frac{dp_{\downarrow}}{dt} = -\gamma p_{\downarrow} + \lambda p_{\uparrow}$$

$General.$

$$\frac{d}{dt} p_a = L_{ab} p_b$$

$$L_{ab} = \begin{pmatrix} -\lambda & \gamma \\ \lambda & -\gamma \end{pmatrix}$$

Quantum Language

OTHER LANGUAGE: CREATION AND ANNIHILATION OPERATOR

$$|0> = |\uparrow> \qquad |1> = |\downarrow>$$

$$d^{+}|0> = |1>$$

$$d\ |1> = |0>$$

$$n = d^{+}d \rightarrow 0, 1$$

$$MAPPING\ \ TO\ \ SPINS:$$

$$S = 1 - 2d^{+}d = 1 - 2n$$

$$S\,|1> = -1\,|1> \qquad S\,|0> = +1\,|0>$$

Mapping

$$\frac{\partial}{\partial t} p(\vec{n},t) = L\, p(\vec{n},t)$$

$$|F(t)> = \sum_{n} p(\vec{n},t)\,|\vec{n}> \Rightarrow \frac{\partial}{\partial t}|F(t)> = \hat{L}\,|F(t)>$$

$$<\vec{n}'|\hat{L}\,|\vec{n}> = L_{n'n}$$

$$d^{+}\,|0> = |1> \quad \lambda - process \quad \uparrow \rightarrow \downarrow$$

$$d\ \ |1> \ = |0> \quad \gamma - process \quad \downarrow \rightarrow \uparrow$$

$$\hat{L}_{f} = \sum_{i} [\gamma(1-d_{i}^{+})d_{i} + \lambda(1-d_{i})d_{i}^{+}]$$

SIR Model

Master Eq. for the probability density

$$p(S,I,R,t) \equiv p(\vec{n},t); \quad \frac{\partial p(n,t)}{\partial t} = L\, p(\vec{n},t)$$

$$n_i = \text{٠,١} \quad i = S,I,R \qquad \left|F(t)\right\rangle = \sum_{\vec{n}} p(\vec{n},t)\left|\vec{n}\right\rangle; \quad \frac{\partial}{\partial t}\left|F\right\rangle = -\hat{H}\left|F\right\rangle \qquad \hat{H} = -\hat{L}$$

$$n_S(i,t) = \text{١}$$

DYNAMICS S -> I -> R

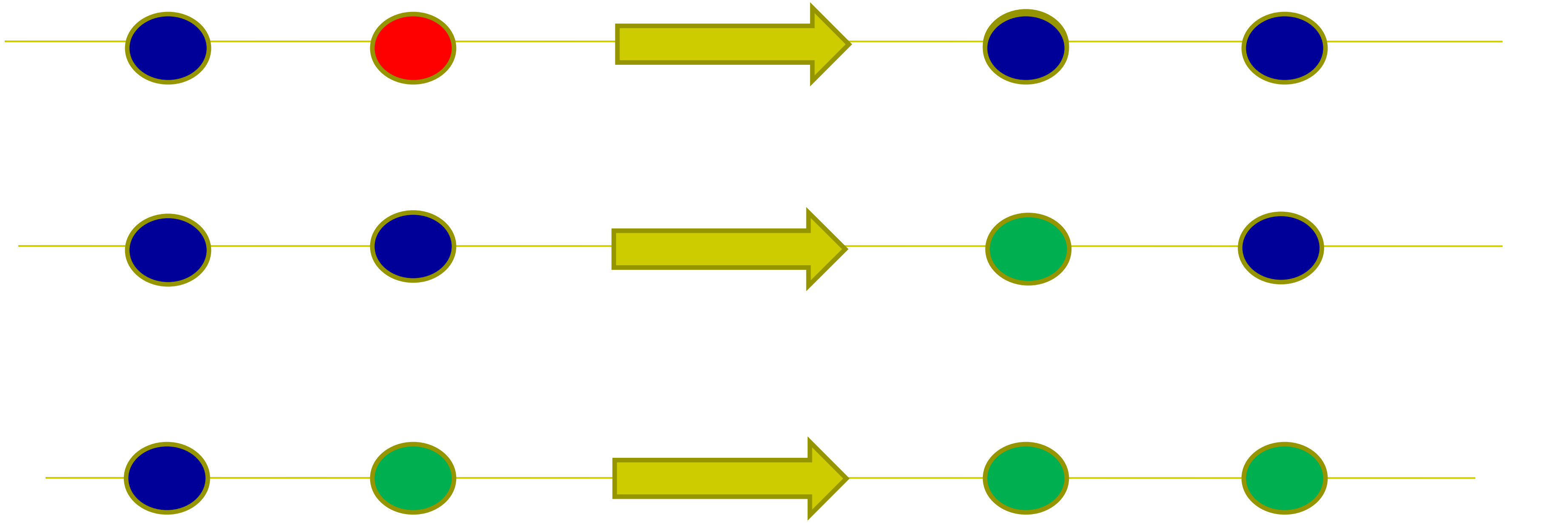
$$I_i + S_j \longrightarrow I_i + I_j$$

$$I_i + I_j \longrightarrow R_i + I_j$$

$$I_i + R_j \longrightarrow R_i + R_j$$

DYNAMICS

S -> I -> R



EVOLUTION OPERATOR

$$-\hat{H} = \beta \sum_i \left[b_{i+1}^{+} a_{i+1} + b_{i-1}^{+} a_{i-1} - A_{i+1}(1 - B_{i+1}) - A_{i-1}(1 - B_{i-1}) \right] B_i$$

$$+\gamma \sum_i (b_i - B_i)$$

Creation of I and annihilation of S

provided lattice site i occupied with I

$$A_l = a_i^{+} a_i \Rightarrow 0,1$$

Particle number operator S state

B Particle number of I state

Evolution Equation

$$\frac{d}{dt} < A_r > = -\beta \left[< A_r B_{r-1} > + < A_r B_{r+1} > \right]$$

$$\frac{d}{dt} < B_r > = \beta \left[< A_r B_{r-1} > + < A_r B_{r+1} > \right] - \gamma < B_r >$$

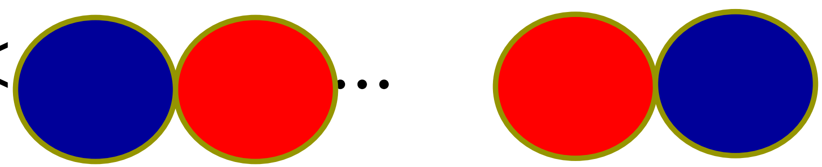
Hierarchy of Equations, MFA decoupling: <AB>=<A><B>

N-Point Cluster Functions

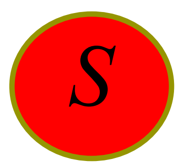
S surrounded by I at the edges

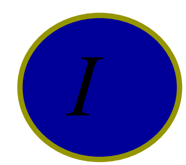
$$H_r(n) = < A_r \, A_{r+1} \quad \dots \quad A_{r+n-1} \, B_{r+n} \quad >$$

$= <$  $>$

$G_r(n) = <$  $>$

$$G_r(n) = < B_{r-1} \, A_r \, \dots . A_{r+n-1} \, B_{r+n} \quad >$$

$A_i = a_i^+ a_i \rightarrow$ 

$B = b_i^+ b_i \rightarrow$ 

N-Point Cluster Functions

$$H_r(n) = < A_r\, A_{r+1} \ldots A_{r+n-1}\, B_{r+n} >$$

$$G_r(n) = < B_{r-1}\, A_r \ldots . A_{r+n-1}\, B_{r+n} >$$

Coupled Equations (Use A B = 0)

$$\frac{d}{dt} H(n) = -[\gamma + \beta] H(n) + \beta [H(n+1) - G(n)]$$

$$\frac{d}{dt} G(n) = -2[\gamma + \beta] G(n) + 2\beta\, G(n+1)$$

N-Point Cluster Functions

$$\frac{d}{dt}H(n) = -[\gamma+\beta]H(n) + \beta[H(n+١) - G(n)]$$

$$\frac{d}{dt}G(n) = -٢[\gamma+\beta]G(n) + ٢\beta G(n+١)$$

$$n_S(i,t) = \langle A_i(t) \rangle \qquad n_I(i,t) = \langle B_i(t) \rangle$$

$$\frac{d}{dt}n_S(t) = -2\beta H(1,t) \qquad \frac{d}{dt}n_I(t) = 2\beta H(1,t) - \gamma n_I(t)$$

N-Point Cluster Functions

$$H(n) = <[n \otimes A][1 \otimes B]>$$

$$G(n) = <[1 \otimes B][n \otimes A][1 \otimes B]>$$

Increase by infection with β

$$<[n+1] \otimes A \quad 1 \otimes B> \rightarrow <[n] \otimes A \quad 1 \otimes B>$$

Recovery of infected individual I ->R with rate γ

Infection of S by the adjacent I with β

$$\frac{d}{dt} H(n) = -[\gamma + \beta] H(n) + \beta [H(n+1) - G(n)]$$

Infection of S at the left border

N-Point Cluster Functions

$$H(n) = <[n \otimes A]\,[1 \otimes B]>$$

$$G(n) = <[1 \otimes B][n \otimes A][1 \otimes B]>$$

$$\frac{d}{dt} G(n) = -2[\gamma + \beta] G(n) + 2\beta\, G(n+1)$$

Coupled equations

$$\frac{d}{dt}H(n) = -[\gamma+\beta]H(n) + \beta[H(n+1) - G(n)]$$

$$\frac{d}{dt}G(n) = -2[\gamma+\beta]G(n) + 2\beta G(n+1)$$

- Competition between growth and reduction processes

- Coupling between H and G

Nontrivial steady state

Solution for arbitrary initial condition

Solution uncorrelated random initial distribution

$$n_S(t) = n_S(0) - 2\beta\tau n_S(0) n_I(0)\left\{\left[\left(1 - \exp\left(-\frac{t}{\tau}\right)\right]\right.\right.$$
$$\left.\times[1 - \beta\tau n_I(0)] + \left[1 - \exp\left(-\frac{2t}{\tau}\right)\right]\frac{\beta\tau n_I(0)}{2}\right\}.$$

$$n_I(t) = n_I(0)[\exp(-\gamma t) + 2\beta\tau n_S(0) f(t)],$$

$$f(t) = \frac{\exp(-t/\tau) - \exp(-\gamma t)}{\gamma\tau - 1}[1 - \beta n_I(0)\tau]$$
$$+ \frac{\beta n_I(0)\tau}{\gamma\tau - 2}[\exp(-2t/\tau) - \exp(-\gamma t)].$$

Solution

Stationary solution

$$\frac{n_S^*}{n_S(0)} = \left[\frac{\gamma + \beta[1 - n_S(0) - n_I(0)]}{\gamma + \beta[1 - n_S(0)]}\right]^2,$$

Relaxation time

$$\tau = \frac{1}{\gamma + \beta[1 - n_S(0)]}.$$

- Due to fluctuations, isolated regions of susceptible individuals evolve
- Finite stationary distribution of the S type even for large population size
- Relaxation time and stationary distribution depends on initial conditions
- Highly nonergodic, far from equilbrium situation
- n_s (t) strictly monotonically decreasing (no S generated)
- n_I (t) exhibit maximum, stationary state n_I =0

MFA

$$\frac{\partial S(t)}{\partial t} = -\beta S(t) I(t)$$

$$\frac{\partial I(t)}{\partial t} = \beta I(t) \left(S(t) - \frac{\gamma}{\beta} \right)$$

$$\frac{\partial R(t)}{\partial t} = \gamma I(t)$$

increase or degrease

Max. if $S(0) > \gamma/\beta$

$S(t') = \gamma/\beta$

$I(t') = I_{max}$

t' different

$$\frac{dR}{dt} = -\frac{dU(R)}{dR}$$

Overdamped motion in a potential

$$U(R) = -\frac{S_0 \gamma^2}{\beta} \exp\left(-\frac{\beta}{\gamma} R \right) + \frac{\gamma R}{2} (R - 2N)$$

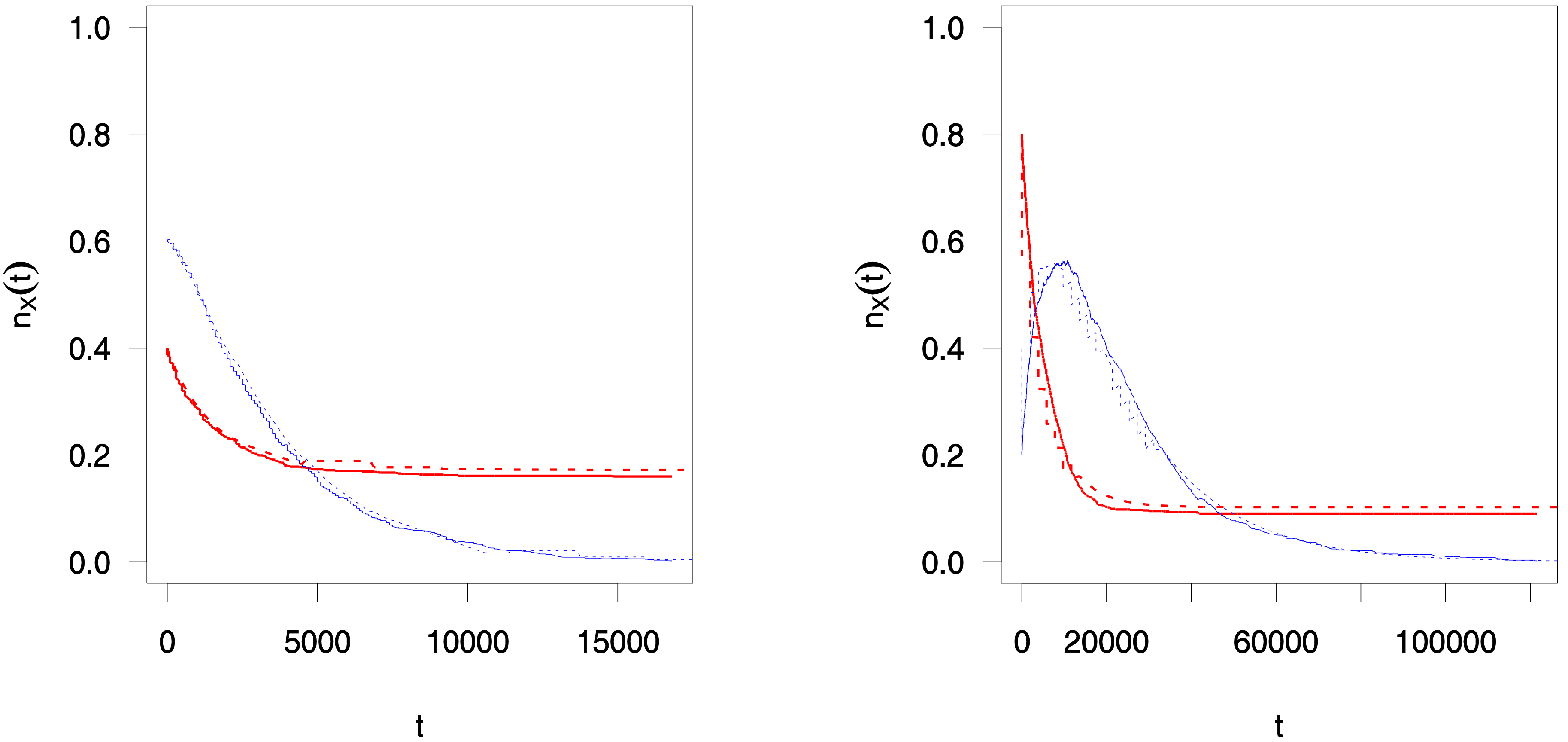
$$S(t \to \infty) \to S^* \approx S_0 \exp\left(-\frac{\beta}{\gamma} N \right)$$

MONTE CARLO SIMULATION

- Initially, each site is occupied independently and randomly by S with $n_s(0)$, I with $n_I(0) = 1-n_s(0)$
- Update: Choose arbitrary site j
- If j occupied by an I, then I decays to R with $\frac{\gamma}{٢\beta+\gamma}$
- If not decay, then with prob ½ adjacent site (left or right)
- Is this site occupied by S, then decay to I with $\frac{2\beta}{2\beta+\gamma}$

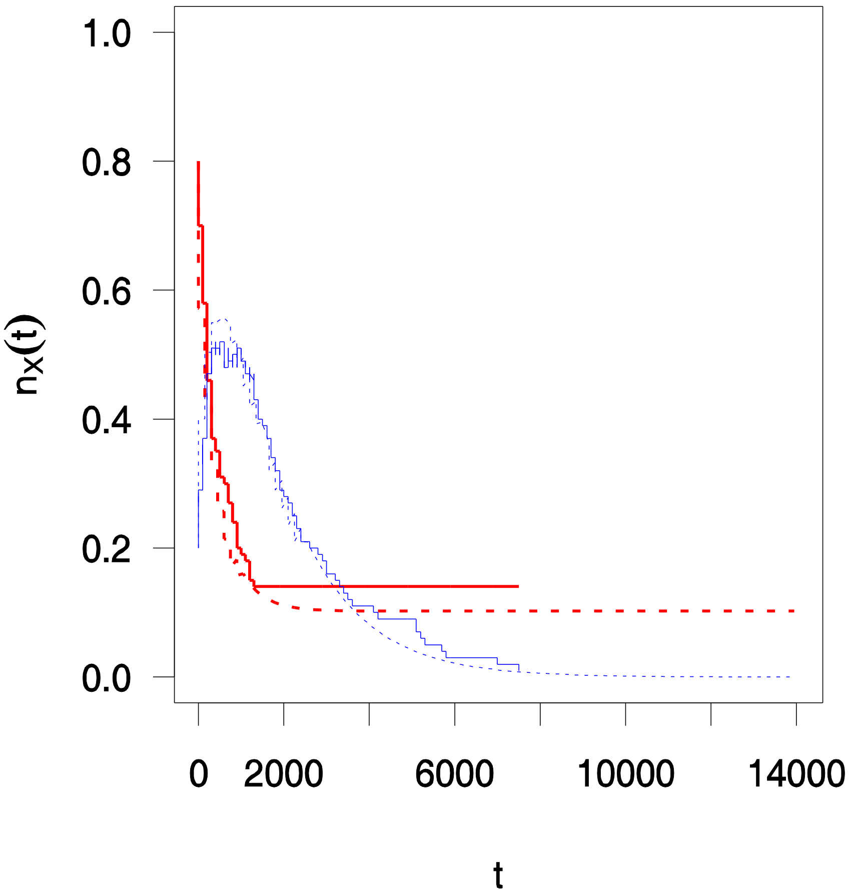
Numerical Results

Solid line: simulation; fixed N = 10^3

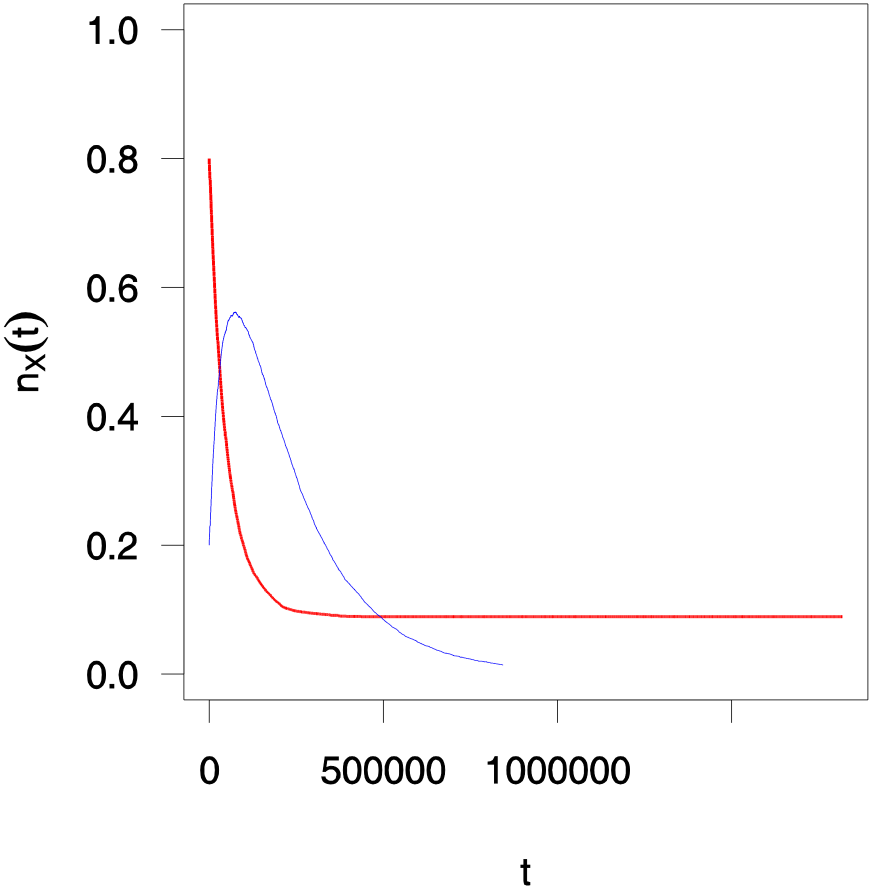


Numerical Results

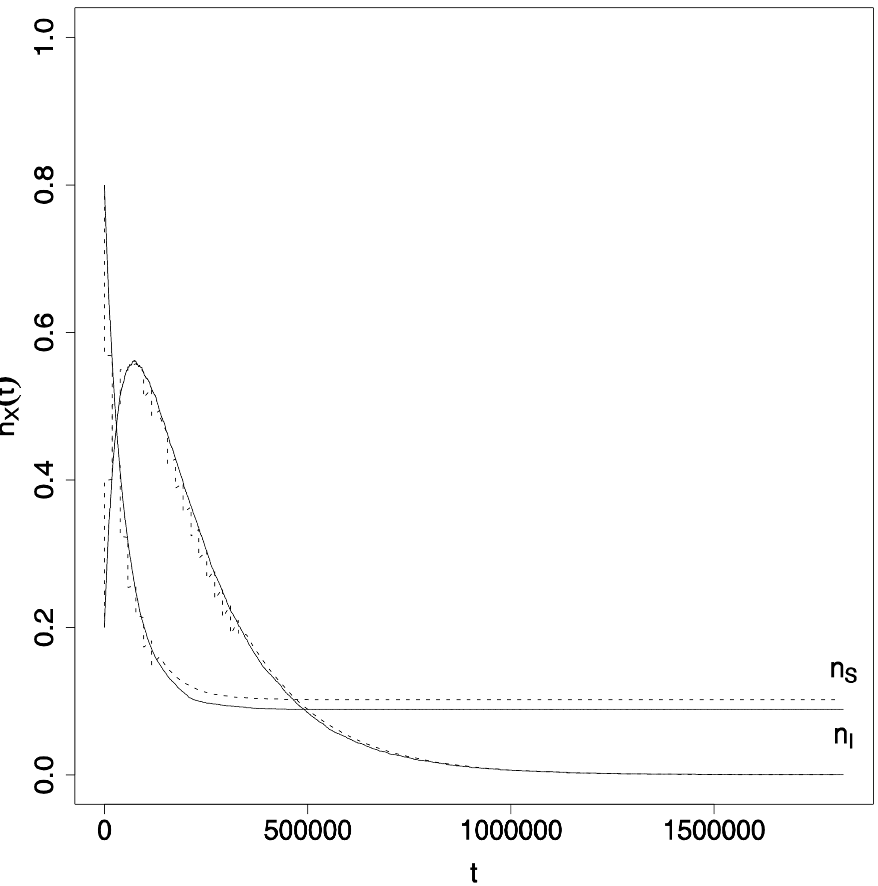
Different N: small N

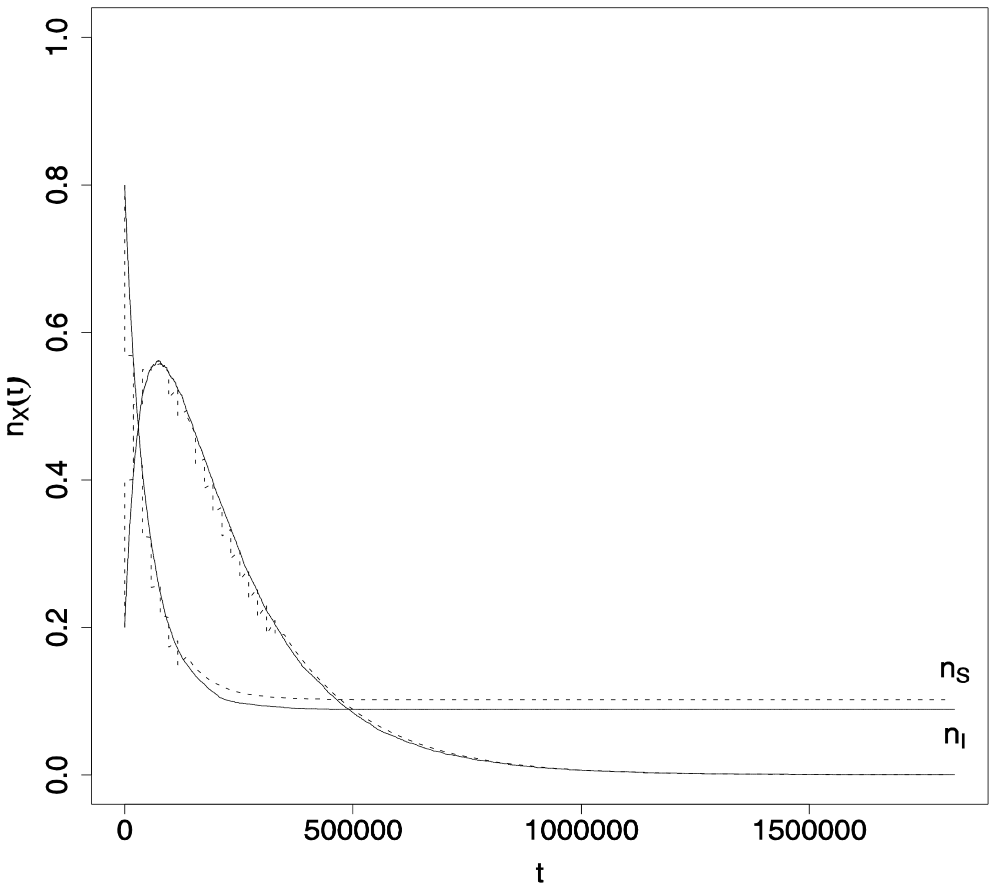


large N (= 10^4) fluctuations irrelvant

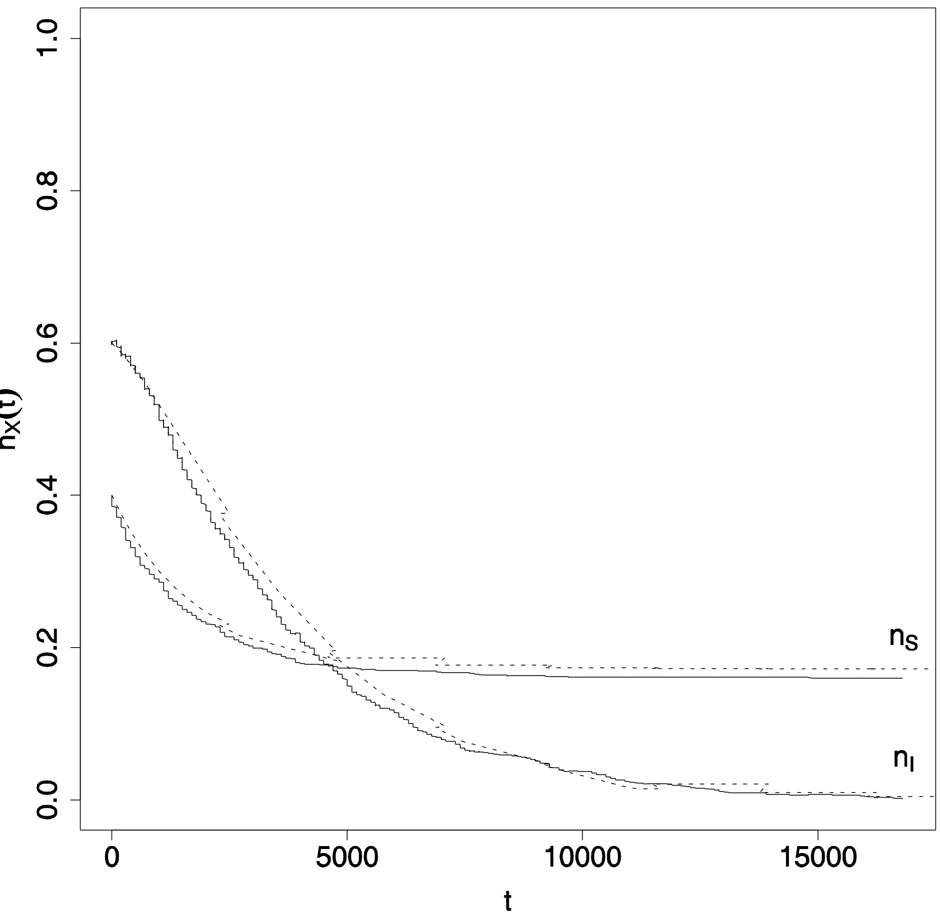


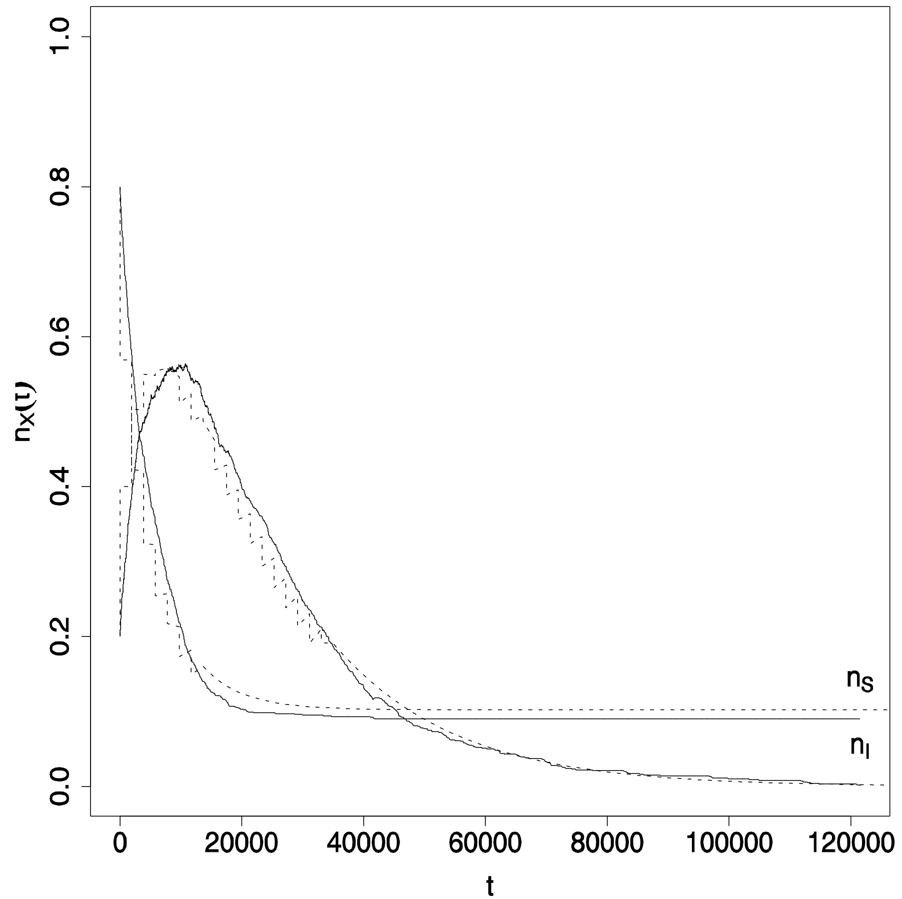
MONTE CARLO SIMULATION





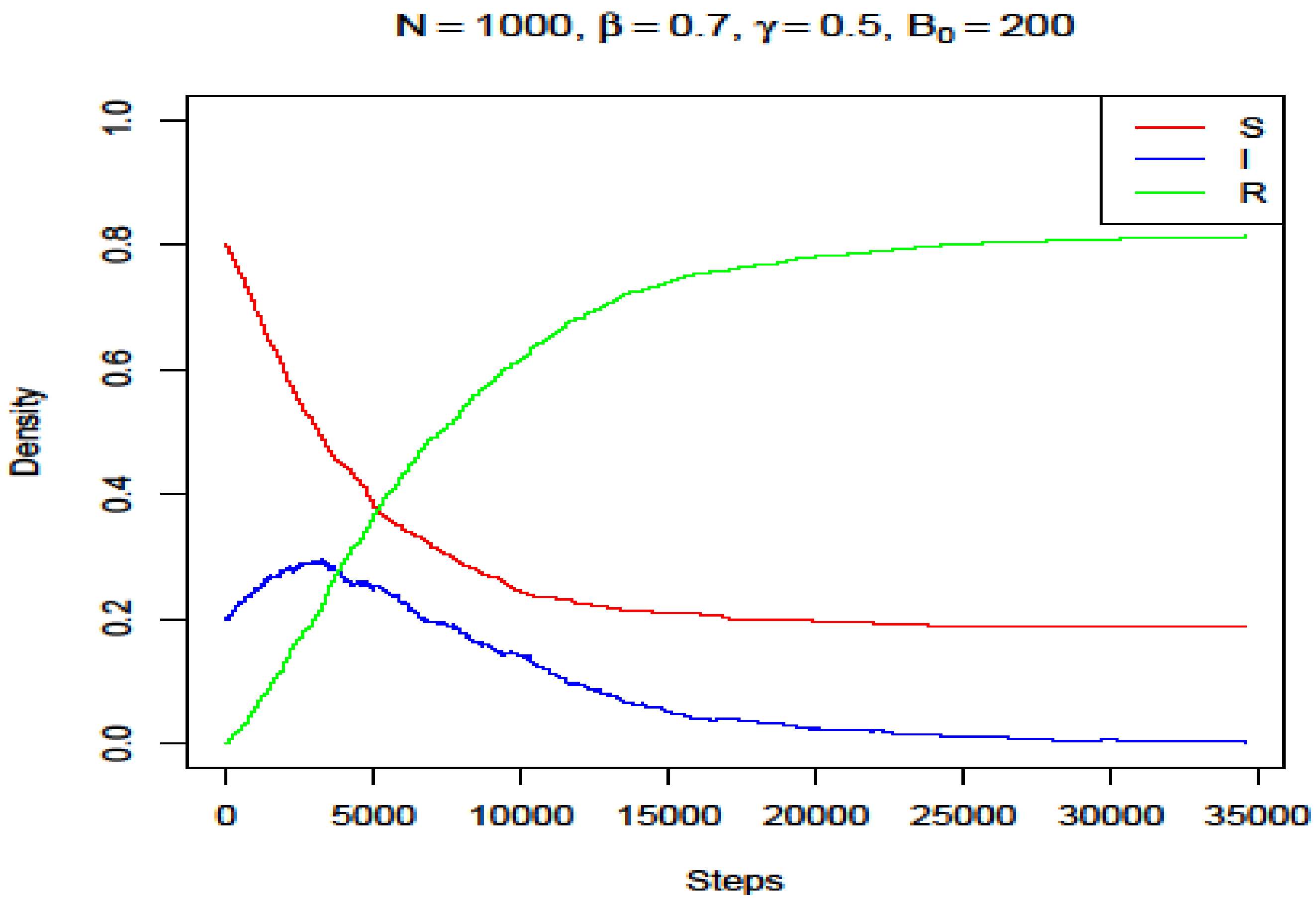
MONTE CARLO SIMULATION



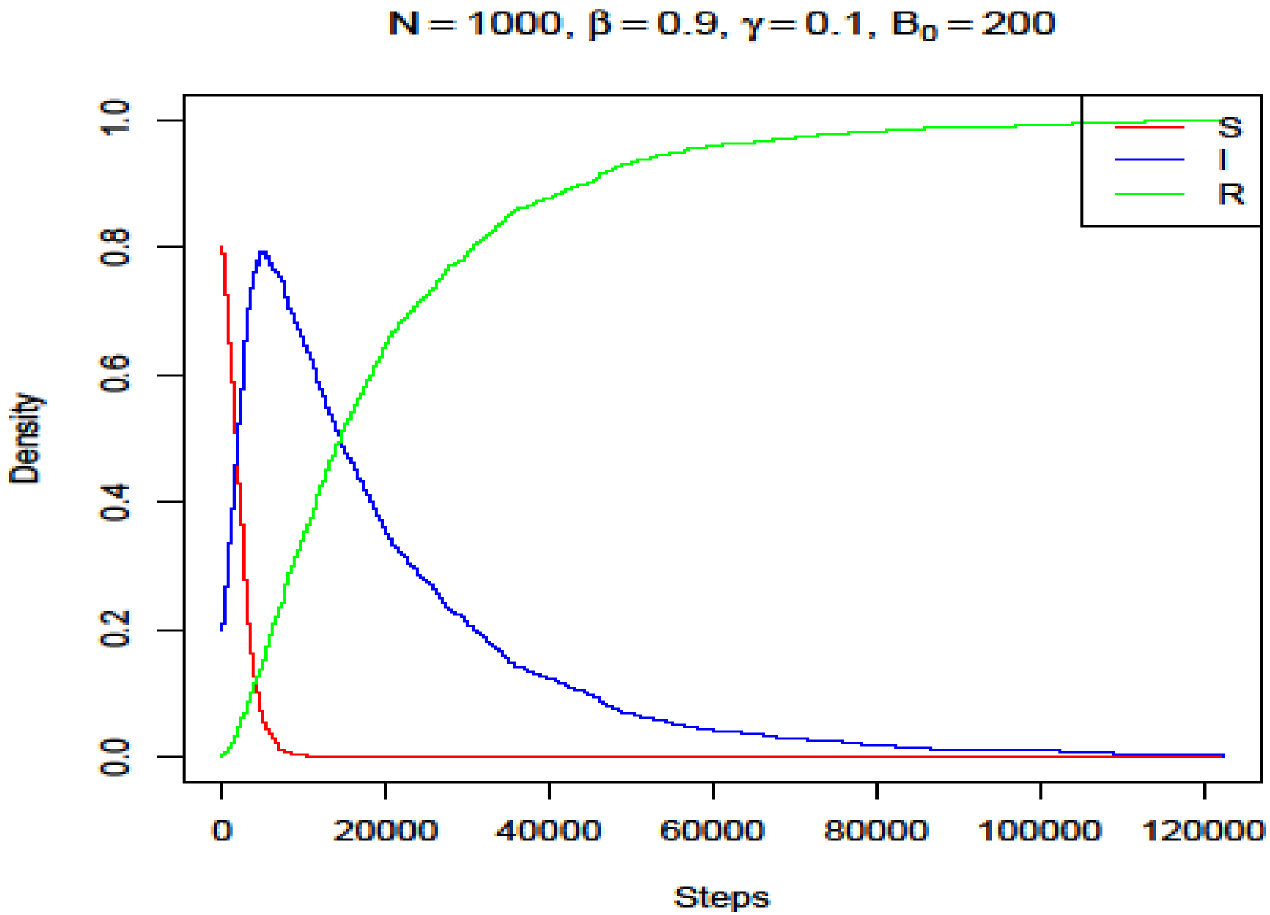


Different parameters (1-D)

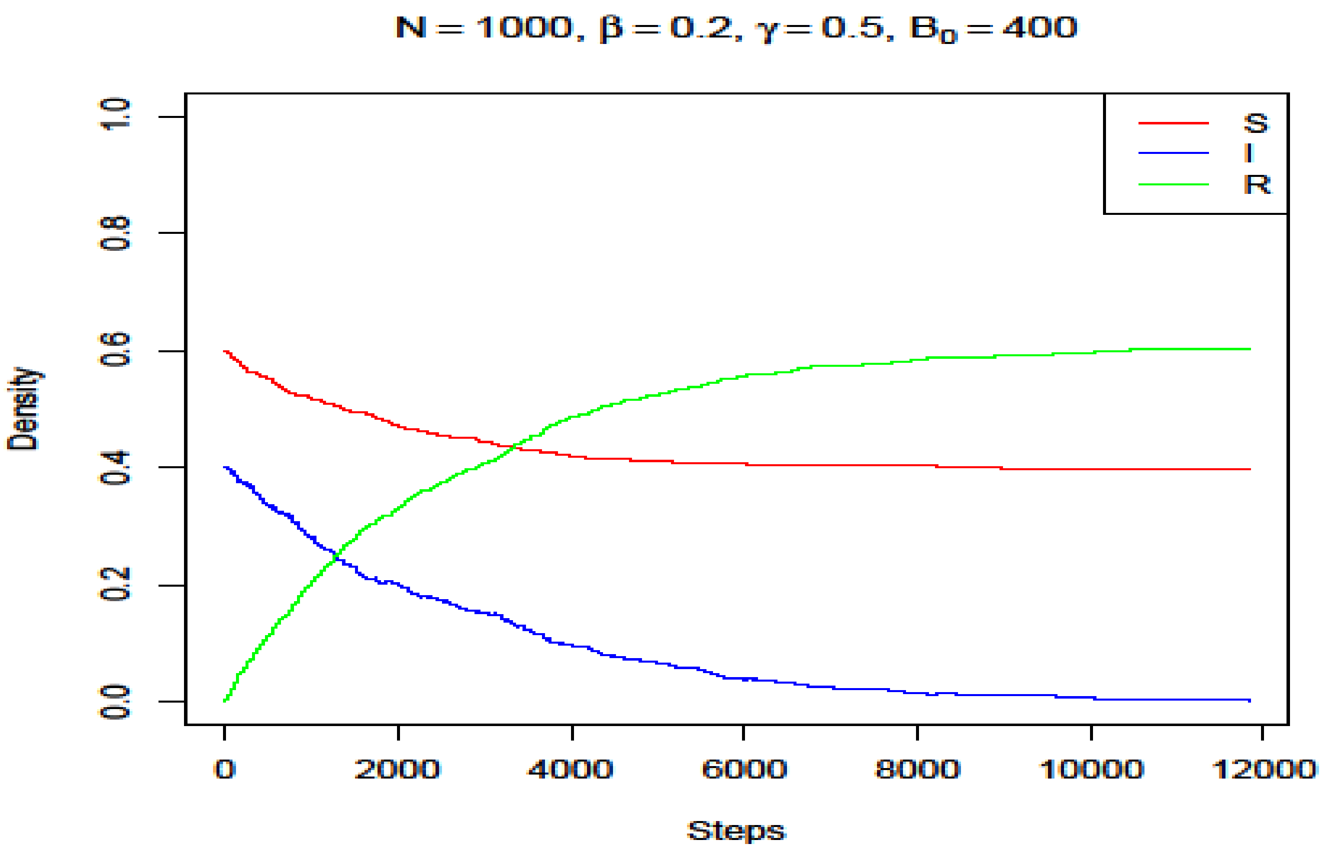
3-D Simulation



3-D Simulation

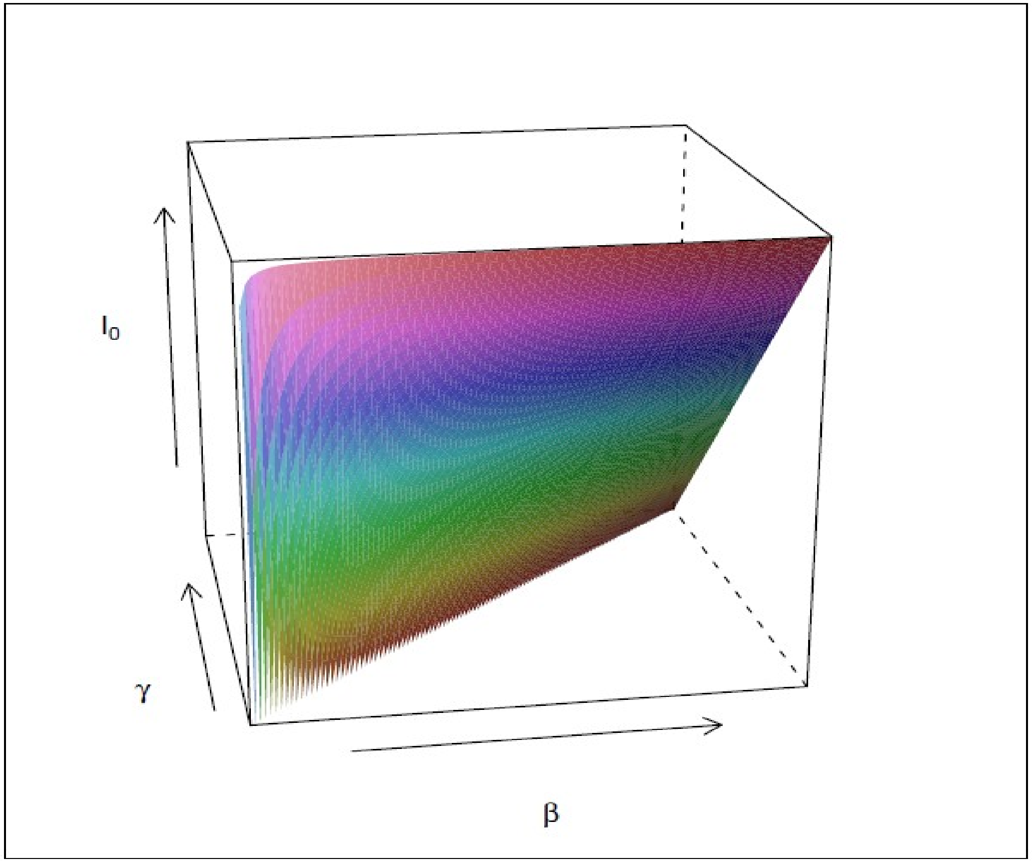


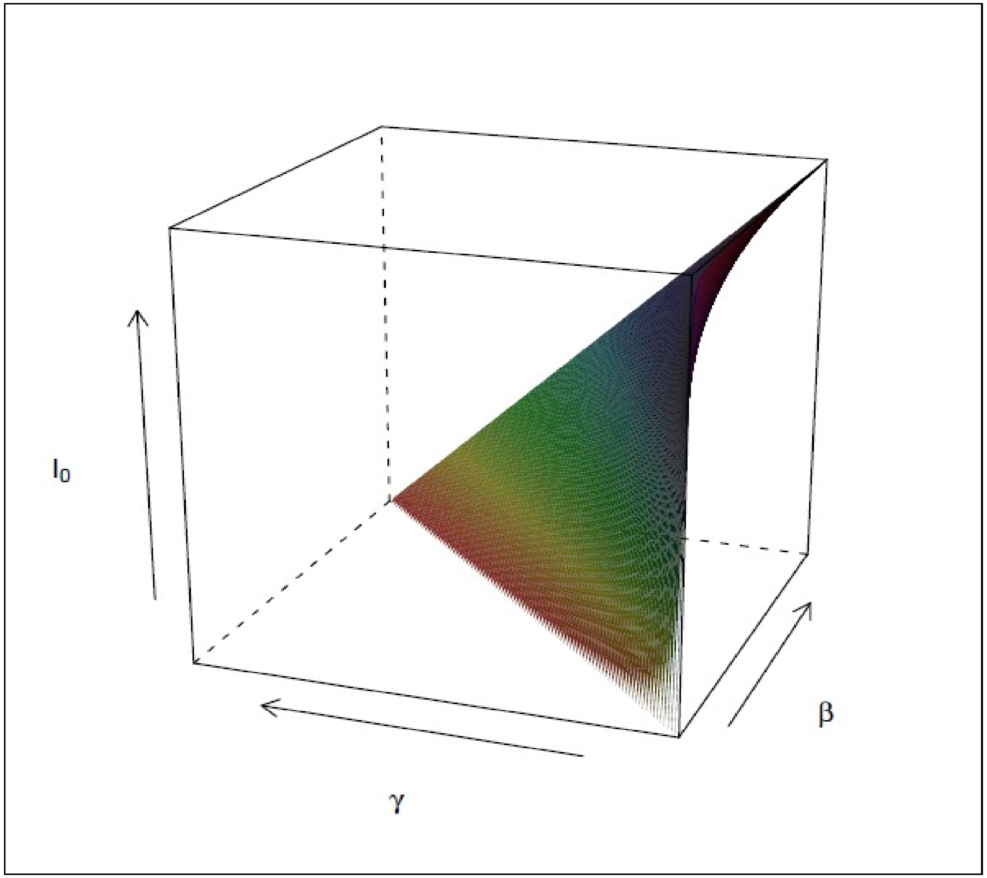
3-D Simulation



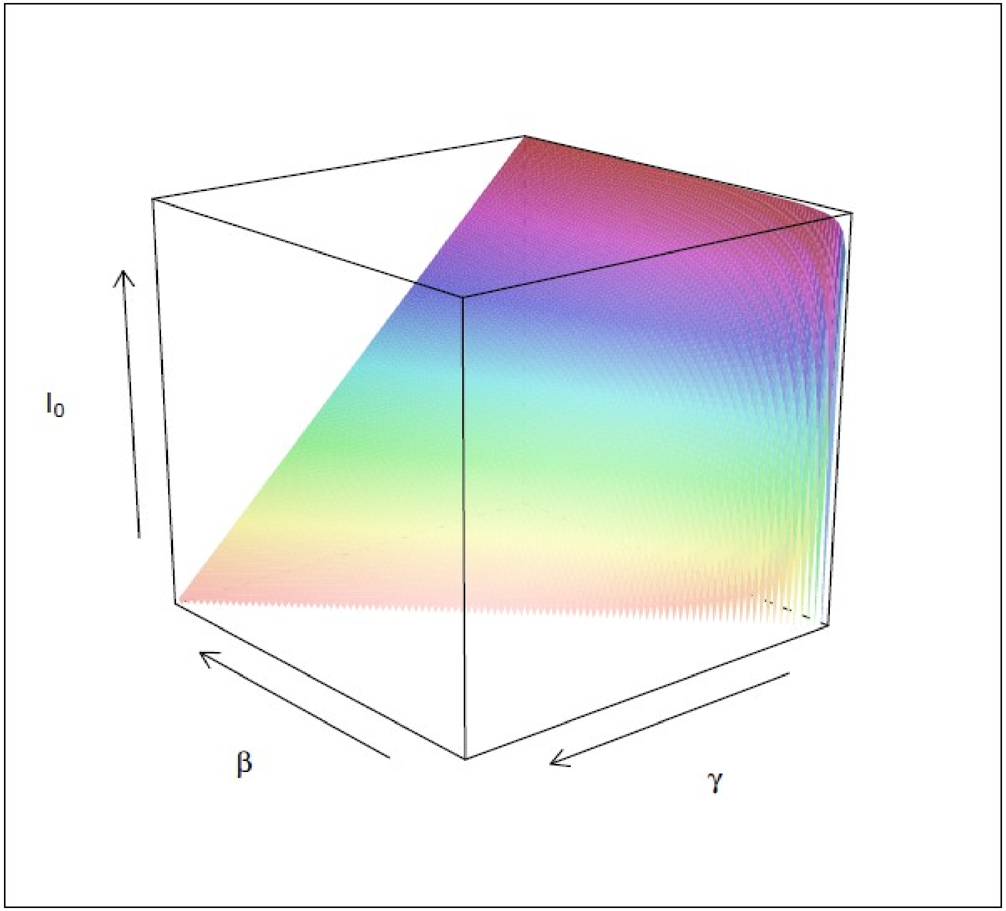
Phase diagram

Initial configuration
Infection rate B
Recovery rate γ





I_0
γ
β



I_0
β
γ

CONCLUSION

- SIMPLIFIED SIR MODEL & STOCHASTIC DYNAMICS
- INCOMPLETE CONTACT & FLUCTUATIONS
- MASTER EQUATION & QUANTUM FORMUL.
- PAULI-OPERATORS & COMMUTE/ANTICOM

EXACT SOLUTION

- ✓ DEPENDING ON PARAMETERS -> DIFFERENT BEHAVIOR
- ✓ INFECTIOUS -> MAXIMUM <-> CONTINUOUS DECAY
- ✓ SIGNIFICANT DIFFERENCE TO MFA
- ✓ COMPARISON <-> MONTE CARLO SIMULATION
- ✓ FLUCTUATIONS IRRELEVANT <-> LARGE POPULATION

PROBLEMS

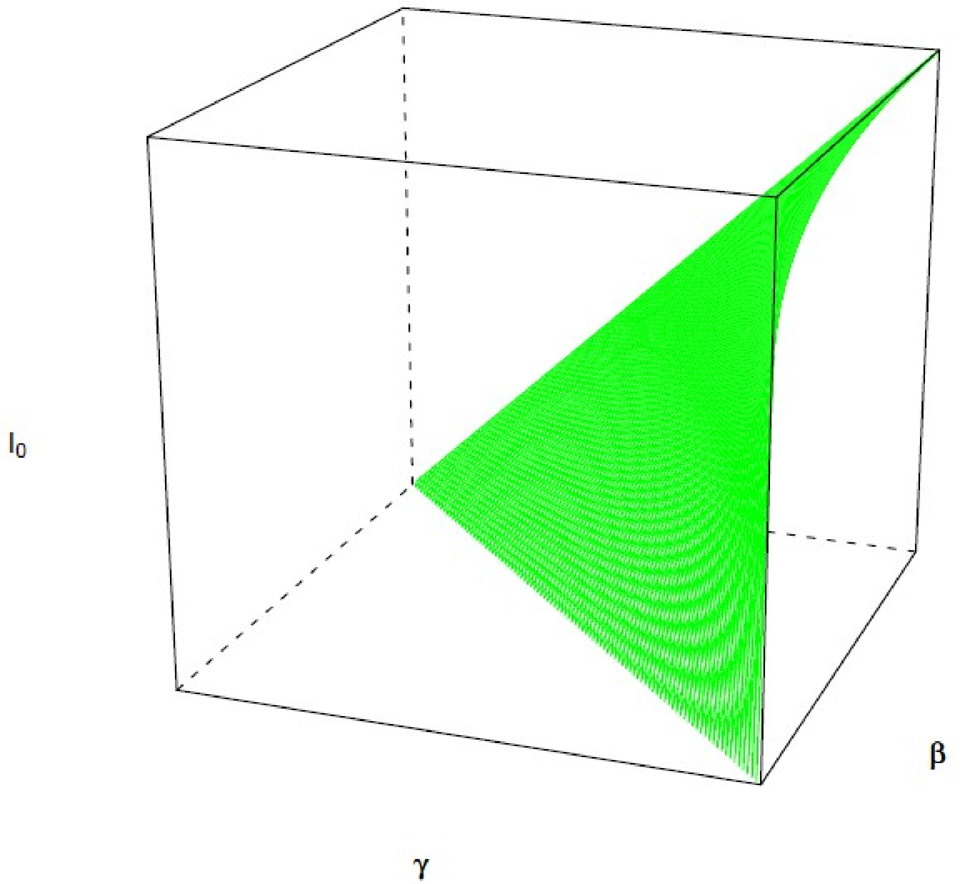
- Hopping of individuals (in particular S)
- Cluster size of I (<BBBBBBB>)
- Empty sites
- Delay time for invection S -> I
- After waiting time R -> S (S generated)
- Immunization
- Higher Dimensions (field theoretical appraoch)
- Critical Dimension
- Scale free networks (higher connectivity)

THE END

Key Words

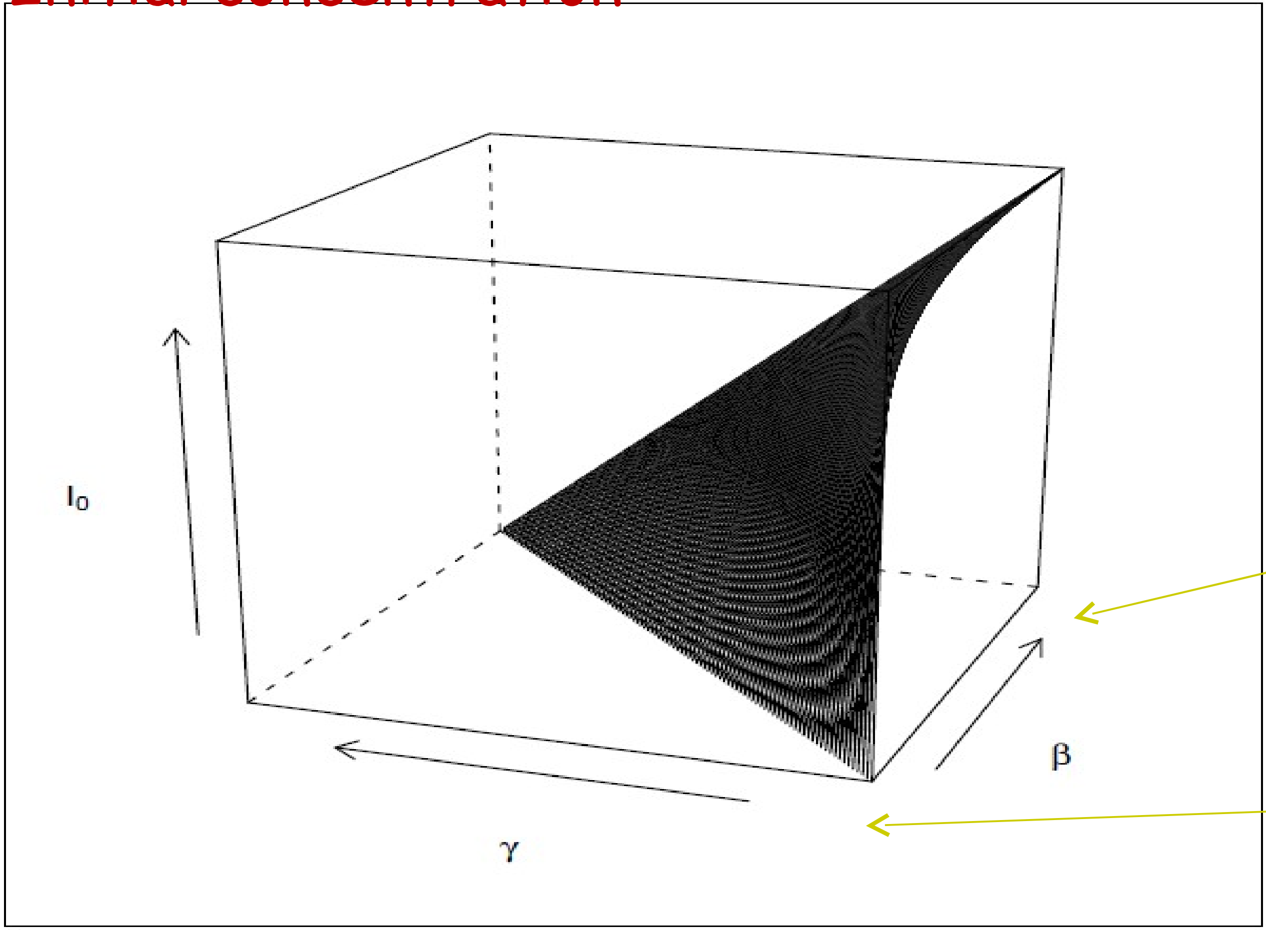
- Uncorrelated random initial conditions: analytical solution
- I population may increase initially before decaying to zero.
- Due to fluctuations, isolated regions of susceptible individuals evolve
- Finite stationary distribution of the S type even for large population size.
- Simulations, Mean-field, Master Equation

‚Phase diagram'



Phase diagram

Initial concentration



Infection rate

Recovery rate